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ABSTRACT

The Kodaira–Thurston manifold is a quotient of a nilpotent Lie group by a cocompact lattice. We compute the family Gromov–Witten invariants which count pseudoholomorphic tori in the Kodaira–Thurston manifold. For a fixed symplectic form the Gromov–Witten invariant is trivial so we consider the twistor family of left-invariant symplectic forms which are orthogonal for some fixed metric on the Lie algebra. This family defines a loop in the space of symplectic forms. This is the first example of a genus one family Gromov–Witten computation for a non-Kähler manifold.

1. Introduction

The enumerative geometry of complex curves in complex manifolds is an old and venerable pursuit [Kle76] which gained momentum in the last fifteen years of the twentieth century for two main reasons. The first of these was Gromov’s paper [Gro85] which explained how to count persistent pseudoholomorphic curves in symplectic manifolds and gave many applications of the existence of pseudoholomorphic curves to symplectic topology. The second was Witten’s recasting [Wit88] of Gromov’s theory in the language of topological sigma models (Gromov–Witten theory, see [RT95, RT97] for a mathematical approach) and the subsequent observation [CdIOGP91] of Candelas–de la Ossa–Green–Parks that string dualities give concrete predictions in enumerative geometry for rational (genus zero) curves in Calabi–Yau three-folds. In the case of higher genus curves, predictions were made by Bershadksy *et al.* [BCOV93]. In many places these predictions have been confirmed by computations: see for example [Zin09] in the genus one case.

The enumerative invariants which emerge have beautiful structural properties, for instance, Bryan and Leung [BL00] computed the Gromov–Witten invariants for the hyper-Kähler sphere of K3 surfaces (a family Gromov–Witten invariant, see §2.3 below) and showed that the generating function for these numbers (the *genus g Gromov–Witten potential*) is

$$\left(q \frac{d}{dq} G_2(q) \right)^g / \Delta(q)$$

where $G_2(q) = -\frac{1}{24} + \sum_{n \geq 1} \sigma_1(n)q^n$, $\sigma_1(n) = \sum_{d|n} d$ is the sum of divisors of n and $\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}$. These are the weight 2 Eisenstein series quasimodular form and the discriminant modular form respectively.

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The physical predictions make use of the special geometry of Calabi–Yau manifolds, but Gromov’s philosophy¹ is that integrability of an almost complex structure is not necessary to have an intricate enumerative theory of holomorphic curves. One might ask what physical predictions hold in the world of non-Kähler symplectic manifolds and whether similarly beautiful formulae can be found for their Gromov–Witten potentials. One large and well-understood class of symplectic manifolds containing non-Kähler examples is the class of symplectic nilmanifolds: compact left-quotients of a nilpotent Lie group equipped with a left-invariant symplectic form. Apart from the higher-dimensional tori these are all non-Kähler [BG88]; the best known is the four-dimensional Kodaira–Thurston manifold [Thu76] which will be the focus of this paper.

At the time of writing, mirror symmetry is not known for the Kodaira–Thurston manifold. In particular, it is not clear to the authors what the genus one partition function of the mirror B-model might be. It would be interesting to compute this and compare it with the Gromov–Witten invariants we calculate here, as a first step towards understanding the conjectures of [BCOV93] in a non-Kähler setting.

The Gromov–Witten invariants of a single symplectic nilmanifold are not very interesting. In many cases (like the Kodaira–Thurston manifold) one can connect a left-invariant symplectic form ω to $-\omega$ along a path of left-invariant symplectic forms. Since Gromov–Witten invariants are invariant under deformations of the symplectic form, if there is a non-zero homology class A with a non-zero Gromov–Witten invariant then there is a J_+ -holomorphic curve representing A (where J_+ is ω -compatible) and a J_- -holomorphic curve representing A (where J_- is $-\omega$ -compatible). Since non-constant J -holomorphic curves have positive ω -area, it follows that $\int_A \omega$ is both positive and negative, a contradiction.

However, if we allow families of symplectic manifolds and of almost complex structures, and if we count holomorphic curves which are holomorphic for some J in the family, then we can still obtain non-zero invariants. These *family Gromov–Witten invariants* have been defined and computed in various places in the literature, for instance, the Bryan–Leung computation mentioned above; we recall the definition in §2.3. Moreover, there are certain natural families of left-invariant symplectic forms on nilmanifolds (the *twistor families*). This paper provides techniques for computing the genus one family Gromov–Witten invariants for the twistor families of symplectic nilmanifolds. We do the complete computation for the Kodaira–Thurston manifold.

DEFINITION 1.1 (Kodaira–Thurston manifold). Let Γ be the group of affine transformations of \mathbf{R}^4 (with coordinates x, y, z, t) generated by three translations (in the y, z and t directions) and the map

$$x \mapsto x + 1 \quad y \mapsto y \quad z \mapsto z + y \quad t \mapsto t.$$

The quotient $\Gamma \backslash \mathbf{R}^4$ is the *Kodaira–Thurston manifold*, K . It is also a left-quotient of a simply-connected nilpotent Lie group N by a cocompact lattice $\Gamma \subset N$; see §4.

The twistor family for K is the circle of symplectic forms

$$\omega_\theta = dt \wedge a_\theta + (dz - xdy) \wedge b_\theta$$

where $a_\theta = \cos \theta dx + \sin \theta dy$ and $b_\theta = -\sin \theta dx + \cos \theta dy$. The second integral homology $H_2(K; \mathbf{Z})$ is of rank 4 and is generated by the homology classes represented by the tori

$$E_{ij} = [\{(x_1, x_2, x_3, x_4) \in K \mid x_i = x_j = 0\}]$$

¹‘What fascinated me even more was the familiar web of algebraic curves in a surface emerging in its full beauty in the softish environment of general (nonintegrable!) almost complex structures. (Integrability had always made me feel claustrophobic.)’ [Gro97].

where $i \in \{1, 2\}$ and $j \in \{3, 4\}$. We also define $E_{i34} \in H_3(K; \mathbf{Z}), i \in \{1, 2\}$ by

$$E_{i34} = [\{(x_1, x_2, x_3, x_4) \in K \mid x_i = 0\}].$$

See § 4.3 for a complete description of the homology and cohomology. We will write

$$[A_{13}, A_{23}, A_{14}, A_{24}]$$

for the homology class $A = \sum A_{ij}E_{ij}$. By Lemma 4.13 we know that if $A \in H_2(K; \mathbf{Z})$ is a homology class represented by a torus then $A_{13}A_{24} = A_{14}A_{23}$. It is helpful to remember that K is a T^2 -bundle over T^2 in two ways:

- (a) there is a projection $(x, y, z, t) \mapsto (x, y)$ whose fibres are Lagrangian with respect to ω_θ for all θ ;
- (b) there is a projection $(x, y, z, t) \mapsto (x, t)$ whose fibres are symplectic for ω_0 .

THEOREM 1.2. *Let K denote the Kodaira–Thurston manifold and W the twistor family of left-invariant symplectic structures on K . If $A = [A_{13}, A_{23}, A_{14}, A_{24}] \in H_2(K; \mathbf{Z})$ is a non-zero homology class and $m = \gcd(A_{13}, A_{23}), n = \gcd(A_{14}, A_{24})$ then*

$$\text{GW}_{1,1}(W, A) = \frac{(m^2 + n^2)\sigma_2(\gcd(m, n))}{\gcd(m, n)^3} (A_{13}E_{134} + A_{23}E_{234}) \in H_3(K; \mathbf{Z})$$

where $\sigma_2(x) = \sum_{d|x} d^2$.

Note that, according to our definition (2.11), family Gromov–Witten invariants should really live in $H_*(K \times W; \mathbf{Z})$. In our situation they all have the form $C \otimes [\star]$ where $[\star] \in H_0(W, \mathbf{Z})$ is the homology class of a point. This is because each connected component of the moduli space of tori which are J_θ -holomorphic for some $\theta \in W$ consists of tori which are J_θ -holomorphic for a fixed θ .

We would like to stress that the moduli spaces of pseudoholomorphic tori we consider are, unusually, odd-dimensional. This can be understood as follows. The index of the Fredholm problem for counting (unmarked) tori in a four-manifold with $c_1 = 0$ is zero. In each space of ω_θ -compatible almost complex structures there is a codimension one ‘wall’ of almost complex structures where the kernel of the Fredholm problem is one-dimensional and there is a one-dimensional cokernel. For all the moduli spaces which contribute to the Gromov–Witten invariant, our one-dimensional family J_θ is transverse to that wall, and so the (one-dimensional) moduli space is regular from the point of view of family Gromov–Witten theory.

Let us spell out the geometric content of this theorem for the specific family J_θ . Since the Gromov–Witten class is in $H_3(K; \mathbf{Z})$, it detects holomorphic curves intersecting a loop in K . Let L be a loop in K and let $A \in H_2(K; \mathbf{Z})$ be as in the statement of the theorem. There is a unique almost complex structure $J_\theta \in W$ (see Lemma 5.7) for which there are J_θ -holomorphic tori representing A . For this J_θ , the pseudoholomorphic tori intersect L at

$$\frac{m^2 + n^2}{\gcd(m, n)^3} \sigma_2(\gcd(m, n)) (A_{13}E_{134} + A_{23}E_{234}) \cap [L]$$

points (counted with multiplicity and signs). The complex structure on the domain torus is allowed to vary but when $m \neq 0$ it is actually constant over each component of the moduli space (Lemma 5.7 again).

Two obvious classes containing holomorphic tori are E_{13} and E_{14} .

Example 1.3. The class E_{14} is represented by the fibres of the projection

$$(x, y, z, t) \mapsto (x, t).$$

These are J_0 -holomorphic tori and all J_0 -holomorphic tori have this form. In this case, $m = 0$ and the j -invariant of these fibres varies in a loop. In fact, these tori are all irregular and do not contribute to the Gromov–Witten invariant, see Lemma 7.6.

Example 1.4. The class E_{13} is represented by sections of the projection

$$(x, y, z, t) \mapsto (x, t).$$

There is an S^1 -family of $J_{\pi/2}$ -holomorphic sections which intersect a loop L in $[L] \cap [E_{134}]$ points (with multiplicity). In fact, we will not deal with these sections explicitly because they do not correspond to *reduced Lie algebra homomorphisms* (see Definition 4.16). Instead we will apply a diffeomorphism of K to move to the homology class $E_{13} + E_{23}$ without affecting the Gromov–Witten computation, using Lemma 4.14 and (4.6). The $J_{\pi/2}$ -holomorphic tori representing E_{13} become $J_{\pi/4}$ -holomorphic tori representing $E_{13} + E_{23}$.

Outline of proof of Theorem 1.2

Theorem 1.2 is proved by reducing the problem to the enumeration of certain homomorphisms $\mathbf{Z}^2 \rightarrow \pi_1(K)$. Let $u : T^2 \rightarrow K$ be a J -holomorphic torus where J is a left-invariant almost complex compatible with ω . We perform the following steps.

Take a lift of u to the universal covers $\tilde{u} : \mathbf{R}^2 \rightarrow N$ and compare it with the unique Lie group homomorphism $H : \mathbf{R}^2 \rightarrow N$ extending the induced map $\pi_1(u) : \mathbf{Z}^2 \rightarrow \pi_1(K)$ on fundamental groups.

The map $(p, q) \mapsto H(p, q)^{-1}\tilde{u}(p, q)$ is then bounded (Corollary 3.13) and we are interested in its logarithm $C : \mathbf{R}^2 \rightarrow \mathfrak{n}$ where \mathfrak{n} is the Lie algebra of N .

The Cauchy–Riemann equations for C imply that C satisfies a second order elliptic system of equations. In Proposition 4.1 we show that this system separates into equations for which the Hopf maximum principle holds [GT01, Theorem 3.1]. We apply this to prove that C is constant. Hence, for all left-invariant J which are compatible with a left-invariant symplectic form, all J -holomorphic tori are of the form ve^C , where v comes from a Lie algebra homomorphism and C is a constant.

Another maximum principle allows us to study the linearised problem. We prove that all moduli spaces are cut out cleanly by the Cauchy–Riemann operator (Theorem 7.1). In § 7.3 we determine which tori are regular and, for non-regular tori, we write down an explicit section of the obstruction bundle, proving that these do not contribute to the Gromov–Witten invariant. In § 7.4, we determine orientations on the moduli spaces.

It remains to count the tori. By applying an automorphism of Γ we reduce ourselves to considering only non-zero homology classes

$$[A_{13}, A_{23}, A_{14}, A_{24}]$$

where $A_{13} = A_{23}$ and $A_{14} = A_{24}$ (Lemma 4.14), for which we can further assume that the homomorphism H has a particularly simple form (Lemma 4.17). This enables us to enumerate the tori and to understand the homology classes represented by the evaluation maps (§ 8).

Generalisations

We carry out the full calculation only for the Kodaira–Thurston manifold K but we formulate the problem for two-step symplectic nilmanifolds in general. The universal cover of K is the only nonabelian symplectic two-step nilpotent Lie group in dimension four. For examples in higher dimensions the main difference is that the Cauchy–Riemann equations are a more complicated elliptic system (with more serious nonlinearities and coupling) and it becomes harder to apply the maximum principle. Our methods can be extended to a limited range of homology classes in certain higher-dimensional examples; the ones constructed in [CFdL85]. For k -step nilmanifolds with $k \geq 3$ the equations are even harder to deal with.

Outline of the paper

In § 2 we explain the classical computations of genus one Gromov–Witten invariants for two-tori and for higher-dimensional tori. We also define the family Gromov–Witten invariants.

In § 3 we introduce two-step nilpotent Lie groups and their twistor families W of symplectic structures. We also write down the Cauchy–Riemann equations for the logarithm of a J -holomorphic torus (J compatible with some $\omega \in W$).

In § 4 we review the Kodaira–Thurston manifold and its basic properties. In particular we show that all tori are descended from right-translates of Lie algebra homomorphisms (Proposition 4.1).

In § 5 we describe the moduli spaces of holomorphic tori.

In § 6 we compute the automorphism groups of the unmarked holomorphic tori in the Kodaira–Thurston manifold.

In § 7 we study the linearised problem, including checking regularity and computing obstruction bundles and orientations.

In § 8 we complete the proof of Theorem 1.2.

Notation

For brevity in our coordinate expressions we will sometimes use the following convention to denote antisymmetrisation of indices:

$$A_{[ij]} = A_{ij} - A_{ji}.$$

For example,

$$\partial_{[p}X_i\partial_{q]}Y_j = \partial_pX_i\partial_qX_j - \partial_qX_i\partial_pX_j.$$

2. Background

We begin by giving an overview of genus one Gromov–Witten theory for the (twistor families of) complex tori, which are precisely the nilmanifolds arising from an abelian Lie group. In doing so we build up in embryonic form many of the ideas we need for the nonabelian case.

2.1 The 2-torus

The space whose genus one Gromov–Witten invariants are easiest to calculate is the two-torus, T^2 . Let $\Lambda_\tau \cong \mathbf{Z}^2$ denote the \mathbf{Z} -lattice in \mathbf{C} spanned by the vectors 1 and $\tau = \tau_1 + i\tau_2$, $\tau_2 > 0$. Let $\Sigma_\tau = \Lambda_\tau \backslash \mathbf{C}$ denote the corresponding complex torus.

LEMMA 2.1. *Any non-constant holomorphic map $f : \Sigma_{\tau'} \rightarrow \Sigma_\tau$ is a covering map (unbranched).*

This is clear because branching increases genus by the Riemann–Hurwitz formula. Therefore, counting holomorphic maps $\Sigma_{\tau'} \rightarrow \Sigma_\tau$ of degree $\ell \geq 1$ amounts to counting holomorphic covering spaces of Σ_τ or, equivalently, sublattices $\Lambda_{\tau'} \subset \Lambda_\tau$ of index ℓ modulo the action of $\text{SL}(2, \mathbf{Z})$ which reparametrises the domain.

LEMMA 2.2. *There are $\sigma_1(\ell) = \sum_{d|\ell} d$ sublattices of Λ_τ of index $\ell \geq 1$, modulo the action of $\text{SL}(2, \mathbf{Z})$.*

Proof. This is standard and we reproduce the argument only for comparison later. A sublattice of index ℓ is specified by a homomorphism $\mathbf{Z}^2 \rightarrow \mathbf{Z}^2$ whose image has index ℓ , that is, a two-by-two integer matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant ℓ . The $\text{SL}(2, \mathbf{Z})$ -action is just right multiplication. Using this right action one can perform the Euclidean algorithm on c and d to ensure that c vanishes. Similarly, one can ensure that $0 \leq b < a$. Now for each $d \mid \ell$ there are d possible matrices up to the action of $\text{SL}(2, \mathbf{Z})$:

$$\begin{pmatrix} d & b \\ 0 & \ell/d \end{pmatrix}, \quad b = 0, \dots, d - 1. \quad \square$$

We define the moduli space

$$\mathcal{M}_{1,1}(\Sigma_\tau, \ell[\Sigma_\tau])$$

to consist of equivalence classes of pairs (u, z) where $u : \Sigma_{\tau'} \rightarrow \Sigma_\tau$ is a holomorphic map of degree $\ell \geq 1$ and $z \in \Sigma_{\tau'}$ is a point. The equivalence relation equates (u, z) with $(u \circ \varphi^{-1}, \varphi(z))$ for any holomorphic automorphism $\varphi : \Sigma_{\tau'} \rightarrow \Sigma_{\tau'}$ for which $u \circ \varphi^{-1} = u$. This has an evaluation map

$$\text{ev} : \mathcal{M}_{1,1}(\Sigma_\tau, \ell[\Sigma_\tau]) \rightarrow \Sigma_\tau, [u, z] \mapsto u(z).$$

LEMMA 2.3. *The map ev has degree $\sigma_1(\ell)$.*

Proof. We have seen that there are $\sigma_1(\ell)$ tori in the moduli space and that all of these are ℓ -fold covering spaces of Σ_τ . Fix one such covering map. If $x \in \Sigma_\tau$ then the preimages of this point under this covering map are all equivalent by the action of the deck group, which acts by holomorphic automorphisms preserving the covering. Hence they represent the same element in the moduli space $\mathcal{M}_{1,1}(\Sigma_\tau, \ell[\Sigma_\tau])$, so the degree of the evaluation map is just the number of covering spaces, $\sigma_1(\ell)$. \square

These curves are all regular in the sense of Gromov–Witten theory; the cokernel of the linearisation is just the quotient of the Dolbeault cohomology group $H^{0,1}(\Sigma_{\tau'}; u^*T\Sigma_\tau)$ by the image of $H^{0,1}(\Sigma_{\tau'}; T\Sigma_{\tau'})$ under pushforward $du : T\Sigma_{\tau'} \rightarrow T\Sigma_\tau$ (this quotient corresponds to allowing τ' to vary):

$$H^{0,1}(\Sigma_{\tau'}; u^*T\Sigma_\tau) / H^{0,1}(\Sigma_{\tau'}; T\Sigma_{\tau'}) \cong H^{0,1}(\Sigma_{\tau'}; T\Sigma_{\tau'}) / H^{0,1}(\Sigma_{\tau'}; T\Sigma_{\tau'}) = 0.$$

The one-point Gromov–Witten class of degree $\ell \geq 1$, genus one curves through a point of T^2 is the pushforward under ev of the fundamental class of the moduli space and is therefore given by

$$\text{GW}_{1,1}(T^2, \ell[T^2]) = \sigma_1(\ell)[T^2].$$

2.2 The $2n$ -torus

The situation for the $2n$ -torus is similar but some of the features it presents are new and will be developed in a more general context later in the paper. For a start, a generic abelian variety contains no closed holomorphic curves, so we know that the Gromov–Witten invariants vanish. However, holomorphic curves persist if we take a family of complex structures and look for curves which are holomorphic with respect to one of the complex structures. This phenomenon,

made precise in § 2.3, is familiar from the case of K3 surfaces [BL00], where an elliptically-fibred K3 contains elliptic curves through every point which disappear if one perturbs the complex structure, but which persist in families which are deformations of the hyper-Kähler two-sphere of complex structures. The Gromov–Witten invariants which count curves representing some second homology class $A \neq 0$ which are J -holomorphic for some J in a fixed finite-dimensional, compact, oriented family are called *family Gromov–Witten invariants*. Note that these J must all be tamed by symplectic forms in order to achieve Gromov compactness, but that the taming form (and even its cohomology class) might depend on J .

The following definition is a natural generalisation of this hyper-Kähler sphere to examples which are not hyper-Kähler.

DEFINITION 2.4 (Twistor family). Let g be an inner product on the vector space \mathbf{R}^{2n} and let \mathfrak{o} be an orientation. The *twistor family* of complex structures is the space of \mathfrak{o} -positive orthogonal complex structures

$$W = \{\psi \in \text{GL}^+(\mathbf{R}^{2n}) \mid \psi^2 = -\text{Id}, g(\psi X, \psi Y) = g(X, Y) \forall X, Y \in \mathbf{R}^{2n}\}.$$

Note that $W \cong \text{SO}(2n)/U(n)$ since $\text{SO}(2n)$ acts transitively on W with stabiliser $U(n)$. Each $\psi \in W$ gives rise to a two-form

$$\omega_\psi(X, Y) := -g(X, \psi Y)$$

and to a bi-invariant Kähler structure (Ω_ψ, J_ψ) on the torus $\mathbf{Z}^{2n} \setminus \mathbf{R}^{2n}$.

The genus one family Gromov–Witten invariants of the twistor family of $2n$ -tori are easy to compute. Let us write $\text{GW}_{1,k}(W, A) \in H_*((T^{2n})^k \times W; \mathbf{Z})$ for the homology class of the evaluation pseudocycle for genus one curves representing the homology class $A \neq 0$ which are J_ψ -holomorphic for some $\psi \in W$ (see § 2.3 for definitions).

Note first that if $\varphi \in \text{SL}(2n, \mathbf{Z})$ is a matrix then φ^*W is the twistor family of φ^*g . Since g and φ^*g can be connected by a path of inner products we see that φ^*W and W are isotopic as families of complex structures. More importantly, the families $\{\omega_\psi\}_{\psi \in W}$ and $\{\omega_{\varphi^*\psi}\}_{\psi \in W}$ of taming symplectic forms are isotopic. The family Gromov–Witten invariants are equivariant under diffeomorphisms φ , so that

$$\varphi_* \text{GW}_{1,k}(W, A) = \text{GW}((\varphi^{-1})^*(W), \varphi_*A),$$

and also unchanged by deformations through tamed families; hence we see that

$$\text{GW}_{1,k}(W, \varphi_*A) = \varphi_* \text{GW}_{1,k}(W, A).$$

Homology classes represented by two-tori are specified by homomorphisms $\rho : \mathbf{Z}^2 \rightarrow \mathbf{Z}^{2n}$ on the level of fundamental groups; we write $[\rho]$ for the corresponding homology class. Two such homomorphisms ρ and ρ' define the same homology class if and only if $\Lambda^2 \rho = \Lambda^2 \rho'$, that is, if all two-by-two minors of ρ and ρ' agree. Acting on the left by an element of $\text{SL}(2n, \mathbf{Z})$, one can assume that

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

The counting of such homomorphisms up to the reparametrisation action of $\mathrm{SL}(2, \mathbf{Z})$ is again performed by the function $\sigma_1(\ell)$ where ℓ is the only nonvanishing two-by-two minor, so $\ell = \ell([\rho])$ is the divisibility of the homology class (the only invariant of the $\mathrm{SL}(2n, \mathbf{Z})$ -action).

Each homomorphism $\rho : \mathbf{Z}^2 \rightarrow \mathbf{Z}^{2n}$ actually defines a 2-plane $\Pi(\rho) \subset \mathbf{R}^{2n}$ which is J_ψ -holomorphic for ψ in a subvariety $W(\rho) \subset W$. This subvariety is diffeomorphic to $\mathrm{SO}(2n-2)/U(n-1)$. Each such 2-plane descends to a J_ψ -holomorphic genus one curve $v : T^2 \rightarrow T^{2n}$ in T^{2n} .

LEMMA 2.5. *All J_ψ -holomorphic curves in the homology class $[\rho]$ are affine translates of v .*

Proof. This is Liouville’s theorem. Let (a, b) be conformal coordinates on T^2 and let $\tilde{u} : \mathbf{R}^2 \rightarrow \mathbf{R}^{2n}$ denote the lift of an arbitrary J_ψ -holomorphic curve u in the homology class $[\rho]$ to the universal cover. The Cauchy–Riemann equations are linear and the complex structure is constant:

$$\partial_b \tilde{u} = \psi \partial_a \tilde{u};$$

hence in each coordinate of \mathbf{R}^{2n} the Laplacian $\Delta \tilde{u}_i = \partial_a^2 \tilde{u}_i + \partial_b^2 \tilde{u}_i = 0$. We also have $\Delta h = 0$ where h is the inclusion of $\Pi \subset \mathbf{R}^{2n}$.

It is easy to see that u and v are homotopic and hence $\tilde{u} - h$ is bounded and harmonic. By the maximum principle it is constant which proves that the two curves are affine translates of one another. \square

This implies that there is precisely one J_ψ -holomorphic curve in the class $[\rho] \neq 0$ through each point for any $\psi \in W(\rho)$. Therefore, the family Gromov–Witten invariant is

$$\mathrm{GW}_{1,1}(W, [\rho]) = \sigma_1(\ell([\rho]))[T^{2n}] \otimes [W(\rho)] \in H_*(T^{2n} \times W; \mathbf{Z}).$$

2.3 Family Gromov–Witten invariants

Family Gromov–Witten invariants have been defined, calculated and used in many places in the literature including [Bus05, Kęd04, KMPS10, LO08, Lee04, Lee06, LL05, LP05, Lu98, MP07, Sei99]. Below, we explain the special cases we require. For more details see [MS04, RT97].

2.3.1 *Setting.* We first set up some notation and assumptions for the rest of this section.

ASSUMPTION 2.6. Let X be a compact, connected, smooth, oriented manifold. Let Ω denote the space of symplectic forms on X . Let B be a compact, smooth, oriented manifold and ω be a family of symplectic structures on X , that is, a map $\omega : B \rightarrow \Omega$. We will assume that $(X, \omega(b))$ is a symplectically aspherical symplectic manifold with $c_1 = 0$. We will denote by $A \in H_2(X; \mathbf{Z})$ a non-zero homology class.

Remark 2.7. Note that the Kodaira–Thurston manifold, which is our main example, is a quotient of a nilpotent Lie group N by a cocompact discrete subgroup equipped with a left-invariant symplectic form. All such examples are aspherical and satisfy $c_1 = 0$. We will specify the family ω in Definition 3.4.

Let \mathcal{J} denote the space of almost complex structures on X .

DEFINITION 2.8. A family of ω -compatible almost complex structures J is a map

$$J : B \rightarrow \mathcal{J}$$

such that $J(b)$ is $\omega(b)$ -compatible for all $b \in B$. We will write $\mathcal{J}(B)$ for the space of families of ω -compatible almost complex structures.

2.3.2 *Complex structures on the torus.* Let (p, q) be coordinates on \mathbf{R}^2 and

$$j_\tau = \begin{pmatrix} -\frac{\tau_1}{\tau_2} & -\left(\frac{\tau_1^2}{\tau_2} + \tau_2\right) \\ \frac{1}{\tau_2} & \frac{\tau_1}{\tau_2} \end{pmatrix}$$

be a complex structure, where $\tau = \tau_1 + i\tau_2$ is an element of the upper half-plane $\mathbf{H} \subset \mathbf{C}$. This descends to the quotient $\mathbf{Z}^2 \backslash \mathbf{R}^2$ and gives a complex torus Σ_τ . Equivalently, we can consider coordinates (a, b) on \mathbf{R}^2 with the complex structure

$$j_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and divide by the lattice $\Lambda_\tau = \langle 1, \tau \rangle$, i.e.

$$\Sigma_\tau = (\mathbf{Z}^2 \backslash \mathbf{R}^2, j_\tau) \stackrel{\Phi_\tau}{\cong} (\Lambda_\tau \backslash \mathbf{R}^2, j_i)$$

where the diffeomorphism is

$$\Phi_\tau \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

or

$$\Phi_\tau^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\tau_1}{\tau_2} \\ 0 & \frac{1}{\tau_2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}.$$

2.3.3 *Moduli space of pseudoholomorphic maps.*

DEFINITION 2.9. Given a non-zero homology class $A \in H_2(X; \mathbf{Z})$ and a family of compatible almost complex structures $J \in \mathcal{J}(B)$, define the space

$$\mathcal{M}_{1,1}(A, J)$$

consisting of equivalence classes of quadruples (u, τ, z, b) where $\tau \in \mathbf{H}$, $b \in B$, $z \in \mathbf{Z}^2 \backslash \mathbf{R}^2$ is a marked point and u is a $(j_\tau, J(b))$ -holomorphic map

$$u : \mathbf{Z}^2 \backslash \mathbf{R}^2 \rightarrow X, \quad d_z u(j_\tau v) = J(b) d_z u(v)$$

such that $u_*([\mathbf{Z}^2 \backslash \mathbf{R}^2]) = A$. We say that two quadruples are equivalent $(u, \tau, z, b) \sim (u', \tau', z', b')$ if there exists a diffeomorphism $\varphi : \mathbf{Z}^2 \backslash \mathbf{R}^2 \rightarrow \mathbf{Z}^2 \backslash \mathbf{R}^2$ such that

$$b = b', \quad u' = u \circ \varphi^{-1}, \quad z' = \varphi(z), \quad j_{\tau'} = \varphi^* j_\tau.$$

Note that, since $g = 1$, $c_1 = 0$ and there is one marked point, the expected dimension of this moduli space is $\dim(B) + 2$. There is also a well-defined evaluation map

$$\text{ev} : \mathcal{M}_{1,1}(A, J) \rightarrow X \times B, \quad \text{ev}(u, z, \tau, b) = (u(z), b).$$

Remark 2.10 (Compactness). To compactify the moduli space of genus one curves with one marked point we consider the moduli space $\overline{\mathcal{M}}_{1,1}(A, J)$ of genus one stable maps to X with one marked point. If X is symplectically aspherical then the domain of a stable map in $\overline{\mathcal{M}}_{1,1}(A, J)$ is necessarily an irreducible smooth genus one curve, so the moduli space $\mathcal{M}_{1,1}(A, J)$ is already compact. To see this, note that if the domain is nodal then there is a sphere component with precisely two special points (that is, points which are either marked or nodal). By stability, this sphere component must be non-constant, but since we are assuming X to be symplectically aspherical a stable map can have no non-constant sphere components.

If J is regular (see Definition 2.13 below) then $\mathcal{M}_{1,1}(A, J)$ is a smooth, compact, oriented, $(\dim B + 2)$ -dimensional manifold and we can define the Gromov–Witten invariant to be the homology class

$$\text{GW}_{1,1}(\omega, A) = \text{ev}_*([\mathcal{M}_{1,1}(A, J)]) \in H_*(X \times B; \mathbf{Z}) \tag{2.11}$$

which is equivariant under diffeomorphisms φ of X :

$$\varphi_* \text{GW}_{1,k}(\omega, A) = \text{GW}_{1,k}((\varphi^{-1})^*\omega, \varphi_*A). \tag{2.12}$$

To define genus one Gromov–Witten invariants properly [RT97] one must study moduli spaces of solutions to the perturbed Cauchy–Riemann equations for a suitable perturbation ν depending on $z \in \Sigma$ and j_τ . We omit further discussion of the general definition because in all our examples, pseudoholomorphic curves are either regular or can be made regular by a perturbation of $J \in \mathcal{J}(B)$.

2.3.4 Regularity and obstructions. Let \mathcal{B} denote the $W^{1,p}$ -completion of the space of smooth maps $u : \mathbf{Z}^2 \setminus \mathbf{R}^2 \rightarrow X$. There is a Banach bundle \mathcal{E} over $B \times \mathbf{H} \times \mathcal{B}$ whose fibre at (b, τ, u) is the L^p -completion

$$L^p\Omega_{j_\tau, J(b)}^{0,1}(\Sigma, u^*TX).$$

There is a section $\overline{\partial} : B \times \mathbf{H} \times \mathcal{B} \rightarrow \mathcal{E}$ given by

$$\overline{\partial}(b, \tau, u) = J(b)du - du \circ j_\tau.$$

If $(b, \tau, u) \in \overline{\partial}^{-1}(0)$ then u is a $(j_\tau, J(b))$ -holomorphic curve and the section has a natural vertical linearisation

$$D_{(b,\tau,u)}\overline{\partial} : T_bB \times T_\tau\mathbf{H} \times W^{1,p}(\Sigma, u^*TX) \rightarrow L^p\Omega_{j_\tau, J(b)}^{0,1}(\Sigma, u^*TX)$$

called the linearised Cauchy–Riemann operator.

DEFINITION 2.13 (Regularity). We say that a family J of ω -compatible almost complex structures is *regular* if for every $(b, \tau, u) \in \overline{\partial}^{-1}(0)$ (with u simple or multiply covered) the linearised Cauchy–Riemann operator $D_{(b,\tau,u)}\overline{\partial}$ is surjective. Equivalently, the section $\overline{\partial}$ vanishes transversely.

If J is regular then we can define Gromov–Witten invariants by (2.11). More generally, we can compute Gromov–Witten invariants using a J which is not regular but for which $\overline{\partial}$ vanishes cleanly.

DEFINITION 2.14 (Cleanliness). We say that a family J of ω -compatible almost complex structures is *clean* if at every point $(b, \tau, u) \in \overline{\partial}^{-1}(0)$ (with u simple or multiply covered) the moduli space $\overline{\partial}^{-1}(0)$ is a smooth manifold with tangent space $\ker(D_{(b,\tau,u)}\overline{\partial})$. Equivalently, $\overline{\partial}$ vanishes cleanly. In this case the cokernels $\text{coker}(D\overline{\partial})$ form a vector bundle over $\overline{\partial}^{-1}(0)$ which we call the *obstruction bundle* and denote by \mathcal{O} .

The following theorem can be proved by a simple modification of the proof of [MS04, Proposition 7.2.3]. The key point is that, since X is symplectically aspherical, there are no nodal genus one stable maps with one marked point, so $\mathcal{M}_{1,1}(A, J)$ is compact (see Remark 2.10).

THEOREM 2.15. *Let (X, ω) be as in Assumption 2.6. If J is a clean family of ω -compatible almost complex structures then the one-point Gromov–Witten invariant is given by*

$$\text{GW}_{1,1}(\omega, A) = \text{ev}_* \text{PD}(e(\mathcal{O}))$$

where PD denotes Poincaré duality and e denotes the Euler class.

2.3.5 Orientations. To really make sense of the fundamental class of the moduli space or of the Euler class of the obstruction bundle one needs orientations. We therefore briefly recall how to orient our moduli spaces when they are clean. Recall that $D\bar{\partial}|_{W^{1,p}(\Sigma, u^*TX)}$ splits as a sum of its (Fredholm) complex linear and (compact) complex antilinear parts. We abuse terminology by calling

$$\frac{1}{2}(D\bar{\partial}(\alpha, \eta, \xi) - \psi D\bar{\partial}(\alpha, \eta, \psi\xi))$$

the complex linear part of $D\bar{\partial}$ (it is only complex linear in ξ).

The linearised Cauchy–Riemann operator is homotopic through Fredholm operators to its complex linear part. There is a *determinant bundle* over the space of Fredholm operators whose fibre at D is the determinant line $\Lambda^{\dim \ker(D)} \ker(D) \otimes \Lambda^{\dim \text{coker}(D)} \text{coker}(D)$. When the moduli space is regular (so that its tangent space at u is the kernel of $D\bar{\partial}$), an orientation of the determinant line is precisely an orientation of the moduli space. Having chosen an orientation on B , the determinant line of a complex linear Cauchy–Riemann operator is canonically oriented and one can transport this orientation along a linear homotopy of operators from $D\bar{\partial}$ to its complex linear part.

When the moduli space is clean rather than regular, an orientation of the determinant line is still all that is needed to define the Euler class of the obstruction bundle.

3. Two-step nilpotent Lie groups

3.1 Generalities

A Lie group N is said to be *k-step nilpotent* if its lower central series

$$N \supset [N, N] \supset [N, [N, N]] \supset \dots$$

reaches the trivial group in k steps. In particular, all iterated Lie brackets of $k + 1$ or more elements vanish. We are interested in two-step nilpotent groups. The main advantage of this class is the simplicity of the Baker–Campbell–Hausdorff formula

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y]\right)$$

for the logarithm of a product.

Henceforth, N will denote a connected, simply connected, two-step nilpotent Lie group of even dimension with Lie algebra \mathfrak{n} . For computational convenience we will implicitly embed N into a real linear group $\text{GL}(V)$ and \mathfrak{n} into $\mathfrak{gl}(V)$ so that we can write $a + b \in \text{GL}(V)$ for $a, b \in N$ and $XY \in \mathfrak{gl}(V)$ for $X, Y \in \mathfrak{n}$. Note that such an embedding exists by Engel’s theorem and that the exponential map $\exp : \mathfrak{n} \rightarrow N$, which thanks to the embedding in $\text{GL}(V)$ we can now write

$$\exp(X) = 1 + X + \frac{1}{2}X^2 + \dots,$$

is a diffeomorphism. We denote its inverse by \log .

Since \mathfrak{n} is a linear space there is a canonical isomorphism $T\mathfrak{n} \cong \mathfrak{n} \times \mathfrak{n}$, so we will write $(X, Y) \in \mathfrak{n} \times \mathfrak{n}$ and $Y \in T_X\mathfrak{n}$ to mean the same thing. There is a canonical map $\pi_N : TN \rightarrow \mathfrak{n}$ defined by

$$\pi_N(X) = L(s^{-1})_*X \quad \text{for } X \in T_sN. \tag{3.1}$$

Here $L(s) : N \rightarrow N$ is the left-multiplication by $s \in N$. Precomposing with $d\exp : T\mathfrak{n} \rightarrow TN$ we get a map $\pi_{\mathfrak{n}} : \mathfrak{n} \times \mathfrak{n} \rightarrow \mathfrak{n}$, explicitly,

$$\pi_{\mathfrak{n}}(X, Y) = L(\exp(-X))_*(d_X \exp)(Y). \tag{3.2}$$

LEMMA 3.3. *We have*

$$\pi_{\mathfrak{n}}(X, Y) = Y - \frac{1}{2}[X, Y].$$

Proof. The Baker–Campbell–Hausdorff formula implies that

$$\exp(-X) \exp(X + tY) = \exp(t(Y - \frac{1}{2}[X, Y]))$$

so

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\exp(X + tY) - \exp(X)}{t} &= \lim_{t \rightarrow 0} \frac{1}{t} \exp(X) \left(\exp\left(t\left(Y - \frac{1}{2}[X, Y]\right)\right) - 1 \right) \\ &= \exp(X) \left(Y - \frac{1}{2}[X, Y] \right). \end{aligned}$$

In $GL(V)$ we know that $L(s)_*v = sv$ so the formula follows. □

3.2 The twistor family of almost Kähler structures

Fix a two-step nilpotent Lie group N as before and endow it with:

- (a) an orientation σ ; and
- (b) a left-invariant metric g (coming from an inner product, also called g , on \mathfrak{n}).

Moreover, let Γ be a cocompact lattice in N ; these always exist if the algebra is defined over \mathbf{Q} .

DEFINITION 3.4. The *twistor family*, denoted by W , is the space of pairs (ω_ψ, ψ) where:

- (a) ψ is a g -orthogonal σ -positive complex structure on \mathfrak{n} ; and
- (b) ω_ψ is the two-form on \mathfrak{n} associated to g and ψ by $g(v, w) = \omega_\psi(v, \psi w)$;

and such that

$$\omega_\psi([X, Y], Z) + \omega_\psi([Y, Z], X) + \omega_\psi([Z, X], Y) = 0.$$

Any pair $(\omega_\psi, \psi) \in W$ yields a left-invariant almost Kähler structure (Ω_ψ, J_ψ) on N . In particular, if $s \in N$, $v \in T_sN$ and $L(s)_*$ denote the differential of left-multiplication by s then

$$J_\psi v = L(s)_*\psi L(s^{-1})_*v. \tag{3.5}$$

By left-invariance these all descend to give almost Kähler structures on $\Gamma \backslash N$. We will often abusively write $\psi \in W$ or $\omega_\psi \in W$ or even $J_\psi \in W$ or $\Omega_\psi \in W$.

This subsumes Definition 2.4 in the case where N is abelian. Notice that W is a subvariety of $SO(2n)/U(n)$, the space of positive orthogonal complex structures on \mathfrak{n} , but may not be a smooth subvariety and it may be empty (we will of course restrict attention to examples where it is nonempty!). If it is not smooth we will restrict attention to some auspicious irreducible component of W which is smooth.

LEMMA 3.6. *Let \mathfrak{z} denote the centre of \mathfrak{n} . For $\psi \in W$,*

$$\psi[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{z}^\perp.$$

If, moreover, $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{t}$ for a one-dimensional subalgebra \mathfrak{t} then

$$\psi\mathfrak{z} = \mathfrak{z}^\perp.$$

Proof. The first assertion follows from the equation

$$g([X, Y], \psi Z) + g([Y, Z], \psi X) + g([Z, X], \psi Y) = 0.$$

When we take $\psi[X, Y] \in \psi[\mathfrak{n}, \mathfrak{n}]$ and $Z \in \mathfrak{z}$, the equation reduces to

$$g(\psi[X, Y], Z) = 0,$$

proving the first claim.

If \mathfrak{t} is one-dimensional then certainly $\psi\mathfrak{t} \subset \mathfrak{t}^\perp$. Moreover, the first claim implies $\psi\mathfrak{t} \subset [\mathfrak{n}, \mathfrak{n}]^\perp$. Hence $\psi\mathfrak{z} = \mathfrak{z}^\perp$. □

3.3 Pseudoholomorphic tori

We have set up conventions for coordinates (p, q) and complex structures j_τ on the torus $\mathbf{Z}^2 \backslash \mathbf{R}^2$ in § 2.3.2. We will write Δ for the Laplacian $\partial_a^2 + \partial_b^2$. Notice that if $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ is a differentiable function then

$$\partial_a f = \partial_p f, \quad \partial_b f = \frac{\partial_q f - \tau_1 \partial_p f}{\tau_2}. \tag{3.7}$$

The Cauchy–Riemann equations. Fix a linear complex structure j_τ on \mathbf{R}^2 and let (a, b) be linear conformal coordinates (so $j_\tau \partial_a = \partial_b$). Consider $\psi \in W$ and the associated left-invariant almost complex structure J_ψ on N .

DEFINITION 3.8. A (j_τ, J_ψ) -holomorphic torus in a nilmanifold $\Gamma \backslash N$ is a map $u : \mathbf{Z}^2 \backslash \mathbf{R}^2 \rightarrow \Gamma \backslash N$ such that

$$J_\psi \circ du = du \circ j_\tau.$$

We will denote by $\pi_1(u) : \mathbf{Z}^2 \rightarrow \Gamma$ the induced map on fundamental groups.

Note that a (j_τ, J_ψ) -holomorphic torus in $\Gamma \backslash N$ lifts to a (j_τ, J_ψ) -holomorphic map between the universal covers

$$\tilde{u} : \mathbf{R}^2 \rightarrow N$$

in one of $\Gamma/\pi_1(u)(\mathbf{Z}^2)$ possible ways. We will fix one such lift.

LEMMA 3.9. *If $w = \log \circ \tilde{u} : \mathbf{R}^2 \rightarrow \mathfrak{n}$ then*

$$\psi(\partial_a w - \frac{1}{2}[w, \partial_a w]) = \partial_b w - \frac{1}{2}[w, \partial_b w] \tag{3.10}$$

which implies

$$\Delta w - \frac{1}{2}[w, \Delta w] = \psi[\partial_a w, \partial_b w] \tag{3.11}$$

where $\Delta = \partial_a^2 + \partial_b^2$.

Proof. The (j_τ, J_ψ) -holomorphic map equation

$$J_\psi(\tilde{u}(a, b))d\tilde{u}(\partial_a) = d\tilde{u}(j_\tau\partial_b)$$

is equivalent to

$$L(\tilde{u}(a, b))_*\psi L(\tilde{u}^{-1}(a, b))_*d\tilde{u}(\partial_a) = d\tilde{u}(\partial_b)$$

because of (3.5). This implies that

$$\psi \circ \pi_N \circ d\tilde{u}(\partial_a) = \pi_N \circ d\tilde{u}(\partial_b)$$

(see (3.1) and (3.2) for the definition of π_N and π_n). Splitting

$$\pi_N = \pi_N \circ d \exp \circ d \log$$

and using the fact that $\pi_N \circ d \exp = \pi_n$, we get a sequence of equations

$$\psi \circ \pi_N \circ d \exp \circ d \log \circ d\tilde{u}(\partial_a) = \pi_N \circ d \exp \circ d \log \circ d\tilde{u}(\partial_b)$$

$$\psi \circ \pi_n \circ dw(\partial_a) = \pi_n \circ dw(\partial_b)$$

$$\psi \circ \pi_n(w, \partial_a w) = \pi_n(w, \partial_b w)$$

and this yields (3.10) thanks to Lemma 3.3. The second order equation follows by cross-differentiating and manipulating (3.10). \square

3.4 Homomorphisms

Let $u : \mathbf{Z}^2 \backslash \mathbf{R}^2 \rightarrow \Gamma \backslash N$ be a map. The induced map $\pi_1(u) : \mathbf{Z}^2 \rightarrow \Gamma$ on fundamental groups extends uniquely to a homomorphism $H : \mathbf{R}^2 \rightarrow N$. To see this, take the images of two generators in \mathbf{Z}^2 : these commute and hence their logarithms commute in the Lie algebra. This means that they span a two-dimensional abelian subalgebra \mathbf{R}^2 . The map H is just the exponential map restricted to this subalgebra. Since H sends \mathbf{Z}^2 into Γ , it descends to a map $v : \mathbf{Z}^2 \backslash \mathbf{R}^2 \rightarrow \Gamma \backslash N$.

LEMMA 3.12. *The maps u and v are freely homotopic.*

Proof. Let \star be a basepoint of $\mathbf{Z}^2 \backslash \mathbf{R}^2$. Freely homotoping u using a path γ joining $u(\star)$ to $v(\star)$ allows us to assume that u and v are maps based at the same point $u(\star) = v(\star)$. The maps $\pi_1(u)$ and $\pi_1(v)$ on fundamental groups are conjugate by construction and this conjugation can be effected by a further free homotopy of u where the base point traces out a loop based at $u(\star)$. That is, after a free homotopy one can assume $\pi_1(u) = \pi_1(v)$. By a based homotopy one can ensure that the maps u and v agree on the 1-skeleton of $\mathbf{Z}^2 \backslash \mathbf{R}^2$ (a wedge of loops). Since $\Gamma \backslash N$ is aspherical (in particular, $\pi_2(\Gamma \backslash N) = 0$), the homotopy can be extended to the 2-skeleton of $\mathbf{Z}^2 \backslash \mathbf{R}^2$. \square

COROLLARY 3.13. *Any lift $\tilde{u} : \mathbf{R}^2 \rightarrow N$ of u differs from H by a bounded amount, i.e. the function $H^{-1}\tilde{u} : \mathbf{R}^2 \rightarrow N$ given by*

$$(p, q) \mapsto H(p, q)^{-1}\tilde{u}(p, q)$$

is bounded.

Proof. After perturbation of the projection u to a map u' based at \star and based-homotopic to v , the map $H^{-1}\tilde{u}'$ descends to a nullhomotopic map $\mathbf{Z}^2 \backslash \mathbf{R}^2 \rightarrow \Gamma \backslash N$ and hence factors as $\mathbf{R}^2 \rightarrow \mathbf{Z}^2 \backslash \mathbf{R}^2 \rightarrow N$. Boundedness of the map upstairs follows from compactness of $\mathbf{Z}^2 \backslash \mathbf{R}^2$. The maps \tilde{u} and \tilde{u}' differ by a bounded perturbation (an equivariant lift of a compact perturbation in $\Gamma \backslash N$). \square

4. The Kodaira–Thurston manifold, K

4.1 Definition

Consider the two-step nilpotent group

$$N = \left\{ \begin{pmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix} : x, y, z, t \in \mathbf{R}, t > 0 \right\}$$

and the lattice Γ consisting of matrices with integer entries. The compact quotient $K = \Gamma \backslash N$ is called the Kodaira–Thurston manifold. The Lie algebra \mathfrak{n} consists of matrices of the form

$$\begin{pmatrix} 0 & x & z & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}$$

and the exponential map is

$$\exp \begin{pmatrix} 0 & x & z & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix} = \begin{pmatrix} 1 & x & z + \frac{xy}{2} & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix}.$$

The commutator subalgebra $[\mathfrak{n}, \mathfrak{n}]$ consists of matrices with $x = y = t = 0$. The centre splits as $\mathfrak{z} = \mathfrak{t} \oplus [\mathfrak{n}, \mathfrak{n}]$ where $\mathfrak{t} = \{x = y = z = 0\}$. We pick a basis for \mathfrak{n} :

$$\begin{aligned} \mathbf{n}_1 &= \partial_y, & \mathbf{n}_2 &= \partial_x \\ \mathbf{n}_3 &= \partial_t, & \mathbf{n}_4 &= \partial_z. \end{aligned}$$

Let us denote by W the twistor family. If $\psi \in W$ then by Lemma 3.6 we know that $\psi(\mathfrak{z}) \subset \mathfrak{z}^\perp$. The complex structure ψ is therefore specified by an isometry $\Psi : \mathfrak{z} \rightarrow \mathfrak{z}^\perp$ which we will think of as a two-by-two special orthogonal matrix (written with respect to the bases $\mathbf{n}_3, \mathbf{n}_4$ and $\mathbf{n}_1, \mathbf{n}_2$). It is not hard to check that any matrix $\Psi \in \text{SO}(2)$ gives an element $\psi \in W$. We will write ψ_θ for the almost complex structure corresponding to the matrix

$$\Psi_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

The tangent space $T_{\psi_\theta} W$ consists of matrices of the form

$$\alpha = r \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}.$$

4.2 Pseudoholomorphic tori and homomorphisms

Given $\psi \in W$, suppose that $u : \mathbf{Z}^2 \backslash \mathbf{R}^2 \rightarrow K$ is a (j, J_ψ) -holomorphic curve for some linear complex structure j on \mathbf{R}^2 . Take (a, b) to be linear j -complex coordinates on \mathbf{R}^2 . Let $\pi_1(u)$ be the induced map on the fundamental group, let H be a homomorphism $\mathbf{R}^2 \rightarrow N$ extending $\pi_1(u)$ and let $\tilde{u} : \mathbf{R}^2 \rightarrow N$ be a lift of u . Denote the logarithms by $w = \log \circ \tilde{u}$ and $h = \log \circ H$. We want to compare \tilde{u} and H so consider $C = \log \circ (H^{-1} \tilde{u})$. By the Baker–Campbell–Hausdorff formula,

$$C = w - h - \frac{1}{2}[h, w].$$

Moreover, since h is a homomorphism, its logarithm is linear (a homomorphism of Lie algebras) and hence $\Delta h = 0$.

PROPOSITION 4.1. *The logarithm C is constant and hence $\tilde{u} = H \exp(C)$ is a right-translate in N of a homomorphism $\mathbf{R}^2 \rightarrow N$. In particular,*

$$\tilde{u} = \exp(h + C + \frac{1}{2}[h, C]).$$

Proof. We decompose the Lie algebra \mathfrak{n} as $\mathfrak{b} \oplus [\mathfrak{n}, \mathfrak{n}] \oplus \psi[\mathfrak{n}, \mathfrak{n}]$. Note that both $\mathfrak{p} = [\mathfrak{n}, \mathfrak{n}]$ and $\mathfrak{q} = \psi[\mathfrak{n}, \mathfrak{n}]$ are one-dimensional. We denote the corresponding components of w by $w_{\mathfrak{b}}, w_{\mathfrak{p}}, w_{\mathfrak{q}}$. We have

$$\begin{aligned} C_{\mathfrak{b}} &= w_{\mathfrak{b}} - h_{\mathfrak{b}} \\ C_{\mathfrak{q}} &= w_{\mathfrak{q}} - h_{\mathfrak{q}} \\ C_{\mathfrak{p}} &= w_{\mathfrak{p}} - h_{\mathfrak{p}} - \frac{1}{2}[h, w]. \end{aligned}$$

Equation (3.11) breaks up into component equations

$$\Delta w_{\mathfrak{b}} = 0 \tag{4.2}$$

$$\Delta w_{\mathfrak{q}} = \psi[\partial_a w, \partial_b w] \tag{4.3}$$

$$\Delta w_{\mathfrak{p}} = \frac{1}{2}[w, \Delta w]. \tag{4.4}$$

Equation (4.2) implies $\Delta C_{\mathfrak{b}} = \Delta w_{\mathfrak{b}} - \Delta h_{\mathfrak{b}} = 0$ and because $C_{\mathfrak{b}}$ is bounded the maximum principle tells us that $C_{\mathfrak{b}}$ is constant. Hence $\partial w_{\mathfrak{b}} = \partial h_{\mathfrak{b}}$ is constant (where ∂ stands for either ∂_a or ∂_b).

Equation (4.3) implies

$$\begin{aligned} \Delta C_{\mathfrak{q}} &= \Delta w_{\mathfrak{q}} - \Delta h_{\mathfrak{q}} \\ &= \psi[\partial_a w, \partial_b w]. \end{aligned}$$

We can expand $w = w_{\mathfrak{b}} + w_{\mathfrak{p}} + w_{\mathfrak{q}}$ in the bracket and ignore the \mathfrak{p} -components since \mathfrak{n} is two-step nilpotent. Furthermore, the term $[\partial_a w_{\mathfrak{q}}, \partial_b w_{\mathfrak{q}}]$ vanishes because \mathfrak{q} is one-dimensional and hence abelian. The term $[\partial_a w_{\mathfrak{b}}, \partial_b w_{\mathfrak{b}}] = [\partial_a h_{\mathfrak{b}}, \partial_b h_{\mathfrak{b}}]$ is constant. The remaining terms are $[\partial_a w_{\mathfrak{q}}, \partial_b h_{\mathfrak{b}}] + [\partial_a h_{\mathfrak{b}}, \partial_b w_{\mathfrak{q}}]$, which are linear first order differential operators with constant coefficients acting on the function $w_{\mathfrak{q}} = C_{\mathfrak{q}} + h_{\mathfrak{q}}$. Therefore,

$$\Delta C_{\mathfrak{q}} = \psi([\partial_a(C_{\mathfrak{q}} + h_{\mathfrak{q}}), \partial_b h_{\mathfrak{b}}] + [\partial_a h_{\mathfrak{b}}, \partial_b(C_{\mathfrak{q}} + h_{\mathfrak{q}})] + [\partial_a h_{\mathfrak{b}}, \partial_b h_{\mathfrak{b}}])$$

is a linear elliptic equation with constant coefficients for $C_{\mathfrak{q}}$. Boundedness of $C_{\mathfrak{q}}$ and the Hopf maximum principle [GT01, Theorem 3.1] imply that $C_{\mathfrak{q}}$ is constant. The crucial observation is that there are no nonlinearities or couplings in (4.3) because \mathfrak{q} is one-dimensional.

We now know that $C_{\mathfrak{b} \oplus \mathfrak{q}} = w_{\mathfrak{b} \oplus \mathfrak{q}} - h_{\mathfrak{b} \oplus \mathfrak{q}}$ is constant and, since h is linear, $\Delta w_{\mathfrak{b} \oplus \mathfrak{q}} = 0$. Equation (4.4) implies that

$$\begin{aligned} \Delta w_{\mathfrak{p}} &= \frac{1}{2}[w, \Delta w] \\ &= \frac{1}{2}[w_{\mathfrak{b} \oplus \mathfrak{q}}, \Delta w_{\mathfrak{b} \oplus \mathfrak{q}}] = 0, \end{aligned}$$

and therefore

$$\begin{aligned} \Delta C_{\mathfrak{p}} &= \Delta w_{\mathfrak{p}} - \Delta h_{\mathfrak{p}} - \frac{1}{2}\Delta[h, w] \\ &= -\frac{1}{2}\Delta[h_{\mathfrak{b} \oplus \mathfrak{q}}, w_{\mathfrak{b} \oplus \mathfrak{q}}] \\ &= -\frac{1}{2}\Delta[h_{\mathfrak{b} \oplus \mathfrak{q}}, h_{\mathfrak{b} \oplus \mathfrak{q}} + C_{\mathfrak{b} \oplus \mathfrak{q}}] = 0 \end{aligned}$$

because h is linear and $[h_{\mathfrak{b} \oplus \mathfrak{q}}, h_{\mathfrak{b} \oplus \mathfrak{q}}] = 0$. Again, the maximum principle implies that $C_{\mathfrak{p}}$ is constant. We have now seen that all components of C are constant. □

4.3 Cohomology and its automorphisms

Consider the left-invariant one-forms

$$dy, dx, dt, \gamma = dz - x dy.$$

The first three one-forms are closed and we denote their cohomology classes by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively. They span $H^1(K; \mathbf{Z}) \subset H^1(K; \mathbf{R})$. The following classes span $H^2(K; \mathbf{Z})$:

$$\begin{aligned} \mathbf{e}_{13} &= [dy \wedge dt], & \mathbf{e}_{23} &= [dx \wedge dt] \\ \mathbf{e}_{14} &= [dy \wedge \gamma], & \mathbf{e}_{24} &= [dx \wedge \gamma]. \end{aligned}$$

Finally, the following classes span $H^3(K; \mathbf{Z})$:

$$\mathbf{e}_{134} = [dy \wedge dt \wedge \gamma], \quad \mathbf{e}_{234} = [dx \wedge dt \wedge \gamma], \quad \mathbf{e}_{124} = [dy \wedge dx \wedge \gamma].$$

We define the dual bases $E_i \in H_1(K; \mathbf{Z})$, $E_{ij} \in H_2(K; \mathbf{Z})$ and $E_{ijk} \in H_3(K; \mathbf{Z})$ for homology, so, for example,

$$\int_{E_{ijk}} \mathbf{e}_{lmn} = \delta_{il} \delta_{jm} \delta_{kn}$$

and we write $A = \sum A_{ij} E_{ij}$, or frequently

$$[A_{13}, A_{23}, A_{14}, A_{24}],$$

for the components of a homology class A .

Remark 4.5. The symplectic form ω_θ corresponding to a rotation matrix Ψ_θ is

$$\omega_\theta = dt \wedge (\cos \theta dx + \sin \theta dy) + \gamma \wedge (-\sin \theta dx + \cos \theta dy),$$

so the ω_θ -symplectic area of A is

$$-(\cos \theta (A_{23} + A_{14}) + \sin \theta (A_{24} - A_{13})).$$

Let $\varphi : \Gamma \rightarrow \Gamma$ be a lattice automorphism. Then by rigidity for nilpotent Lie groups [VGS00, Theorem 2.7] we know that φ extends uniquely to an automorphism of N . As we observed in the case of the $2n$ -torus, the left-invariant metrics φ^*g and g are isotopic through left-invariant metrics and hence the corresponding twistor families of symplectic forms are deformation equivalent. Deformation invariance of Gromov–Witten invariants applied to (2.12) implies

$$\text{GW}_{1,k}(W, \varphi_*A) = \varphi_* \text{GW}_{1,k}(W, A) \tag{4.6}$$

for any $\varphi \in \text{Aut}(\Gamma)$.

LEMMA 4.7. *There is a homomorphism $\text{SL}(2, \mathbf{Z}) \rightarrow \text{Aut}(\Gamma)$ which projects to the standard action of $\text{SL}(2, \mathbf{Z})$ on $\Gamma/Z(\Gamma) \cong \mathbf{Z}^2$.*

Proof. The homomorphism is defined on generators by

$$\sigma_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \varphi_1, \quad \sigma_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \varphi_2$$

where

$$\varphi_1 \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -y \\ x \\ z - xy \\ t \end{pmatrix}$$

and

$$\varphi_2 \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x + y \\ y \\ z + \frac{y(y + 1)}{2} \\ t \end{pmatrix}.$$

Since the projection $\Gamma \rightarrow \Gamma/Z(\Gamma)$ is given by $(x, y, z, t) \mapsto (x, y)$ we see that the maps induced on the quotient are precisely σ_1 and σ_2 . □

The action of φ_i on the second homology is

$$(\varphi_i)_*[A_{13}, A_{23}, A_{14}, A_{24}] = [\sigma_i(A_{12}, A_{23}), \sigma_i(A_{14}, A_{24})]. \tag{4.8}$$

4.4 The homology classes of tori

Let $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ be a Lie algebra homomorphism and write

$$h(p, q) = \sum_{i=1}^4 h_i(p, q)\mathbf{n}_i$$

where $h_i(p, q)$ are its linear coordinate functions. Its exponential is

$$H = \exp(h) = \begin{pmatrix} 1 & h_2 & h_4 + \frac{1}{2}h_1h_2 & 0 \\ 0 & 1 & h_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{h_3} \end{pmatrix}. \tag{4.9}$$

Since \mathbf{R}^2 is an abelian Lie algebra,

$$[\partial_p h, \partial_q h] = \partial_{[p} h_2 \partial_{q]} h_1 = 0. \tag{4.10}$$

The map $H : \mathbf{R}^2 \rightarrow N$ will descend to a closed torus in K if

$$\exp(h(1, 0)), \exp(h(0, 1)) \in \Gamma.$$

Equivalently, the derivatives $\partial_p h_i, \partial_q h_i$ for $i = 1, 2, 3$, $\partial_p h_4 + \frac{1}{2}\partial_p h_1 \partial_p h_2$ and $\partial_q h_4 + \frac{1}{2}\partial_q h_1 \partial_q h_2$ must be integers.

DEFINITION 4.11. If $H = \exp(h)$ descends to a closed torus with homology class A then we say h represents the homology class A and we write $[h] = A$. Equivalently, if $h = \log \circ H$ where $H : \mathbf{R}^2 \rightarrow N$ is the unique homomorphic extension of a lattice homomorphism $\rho : \mathbf{Z}^2 \rightarrow \Gamma$ then we can write $[\rho] := [h]$.

LEMMA 4.12. *Let $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ be a Lie algebra homomorphism such that $H = \exp \circ h$ descends to a closed genus one curve in K . Then $[h] = A$ where*

$$A_{ij} = \partial_{[p} h_i \partial_{q]} h_j.$$

Proof. By considering closed invariant two-forms on N we get an isomorphism [VGS00, Corollary 4.7]

$$H^2(K; \mathbf{R}) \cong H^2_{\text{Lie}}(\mathfrak{n}).$$

Similarly, we have an isomorphism $H^2(T^2; \mathbf{R}) \cong H^2_{\text{Lie}}(\mathbf{R}^2) \cong \mathbf{R}$. In terms of these isomorphisms, the pullback map $H^2(K; \mathbf{R}) \rightarrow H^2(T^2; \mathbf{R})$ is just the pullback in Lie algebra cohomology induced by the homomorphism h . This pullback is induced by the map

$$\Lambda^2 h^\vee : \Lambda^2 \mathfrak{n}^\vee \rightarrow \Lambda^2 \mathbf{R}^2$$

which simply takes the two-by-two minors of the matrix representing h , whose rows are $(\partial_p h_i, \partial_q h_i)$.

The subspace of $\Lambda^2 \mathfrak{n}^\vee$ spanned by \mathbf{e}_{ij} consists of Lie cochains and projects isomorphically to $H^2_{\text{Lie}}(\mathfrak{n})$. The coefficients A_{ij} of A are precisely the pullbacks of these forms to $H^2_{\text{Lie}}(\mathbf{R}) \cong \mathbf{R}$ and these are just the minors of the transpose of the matrix whose columns are $\partial_p h$ and $\partial_q h$:

$$A_{ij} = h^*[\mathbf{e}_{ij}] = \partial_{[p} h_i \partial_{q]} h_j. \quad \square$$

Note that it is not immediately obvious why $\partial_{[p} h_1 \partial_{q]} h_4$ is an integer (though it follows from the lemma).

LEMMA 4.13. *If $A = [h]$ for some homomorphism $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ then*

$$A_{13}A_{24} = A_{14}A_{23}.$$

Proof. There is a commutative diagram of Plücker maps.

$$\begin{array}{ccc} \mathbf{R}^2 \times \mathbf{R}^2 & \xrightarrow{\wedge} & \Lambda^2 \mathbf{R}^2 \cong \mathbf{R} \\ h \times h \downarrow & & \downarrow \Lambda^2 h \\ \mathfrak{n} \times \mathfrak{n} & \xrightarrow{\wedge} & \Lambda^2 \mathfrak{n} \end{array}$$

The image of $(\partial_p, \partial_q) \in \mathbf{R}^2 \times \mathbf{R}^2$ in $\Lambda^2 \mathfrak{n}$ is the sextuple of two-by-two minors $D_{ij} = \partial_{[p} h_i \partial_{q]} h_j$ of the matrix of h . This sits inside the Plücker quadric

$$D_{12}D_{34} - D_{13}D_{24} + D_{14}D_{23} = 0.$$

However, $D_{12} = 0$ because \mathbf{R}^2 is abelian, and Lemma 4.12 implies that $D_{ij} = A_{ij}$ for $i = 1, 2, j = 3, 4$. □

LEMMA 4.14. *If $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ is a homomorphism with $[h] = A$ then there is an automorphism φ of Γ such that $\varphi_* A = [m, m, n, n]$ where*

$$m = \gcd(A_{13}, A_{23}), \quad n = \gcd(A_{14}, A_{24}).$$

Defining $a = A_{13}/m, b = A_{23}/m$, we see from Lemma 4.13 that

$$a = A_{14}/n, \quad b = A_{24}/n, \quad \gcd(a, b) = 1.$$

Proof. By the action of $\text{SL}(2, \mathbf{Z}) \subset \text{Aut}(\Gamma)$ on $H_2(K; \mathbf{Z})$ described in (4.8) we can move the pair (A_{13}, A_{23}) by some $\varphi \in \text{Aut}(\Gamma)$ until it coincides with $(\gcd(A_{13}, A_{23}), \gcd(A_{13}, A_{23}))$. Since $A_{13}A_{24} = A_{14}A_{23}$, the same φ will take (A_{14}, A_{24}) to $(\gcd(A_{14}, A_{24}), \gcd(A_{14}, A_{24}))$. □

We now consider linear reparametrisations of the torus, that is, $SL(2, \mathbf{Z})$ acting on the (p, q) -plane. We have this freedom when counting pseudoholomorphic tori because we specified the complex structure j_τ by giving a point $\tau \in \mathbf{H}$ in the upper-half plane; we are therefore over-counting each torus infinitely often, once for each point in \mathbf{H} giving a diffeomorphic complex structure. The $SL(2, \mathbf{Z})$ -reparametrisation precisely removes this ambiguity. The effect of $\Phi \in SL(2, \mathbf{Z})$ on $\sum_{i=1}^2 \partial_p h_i \mathbf{n}_i$ and $\sum_{i=1}^2 \partial_q h_i \mathbf{n}_i$ is to act on the right:

$$\begin{pmatrix} \partial_p h_1 & \partial_q h_1 \\ \partial_p h_2 & \partial_q h_2 \end{pmatrix} \Phi.$$

DEFINITION 4.15. We write $[h]_{SL}$ for the $SL(2, \mathbf{Z})$ -equivalence class of Lie algebra homomorphisms containing h . Note that the homology class $[h]$ depends only on h . The notion of $SL(2, \mathbf{Z})$ -equivalence also makes sense for lattice homomorphisms $\rho : \mathbf{Z}^2 \rightarrow \Gamma$ by extending them to Lie group homomorphisms and taking the logarithm, and we write $[\rho]_{SL}$ for the equivalence class.

DEFINITION 4.16. We say a homomorphism $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ is *reduced* if $\partial_p h_1 = \partial_p h_2 = 0$. Equivalently, the matrix of derivatives of h is

$$\begin{pmatrix} 0 & \partial_q h_1 \\ 0 & \partial_q h_2 \\ \partial_p h_3 & \partial_q h_3 \\ \partial_p h_4 & \partial_q h_4 \end{pmatrix}.$$

We say that h is *fully reduced* if, moreover,

$$0 \leq \partial_q h_3 < \partial_p h_3.$$

The notion of reduced homomorphism also makes sense for lattice homomorphisms $\rho : \mathbf{Z}^2 \rightarrow \Gamma$ by extending them to Lie group homomorphisms and taking the logarithm.

LEMMA 4.17. For a homology class $A \neq 0$ with $m = A_{13} = A_{23}$, $n = A_{14} = A_{24}$, any Lie algebra homomorphism $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ with $[h] = A$ is $SL(2, \mathbf{Z})$ -equivalent to a reduced homomorphism with $\partial_q h_1 = \partial_q h_2 \neq 0$. If, moreover, $m \neq 0$, then h is $SL(2, \mathbf{Z})$ -equivalent to a unique fully reduced homomorphism.

Proof. Since \mathbf{R}^2 is an abelian Lie algebra we have

$$0 = [\partial_p h, \partial_q h] = \partial_{[p} h_2 \partial_{q]} h_1 \mathbf{n}_1.$$

If $h_1 \neq 0$ this implies that the top two rows

$$\begin{pmatrix} \partial_p h_1 & \partial_q h_1 \\ \partial_p h_2 & \partial_q h_2 \end{pmatrix}$$

of the homomorphism h are linearly dependent. Using the right $SL(2, \mathbf{Z})$ -action we can ensure that $\partial_p h_1 = \partial_p h_2 = 0$. Now we have

$$\begin{aligned} m &= -\partial_q h_1 \partial_p h_3 = -\partial_q h_2 \partial_p h_3 \\ n &= -\partial_q h_1 \partial_p h_4 = -\partial_q h_2 \partial_p h_4 \end{aligned}$$

and since one of these two quantities is non-zero we know that $\partial_q h_1 = \partial_q h_2 \neq 0$.

We have a residual right $SL(2, \mathbf{Z})$ -action by matrices of the form

$$\begin{pmatrix} \pm 1 & \star \\ 0 & \pm 1 \end{pmatrix},$$

since these preserve the condition $\partial_p h_1 = \partial_p h_2 = 0$. We know that $\partial_p h_3 \neq 0$ because m is assumed to be non-zero. Using the action of these matrices and this fact we can obtain $\partial_p h_3 > 0$ and implement the Euclidean algorithm on $\partial_q h_3$ to ensure that $0 \leq \partial_q h_3 < \partial_p h_3$. \square

Note that for a reduced homomorphism, $\partial_q h_1 = \partial_q h_2$ divides the greatest common divisor $\gcd(m, n)$. It is easy to check that the most general reduced homomorphism giving the numbers m and n is

$$\begin{aligned} \partial_q h_1 &= \partial_q h_2 = -\operatorname{sgn}(m)d \\ \partial_p h_3 &= \frac{|m|}{d} \\ \partial_p h_4 &= -\frac{n}{\operatorname{sgn}(m)d} \end{aligned}$$

for a positive divisor d of $\gcd(m, n)$, where $\operatorname{sgn}(m)$ denotes the sign of m . In matrix form this looks like

$$\begin{pmatrix} 0 & -\operatorname{sgn}(m)d \\ 0 & -\operatorname{sgn}(m)d \\ \frac{|m|}{d} & \partial_q h_3 \\ -\frac{n}{\operatorname{sgn}(m)d} & \partial_q h_4 \end{pmatrix}. \tag{4.18}$$

5. Pseudoholomorphic tori in K

We have seen (Proposition 4.1) that if $\psi \in W$ then all (j, J_ψ) -holomorphic tori in K are quotients of maps $\mathbf{R}^2 \rightarrow N$ of the form

$$\exp(h + C + \frac{1}{2}[h, C]) \tag{5.1}$$

where $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ is a Lie algebra homomorphism and $C \in \mathfrak{n}$ is a constant. We know that if h descends to a closed genus one curve then the derivatives

$$\partial_p h_i, \partial_q h_i \quad \text{for } i = 1, 2, 3, \partial_p h_4 + \frac{1}{2}\partial_p h_1 \partial_p h_2, \partial_q h_4 + \frac{1}{2}\partial_q h_1 \partial_q h_2$$

are integers. The problem is now to enumerate these tori modulo the reparametrisation action of $SL(2, \mathbf{Z})$ on \mathbf{R}^2 .

In light of Lemma 4.14 we will restrict attention to non-zero homology classes with $A_{13} = A_{23}$ and $A_{14} = A_{24}$ without loss of generality, and by reparametrising as in Lemma 4.17 we can assume $\partial_p h_1 = \partial_p h_2 = 0$ (i.e. h is reduced). Using the usual decomposition $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z}$ of the Lie algebra into the centre \mathfrak{z} and its orthogonal complement, we can write this assumption as

$$\partial_p h_{\mathfrak{a}} = 0.$$

By (3.7), this is equivalent to

$$\partial_{\mathfrak{a}} h_{\mathfrak{a}} = 0.$$

The expected dimension of genus g curves in K which are J_ψ -holomorphic for some $\psi \in W$ is

$$4 \cdot (1 - g) + 6g - 6 + \dim W$$

since $c_1(K) = 0$. For tori ($g = 1$) the expected dimension is $\dim W = 1$. When a marked point is added we get

$$\text{virdim } \mathcal{M}_{1,1}(W, A) = 3.$$

Our task in this section is to write down the moduli space and compute its dimension.

5.1 Defining moduli spaces

Let $A = [m, m, n, n] \in H_2(K, \mathbf{Z})$ be a non-zero homology class and define the space of maps

$$\mathcal{M}(W, A) = \left\{ (u, \tau, \psi) \left| \begin{array}{l} \tau \in \mathbf{H}, \psi \in W, \\ u : T^2 \rightarrow K \text{ is } (j_\tau, J_\psi)\text{-holomorphic,} \\ u_*[T^2] = A. \end{array} \right. \right\}$$

Let $\rho : \mathbf{Z}^2 \rightarrow \Gamma$ be a homomorphism with $[\rho] = A$ and let $H : \mathbf{R}^2 \rightarrow N$ be its unique homomorphic extension. Define

$$\mathcal{M}_\rho(W) = \{(He^C, \tau, \psi) \in \mathcal{M}(W, A) \text{ for some } C \in \mathfrak{n}\}$$

and note that

$$\mathcal{M}(W, A) = \coprod_{[\rho]=A} \mathcal{M}_\rho(W).$$

We also define

$$\begin{aligned} \mathcal{M}^{\text{red}}(W, A) &= \coprod_{[\rho]=A, \rho \text{ reduced}} \mathcal{M}_\rho(W) \\ \mathcal{M}^{\text{ful}}(W, A) &= \coprod_{[\rho]=A, \rho \text{ fully reduced}} \mathcal{M}_\rho(W). \end{aligned}$$

The one-point moduli space is given by

$$\mathcal{M}_{1,1}(W, A) = \mathcal{M}(W, A) \times_{\text{Aff}(T^2)} T^2$$

where $\text{Aff}(T^2) = \text{SL}(2, \mathbf{Z}) \ltimes T^2$ is the group of affine reparametrisations of T^2 . Here, $\varphi \in \text{Aff}(T^2)$ acts by

$$\varphi(u, \tau, \psi, z) = (u \circ \varphi^{-1}, \varphi(\tau), \psi, \varphi(z))$$

where the action of $\text{SL}(2, \mathbf{Z})$ on \mathbf{H} is the standard one. We will also write

$$\mathcal{M}_{1,1}(W, [\rho]_{\text{SL}}) := \mathcal{M}_\rho(W) \times_{T^2} T^2.$$

LEMMA 5.2. *The action of $\text{Aff}(T^2)$ on $\mathcal{M}(W, A) \times T^2$ is free.*

Proof. Suppose (u, τ, ψ, z) is fixed by φ . Let v denote the simple curve underlying u and let $\pi : \Sigma_\tau \rightarrow \Sigma_{\tau'}$ denote the holomorphic covering space such that $u = v \circ \pi$. The curve v has an open set $V \subset \Sigma_{\tau'}$ of injective points. Let $x \in V$. An automorphism φ of u satisfies $u(\varphi x') = u(x')$ for any $x' \in \pi^{-1}(x)$. This implies that $\pi \circ \varphi = \pi$ on $\pi^{-1}(V)$ and hence everywhere, so φ is a deck transformation (a translation). However, $\varphi(z) = z$, so $\varphi = \text{Id}$. \square

We first divide out by translations. The space $\mathcal{M}(W, A) \times_{T^2} T^2$ has a residual $\mathrm{SL}(2, \mathbf{Z})$ -action. The following lemma is immediate from Lemma 4.17.

LEMMA 5.3. *Every $\mathrm{SL}(2, \mathbf{Z})$ -orbit of $\mathcal{M}(W, A) \times_{T^2} T^2$ contains a point of*

$$\mathcal{M}^{\mathrm{red}}(W, A) \times_{T^2} T^2.$$

If $m \neq 0$ then every orbit contains a unique point of

$$\mathcal{M}^{\mathrm{ful}}(W, A) \times_{T^2} T^2. \quad \square$$

5.2 Describing $\mathcal{M}_\rho(W, A)$

LEMMA 5.4. *Suppose that h is a reduced Lie algebra homomorphism and $C \in \mathfrak{n}$ is a constant. The Cauchy–Riemann equations for*

$$\exp(h + C + \frac{1}{2}[h, C])$$

become

$$\partial_b h_\mathfrak{a} = \psi \partial_a h_\mathfrak{z} \tag{5.5}$$

$$\partial_b h_\mathfrak{z} = [C_\mathfrak{a}, \partial_b h_\mathfrak{a}]. \tag{5.6}$$

Proof. By Lemma 3.6, $\psi(\mathfrak{a}) = \mathfrak{z}$. If $w = h + C + \frac{1}{2}[h, c]$ then we have

$$\begin{aligned} w_\mathfrak{a} &= h_\mathfrak{a} + C_\mathfrak{a} \\ w_\mathfrak{z} &= h_\mathfrak{z} + C_\mathfrak{z} + \frac{1}{2}[h_\mathfrak{a}, C_\mathfrak{a}]. \end{aligned}$$

Taking the \mathfrak{a} -part of the Cauchy–Riemann equation (3.10) and using the fact that $\partial_p h_\mathfrak{a} = \partial_a h_\mathfrak{a} = 0$ (h is reduced) gives (5.5). Taking the \mathfrak{z} -part of (3.10) gives

$$\begin{aligned} 0 &= \partial_b h_\mathfrak{z} + \frac{1}{2}[\partial_b h_\mathfrak{a}, C_\mathfrak{a}] - \frac{1}{2}[h_\mathfrak{a} + C_\mathfrak{a}, \partial_b h_\mathfrak{a}] \\ &= \partial_b h_\mathfrak{z} + [\partial_b h_\mathfrak{a}, C_\mathfrak{a}]. \end{aligned}$$

This last step uses the fact that $[h_\mathfrak{a}, \partial_b h_\mathfrak{a}] = 0$, which follows because $h_\mathfrak{a} = a\partial_a h_\mathfrak{a} + b\partial_b h_\mathfrak{a} = b\partial_b h_\mathfrak{a}$. \square

LEMMA 5.7. *Fix a Lie algebra homomorphism $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ with $h = \sum_{i=1}^4 h_i \mathbf{n}_i$ such that $\partial_p h_\mathfrak{a} = 0$ and $\exp(h(\mathbf{Z}^2)) \subset \Gamma$. We list all possibilities for C , τ and ψ such that*

$$h + C + \frac{1}{2}[h, C]$$

is the logarithm of a J_ψ -holomorphic torus:

- (a) the constant $C_\mathfrak{z}$ is arbitrary;
- (b) the number τ_2 satisfies

$$\tau_2 = \frac{\|\partial_q h_\mathfrak{a}\|}{\|\partial_p h_\mathfrak{z}\|};$$

- (c) the complex structure ψ is specified by the unique matrix $\Psi \in \mathrm{SO}(2)$ which rotates

$$\begin{pmatrix} \partial_p h_\mathfrak{z} \\ \partial_p h_\mathfrak{a} \end{pmatrix} \quad \text{to} \quad \frac{1}{\tau_2} \begin{pmatrix} \partial_q h_\mathfrak{z} \\ \partial_q h_\mathfrak{a} \end{pmatrix}; \tag{5.8}$$

(d) the components of C_a satisfy

$$C_2\partial_q h_1 - C_1\partial_q h_2 = \partial_q h_4 - \tau_1\partial_p h_4. \tag{5.9}$$

Moreover:

- (a) if $\partial_p h_3 \neq 0$ we have $\tau_1 = \partial_q h_3 / \partial_p h_3$;
- (b) if $\partial_p h_3 = 0$ then necessarily $\partial_q h_3 = 0$ and τ_1 is arbitrary.

Proof. The constant C_3 is arbitrary because it does not enter into (5.5) and (5.6). Taking the norm of (5.5) gives

$$\|\partial_b h_a\| = \|\partial_a h_3\|$$

because ψ is orthogonal. Using (3.7) we have

$$\|\partial_b h_a\| = \left\| \frac{\partial_q h_a - \tau_1 \partial_p h_a}{\tau_2} \right\| = \frac{\|\partial_q h_a\|}{\tau_2}$$

since $\partial_p h_a = \partial_a h_a = 0$. This gives the formula for τ_2 . Having fixed τ_2 , (5.5) becomes precisely the desired condition on ψ .

The equation for C_1 and C_2 is simply the \mathfrak{n}_4 -component of (5.6). Finally, the \mathfrak{n}_3 component is

$$\partial_b h_3 = 0$$

since \mathfrak{n}_3 is orthogonal to the commutator subalgebra. Using the fact (3.7) that

$$\partial_b h_3 = \frac{\partial_q h_3 - \tau_1 \partial_p h_3}{\tau_2}$$

we obtain the required dichotomy for τ_1 when $\partial_p h_3$ is either zero or non-zero. □

Note that, unlike C_a , the quantity $C_2\partial_q h_1 - C_1\partial_q h_2$ is invariant under translations of the (p, q) -plane.

COROLLARY 5.10. *Let $\rho : \mathbf{Z}^2 \rightarrow \Gamma$ be a homomorphism, H be its homomorphic extension to $\mathbf{R}^2 \rightarrow N$ and $h = \log H$. When $\partial_p h_3 \neq 0$, the moduli space $\mathcal{M}_\rho(W)$ consists of maps u of the form*

$$He^{C_0+D}$$

where

$$C_0 = \frac{\partial_q h_4 - \tau_1 \partial_p h_4}{\partial_q h_1} \mathfrak{n}_2 \tag{5.11}$$

and D is any element \mathfrak{n} satisfying $[D, h] = 0$.

Proof. By (5.9) we know that if He^C is in the moduli space $\mathcal{M}_\rho(W)$ then C must solve

$$C_2\partial_q h_1 - C_1\partial_q h_2 = \partial_q h_4 - \tau_1\partial_p h_4.$$

The vector $C_0 \in \mathfrak{n}$ given in (5.11) is a particular solution of this inhomogeneous equation. Therefore, $D = C - C_0$ satisfies the corresponding homogeneous equation,

$$D_2\partial_q h_1 - D_1\partial_q h_2 = 0,$$

which is equivalent to $[D, h] = 0$. Note that ψ and τ are determined by the homomorphism ρ . □

LEMMA 5.12. Suppose that $\rho : \mathbf{Z}^2 \rightarrow \Gamma$ is reduced and that $(He^{C_0+D}, \tau, \psi) \in \mathcal{M}_\rho(W)$ as in Corollary 5.10. By reparametrising $(p, q) \mapsto (p + \delta_p, q + \delta_q)$ we can assume that

$$D \cdot \partial_q h_a = D \cdot \partial_p h_3 = 0.$$

In particular, we can ensure that $D_3 \perp \partial_p h_3$ and $D_a = 0$.

Proof. The logarithm of He^{C_0+D} is $h + C_0 + D + \frac{1}{2}[h, C_0]$ since $[h, D] = 0$. We can absorb the reparametrisation of h by $(p, q) \mapsto (p + \delta_p, q + \delta_q)$ into the constant D , which becomes

$$D + \delta_p(\partial_p h + \frac{1}{2}[\partial_p h, C_0]) + \delta_q(\partial_q h + \frac{1}{2}[\partial_q h, C_0]).$$

First, pick δ_q to solve the equation

$$\delta_q(\partial_q h + \frac{1}{2}[\partial_q h, C_0]) \cdot \partial_q h_a + D \cdot \partial_q h_a = 0.$$

This is possible since $(\partial_q h + \frac{1}{2}[\partial_q h, C_0]) \cdot \partial_q h_a = \|\partial_q h_a\|^2 \neq 0$ and it ensures that $D' \cdot \partial_q h_a = 0$, where D' is the new constant after the reparametrisation by $(0, \delta_q)$. Next, remember that since ρ is reduced, $\partial_p h_a = 0$. Let δ_p be the solution of

$$\delta_q(\partial_p h + \frac{1}{2}[\partial_p h, C_0]) \cdot \partial_p h_3 + D' \cdot \partial_p h_3 = 0;$$

this is possible since $(\partial_p h + \frac{1}{2}[\partial_p h, C_0]) \cdot \partial_p h_3 = \|\partial_p h_3\|^2 \neq 0$. Reparametrising D' by $(\delta_p, 0)$ gives D'' satisfying $D'' \cdot \partial_p h_3 = 0$. Note that $D'' \cdot \partial_q h_a$ is still zero because $\partial_p h_a = 0$, so this condition is not affected by the second reparametrisation.

Now assume that we have reparametrised and relabelled so that D satisfies the equations

$$D \cdot \partial_q h_a = D \cdot \partial_p h_3 = 0.$$

To see that this gives $D_a = 0$, note that $[D, h] = [D_a, h_a] = [D_a, q\partial_q h_a] = 0$ (because h is reduced). This is a linear equation for a vector $D_a \in \mathfrak{a}$ and $\dim(\mathfrak{a}) = 2$, so D_a is a multiple of $\partial_q h_a$. But we have just seen that $D_a \perp \partial_q h_a$. The second equation implies $D_3 \perp \partial_p h_3$. \square

5.3 Describing $\mathcal{M}_{1,1}(W, A)$: fully reduced case

We know by definition that $\mathcal{M}_{1,1}(W, A) = \mathcal{M}(W, A) \times_{\text{Aff}(T^2)} T^2$ and $\mathcal{M}(W, A) = \coprod_{[\rho]=A} \mathcal{M}_\rho(W)$. By Lemma 5.3, we know that if $m = A_{13} \neq 0$ then there is a unique, fully reduced ρ in the $\text{SL}(2, \mathbf{Z})$ -orbit of homomorphisms representing the class $[A]$, and hence

$$\begin{aligned} \mathcal{M}_{1,1}(W, A) &= \mathcal{M}^{\text{ful}}(W, A) \times_{T^2} T^2 \\ &= \coprod_{\rho \text{ reduced}} \mathcal{M}_\rho(W) \times_{T^2} T^2. \end{aligned}$$

By Lemma 5.12, we know that a local slice of the moduli space $\mathcal{M}_\rho(W) \times_{T^2} T^2$ (when ρ is fully reduced) is given by

$$(He^{C_0+D(\lambda)}, \tau, \psi, z)$$

where $z \in T^2$ is arbitrary and

$$D(\lambda) = \lambda \begin{pmatrix} 0 \\ 0 \\ \partial_p h_4 \\ -\partial_p h_3 \end{pmatrix} \tag{5.13}$$

for $\lambda \in \mathbf{R}$. In fact, this descends to a global description of the moduli space when we observe that $D(\lambda)$ is central, and hence

$$He^{C_0+D(\lambda)} = e^{D(\lambda)} He^{C_0},$$

which gives the same pseudoholomorphic torus if and only if $e^{D(\lambda)} \in \Gamma$. This occurs precisely when λ is a multiple of $1/\gcd(\partial_p h_3, \partial_p h_4)$.

COROLLARY 5.14. *If $A = \sum A_{ij} E_{ij} \in H_2(K; \mathbf{Z})$ is a homology class with $A_{13} \neq 0$ then the moduli space $\mathcal{M}_{1,1}(W, A)$ is smooth. It is a union of components labelled by fully reduced homomorphisms $\rho : \mathbf{Z}^2 \rightarrow \Gamma$, each component consisting of equivalence classes*

$$[(He^{C_0+D(\lambda)}, z)], \quad \lambda \in \left[0, \frac{1}{\gcd(\partial_p h_3, \partial_p h_4)}\right], \quad z \in T^2$$

where:

- (a) $H : \mathbf{R}^2 \rightarrow N$ is the unique homomorphic extension of ρ and h is its logarithm;
- (b) $D(\lambda)$ is defined by (5.13);
- (c) C_0 is defined by (5.11);
- (d) the equivalence relation equates (u, z) with $(u \circ \varphi^{-1}, \varphi(z))$ for a translation $\varphi : T^2 \rightarrow T^2$ of the domain such that $u \circ \varphi^{-1} = u$.

In particular, the tangent space at (u, τ, ψ, z) comprises the vectors

$$(D(\lambda), V) \in \mathfrak{z} \oplus T_z \Sigma_\tau, \quad \lambda \in \mathbf{R}.$$

The moduli space has dimension three (the expected dimension).

Arguing as in the proof of Lemma 5.2, we see that if u is a torus and v is the underlying simple torus, so that $u = v \circ \pi$ for some holomorphic covering map π , then the size of the equivalence class $[(u, z)]$ is the order of the deck transformation group of this cover.

5.4 Describing $\mathcal{M}_{1,1}(W, A)$: the general case

Suppose now that $m = A_{13} = 0$. Since $A \neq 0$ we know that $n \neq 0$.

By Lemma 5.3 we know that any ρ with $[\rho] = A$ can be conjugated via the action of $SL(2, \mathbf{Z})$ to a reduced homomorphism. For a reduced ρ , the subgroup $\text{Stab}(\rho) \subset \text{Aff}(T^2)$ of affine reparametrisations of \mathbf{R}^2 fixing ρ is generated by the subgroup T^2 of translations and the group isomorphic to $\mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z}) \subset SL(2, \mathbf{Z})$ consisting of matrices of the form

$$\begin{pmatrix} \pm 1 & \star \\ 0 & \pm 1 \end{pmatrix}.$$

Lemma 5.3 can be rephrased as

$$\mathcal{M}_{1,1}(W, A) = \mathcal{M}^{\text{red}}(W, A) \times_{\text{Stab}(\rho)} T^2.$$

Each component is diffeomorphic to

$$\mathcal{M}_\rho(W, A) \times_{T^2} T^2$$

for some reduced ρ .

LEMMA 5.15. *In the case $m = 0, n \neq 0$, the moduli space is a smooth manifold of dimension four and the tangent space at $(u = He^C, \tau, \psi, z)$ comprises triples $(D_3\mathbf{n}_3, \eta_1, V) \in \mathfrak{z} \oplus \text{Re}(T_\tau\mathbf{H}) \oplus T_z\Sigma_\tau$.*

Proof. Once again we let $C_0 = (\partial_q h_4 - \tau_1 \partial_p h_4) / \partial_q h_1$, but remember that in this moduli space τ_1 is allowed to vary, so C_0 is arbitrary. As in the proof of Lemma 5.12, we may still reparametrise so that $D_a = 0$ and $D_3 \cdot \partial_p h_3 = 0$. Since $m = 0$, $\partial_p h_3 = 0$ and hence $D = D_3\mathbf{n}_3$. Therefore, τ_1, D_3, p and q are local coordinates on the moduli space. \square

6. Automorphisms

We observed in Lemma 5.2 that a holomorphic map u from a torus with one marked point z has no nontrivial holomorphic automorphisms. If we consider only unmarked curves then the automorphism group, $\text{Aut}(u)$, of a multiply-covered curve $u = v \circ \pi$ is precisely the deck transformation group of the holomorphic covering π . For the one-point moduli space, the size of this automorphism group becomes the size of the equivalence classes in Corollary 5.14. Therefore we must now compute $|\text{Aut}(u)|$.

LEMMA 6.1. *If $A = [m, m, n, n] \in H_2(K; \mathbf{Z})$ is a non-zero homology class and $u = e^h e^{C_0+D}$ is a holomorphic torus as in Corollary 5.10, where $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ is a Lie algebra homomorphism of the form given in (4.18), then*

$$|\text{Aut}(u)| = \text{gcd}(\text{gcd}(m, n), (mk + n\ell)/d)$$

where $k = \partial_q h_4 + \partial_q h_1 \partial_q h_2 / (2 \text{gcd}(\partial_q h_1, \partial_q h_2))$ and $\ell = \partial_q h_3$.

Proof. Suppose that $\pi_1(u) : \mathbf{Z}^2 \rightarrow \Gamma$ is the (reduced) homomorphism on fundamental groups. We write $\pi : \Gamma \rightarrow \Gamma/Z(\Gamma) \cong \mathbf{Z}^2$ for the projection and

$$\pi_1(u)(1, 0) = \begin{pmatrix} 1 & b_1 & d_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix} \oplus c_1, \quad \pi_1(u)(0, 1) = \begin{pmatrix} 1 & b_2 & d_2 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \oplus c_2.$$

Since $\pi_1(u)$ is reduced, $a_1 = b_1 = 0$ and $a_2 \neq 0$, which implies that $\pi \circ \pi_1(u)$ lands in the cyclic subgroup $\iota : \mathbf{Z} \hookrightarrow \mathbf{Z}^2$ generated by (\bar{b}_2, \bar{a}_2) , where

$$\bar{a}_2 = \frac{a_2}{\text{gcd}(a_2, b_2)} \quad \text{and} \quad \bar{b}_2 = \frac{b_2}{\text{gcd}(a_2, b_2)}.$$

If v is the simple torus underlying u then $\pi \circ \pi_1(v)$ also lands in this subgroup and hence the image of $\pi_1(v)$ is contained in the preimage $\pi^{-1}\iota(\mathbf{Z})$. We have

$$\pi^{-1}(\iota(\mathbf{Z})) = \left\{ \begin{pmatrix} 1 & q\bar{b}_2 & \mathbf{Z} \\ 0 & 1 & q\bar{a}_2 \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathbf{Z} \mid q \in \mathbf{Z} \right\} \cong \mathbf{Z}^3.$$

The isomorphism with \mathbf{Z}^3 is

$$(q, r, s) \mapsto \begin{pmatrix} 1 & q\bar{b}_2 & s + \frac{q(q-1)\bar{a}_2\bar{b}_2}{2} \\ 0 & 1 & q\bar{a}_2 \\ 0 & 0 & 1 \end{pmatrix} \oplus r.$$

Since $\pi_1(u) : \mathbf{Z}^2 \rightarrow \pi^{-1}(\iota(\mathbf{Z})) \cong \mathbf{Z}^3$ is given by the matrix

$$\begin{pmatrix} 0 & & \gcd(a_2, b_2) \\ c_1 & & c_2 \\ d_1 & d_2 - \frac{\gcd(a_2, b_2)(\gcd(a_2, b_2) - 1)}{2} & \bar{a}_2 \bar{b}_2 \end{pmatrix},$$

the maximal sublattice Λ of \mathbf{Z}^3 (and hence of Γ) containing $\iota(\mathbf{Z}^2)$ as a finite-index sublattice has

$$[\Lambda : \iota(\mathbf{Z}^2)] = \gcd(M_1, M_2, M_3)$$

where M_1, M_2, M_3 are the two-by-two minors of this matrix. One can see this by putting the matrix into Smith normal form.

In terms of the integer derivatives of the underlying Lie algebra homomorphism,

$$\begin{aligned} c_1 &= \partial_p h_3, & d_1 &= \partial_p h_4 \\ a_2 &= \partial_q h_1, & b_2 &= \partial_q h_2 \\ c_2 &= \partial_q h_3, & d_2 &= \partial_q h_4 + \frac{1}{2} \partial_q h_1 \partial_q h_2, \end{aligned}$$

so $|\text{Aut}(u)|$ is equal to the greatest common divisor of

$$\gcd(\partial_p h_3, \partial_p h_4) \gcd(\partial_q h_1, \partial_q h_2)$$

and

$$\partial_p h_3 \left(\partial_q h_4 + \frac{\partial_q h_1 \partial_q h_2}{2 \gcd(\partial_q h_1, \partial_q h_2)} \right) - \partial_p h_4 \partial_q h_3.$$

When h has the form given in (4.18), this expression reduces to

$$\begin{aligned} &\gcd \left(\gcd \left(\frac{m}{d}, \frac{m}{d} \right) d, \frac{|m|}{d} k + \frac{n}{\text{sgn}(m)d} \ell \right) \\ &= \gcd(\gcd(m, n), (mk + n\ell)/d) \end{aligned}$$

where $k = \partial_q h_4 + \partial_q h_1 \partial_q h_2 / (2 \gcd(\partial_q h_1, \partial_q h_2))$ and $\ell = \partial_q h_3$. □

7. The linearised Cauchy–Riemann operator

The aim of this section is to prove the following theorem.

THEOREM 7.1. *If $A \in H_2(K; \mathbf{Z})$ is a non-zero homology class with $A_{13} = A_{23}$, $A_{14} = A_{24}$ then $\mathcal{M}_{1,1}(W, A)$ is clean (see Definition 2.14). Moreover, if $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ is a reduced homomorphism (i.e. $\partial_p h_a = 0$) with $[h] = A$ and $\partial_p h_3 = 0$ (which happens if and only if $A_{13} = 0$) then there is a nonvanishing section of the obstruction bundle over the moduli space $\mathcal{M}_{1,1}(W, [h]_{\text{SL}})$ and hence these holomorphic tori do not contribute to the Gromov–Witten invariant in the class A .*

By Lemma 4.17, we may always assume that h is reduced.

7.1 The setup for Fredholm theory

Fix a homomorphism $\rho : \mathbf{Z}^2 \rightarrow \Gamma$. Let $W_\rho^{1,\ell}(\mathbf{R}^2, \mathfrak{n})$ be the $W^{1,\ell}$ -completion of the space of smooth maps $w : \mathbf{R}^2 \rightarrow \mathfrak{n}$ which are ρ -equivariant in the sense that

$$\exp(w(\gamma + z)) = \rho(\gamma) \exp(w(z)) \quad \text{for all } \gamma \in \mathbf{Z}^2.$$

Then $\mathcal{B} := W \times \mathbf{H} \times W_\rho^{1,\ell}(\mathbf{R}^2, \mathfrak{n})$ is a Banach manifold whose tangent space at (ψ, τ, w) is the vector space

$$T_\psi W \oplus T_\tau \mathbf{H} \oplus W_\rho^{1,\ell}(\mathbf{R}^2, w^*T\mathfrak{n})$$

where the subscript ρ denotes equivariant sections. Define the Banach bundle \mathcal{E} over \mathcal{B} whose fibre over (ψ, τ, w) is the L^ℓ -completion of

$$\Omega_{\psi,\tau,\rho}^{0,1}(\mathbf{R}^2, w^*T\mathfrak{n})$$

where $\Omega_{\psi,\tau,\rho}^{(0,1)}(\mathbf{R}^2, w^*T\mathfrak{n})$ denotes the space of smooth ρ -equivariant one-forms on \mathbf{R}^2 with values in $w^*T\mathfrak{n}$ which are anticomplex with respect to the almost complex structures $(j_\tau, w^*\psi)$. The $\bar{\partial}$ -operator

$$\bar{\partial}(\psi, \tau, w) = \psi(\partial_a w - \frac{1}{2}[w, \partial_a w]) - \partial_b w + \frac{1}{2}[w, \partial_b w]$$

gives a section of this bundle whose zero-set comprises the logarithms of tori $\mathbf{Z}^2 \setminus \mathbf{R}^2 \rightarrow K$ in the given homotopy class which are (j_τ, J_ψ) -holomorphic for some $\psi \in W$ and some $\tau \in \mathbf{H}$.

The first aim is to understand the kernel of the linearised $\bar{\partial}$ -operator.

PROPOSITION 7.2. *The linearised $\bar{\partial}$ -operator at a pseudoholomorphic torus (ψ, τ, w) is an operator*

$$D_{(\psi,\tau,w)} \bar{\partial} : T_\psi W \oplus T_\tau \mathbf{H} \oplus W_\rho^{1,\ell}(\mathbf{R}^2, w^*T\mathfrak{n}) \rightarrow L^\ell \Omega_{\psi,\tau,\rho}^{(0,1)}(\mathbf{R}^2, w^*T\mathfrak{n}).$$

If we define $D(\alpha, \eta, \xi) := D_{(\psi,\tau,w)} \bar{\partial}(\alpha, \eta, \xi)(\partial_a)$ (which determines the whole operator $D\bar{\partial}$ since this takes values in $(0, 1)$ -forms) then D is given by

$$\begin{aligned} D(\alpha, \eta, \xi) &= \psi \partial_a \xi - \partial_b \xi + \alpha \left(\partial_a w - \frac{1}{2}[w, \partial_a w] \right) + \frac{1}{\tau_2} (\eta_1 \partial_a w + \eta_2 \partial_b w) \\ &\quad - \frac{1}{2} \psi([\xi, \partial_a w] + [w, \partial_a \xi]) + \frac{1}{2}([\xi, \partial_b w] + [w, \partial_b \xi]). \end{aligned} \tag{7.3}$$

Proof. The equation to be linearised is

$$\bar{\partial}(\psi, \tau, w) = \psi(\partial_a w - \frac{1}{2}[w, \partial_a w]) - \partial_b w + \frac{1}{2}[w, \partial_b w].$$

The only nonobvious part of the computation is the effect of an infinitesimal variation η of τ :

$$D(0, \eta, 0) = \frac{1}{\tau_2} (\eta_1 \partial_a w + \eta_2 \partial_b w).$$

To see this, recall that in the coordinates (a, b) the complex structure is simply $j_i = \Phi_\tau j_\tau \Phi_\tau^{-1}$. If η is an infinitesimal variation of τ then the infinitesimal variation of j_i is computed with respect to the same coordinates (a, b) by

$$\delta_\eta j_i = \Phi_\tau \delta_\eta j_\tau \Phi_\tau^{-1}$$

since Φ_τ is just a change of coordinate matrix and hence is not affected by the variation. We compute

$$\delta_\eta j_i = \Phi_\tau \delta_\eta j_\tau \Phi_\tau^{-1} = \frac{1}{\tau_2} \begin{pmatrix} -\eta_1 & -\eta_2 \\ -\eta_2 & \eta_1 \end{pmatrix}. \quad \square$$

7.2 Equivariance

We now examine more carefully the equivariance condition

$$\exp(w(\gamma + z)) = \rho(\gamma) \exp(w(z)) \quad \text{for all } \gamma \in \mathbf{Z}^2$$

and its linearisation. By the Baker–Campbell–Hausdorff formula we have

$$w(\gamma + z) = \log \rho(\gamma) + w(z) + \frac{1}{2}[\log \rho(\gamma), w(z)].$$

If ξ is an infinitesimal deformation of w as an equivariant map then

$$\xi(\gamma + z) = \xi(z) + \frac{1}{2}[\log \rho(\gamma), \xi(z)].$$

In particular we see that $\xi_{\mathfrak{a}}$ is \mathbf{Z}^2 -invariant and hence bounded. The combination $\xi_{\mathfrak{z}} - \frac{1}{2}[w, \xi_{\mathfrak{a}}]$ is also \mathbf{Z}^2 -invariant:

$$\begin{aligned} \xi_{\mathfrak{z}}(\gamma + z) - \frac{1}{2}[w(\gamma + z), \xi_{\mathfrak{a}}(\gamma + z)] &= \xi_{\mathfrak{z}}(z) + \frac{1}{2}[\log \rho(\gamma), \xi_{\mathfrak{a}}(z)] - \frac{1}{2}[\log \rho(\gamma) + w(z), \xi_{\mathfrak{a}}(z)] \\ &= \xi_{\mathfrak{z}}(z) - \frac{1}{2}[w(z), \xi_{\mathfrak{a}}(z)]. \end{aligned}$$

7.3 Regularity and obstructions

Remember that $w = h + C + \frac{1}{2}[h, C]$ where h is a Lie algebra homomorphism and C is a constant (in particular, second derivatives of w vanish). Cross-differentiating

$$D(\alpha, \eta, \xi) = 0$$

using (7.3) we get

$$\begin{aligned} 0 &= \psi \partial_a^2 \xi - \partial_a \partial_b \xi - \frac{1}{2} \psi [w_{\mathfrak{a}}, \partial_a^2 \xi_{\mathfrak{a}}] + \frac{1}{2}([\partial_{[a} \xi_{\mathfrak{a}}, \partial_{b]} w_{\mathfrak{a}}] + [w_{\mathfrak{a}}, \partial_a \partial_b \xi_{\mathfrak{a}}]) \\ 0 &= \psi \partial_a \partial_b \xi - \partial_b^2 \xi + \frac{1}{2} \psi ([\partial_{[a} \xi_{\mathfrak{a}}, \partial_{b]} w_{\mathfrak{a}}] - [w_{\mathfrak{a}}, \partial_a \partial_b \xi_{\mathfrak{a}}]) + \frac{1}{2} [w_{\mathfrak{a}}, \partial_b^2 \xi_{\mathfrak{a}}] + \frac{1}{2} \alpha ([\partial_a w, \partial_b w]). \end{aligned}$$

Since h is a homomorphism from an abelian Lie algebra,

$$[\partial_a w, \partial_b w] = [\partial_a h, \partial_b h] = 0.$$

The equations then give

$$\Delta \xi - \frac{1}{2}[w_{\mathfrak{a}}, \Delta \xi_{\mathfrak{a}}] - \psi [\partial_{[a} \xi_{\mathfrak{a}}, \partial_{b]} w_{\mathfrak{a}}] = 0.$$

The \mathfrak{a} - and \mathfrak{z} -parts of this equation are

$$\begin{aligned} \Delta \xi_{\mathfrak{a}} &= \psi [\partial_a \xi_{\mathfrak{a}}, \partial_b w_{\mathfrak{a}}] \\ \Delta \xi_{\mathfrak{z}} &= \frac{1}{2} [w_{\mathfrak{a}}, \Delta \xi_{\mathfrak{a}}]. \end{aligned}$$

PROPOSITION 7.4. *Suppose that $\psi \in W$ and that $w = h + C + \frac{1}{2}[h, C]$ is the logarithm of a (j_{τ}, J_{ψ}) -holomorphic curve in K with linearised Cauchy–Riemann operator D . Then, if $D(\alpha, \eta, \xi) = 0$ then $\xi_{\mathfrak{a}}$ and $\xi_{\mathfrak{z}} - \frac{1}{2}[w, \xi_{\mathfrak{a}}]$ are constant.*

Proof. Split the Lie algebra \mathfrak{n} as in the proof of Proposition 4.1 into $\mathfrak{b} \oplus [\mathfrak{n}, \mathfrak{n}] \oplus \psi[\mathfrak{n}, \mathfrak{n}]$. We see that

$$\Delta \xi_{\mathfrak{b}} = 0$$

and $\xi_{\mathfrak{b}}$, being periodic, is constant. Next,

$$\Delta \xi_{\mathfrak{q}} = \psi [\partial_a \xi_{\mathfrak{q}}, \partial_b w_{\mathfrak{b}}],$$

which is a linear elliptic equation with constant coefficients for the single bounded quantity $\xi_{\mathfrak{q}}$, which is therefore constant. Finally,

$$\Delta \xi_{\mathfrak{p}} = \frac{1}{2} [w_{\mathfrak{b}}, \Delta \xi_{\mathfrak{q}}] = 0$$

and so $\xi_{\mathfrak{p}} - \frac{1}{2}[w, \xi_{\mathfrak{a}}]$ is harmonic and bounded. □

To prove Theorem 7.1 we compute the kernel of D and then compare with the computation of the tangent spaces of the moduli space in Corollary 5.14. By Lemma 4.17, we assume without loss of generality that $\partial_p h_a = 0$. Taking the \mathfrak{a} -part of the linearised equation and using the facts that

$$\begin{aligned} \alpha(\mathfrak{z}) &\subset \mathfrak{a} & \alpha(\mathfrak{a}) &\subset \mathfrak{z} \\ \partial_a \xi_a &= 0 & \partial_b \xi_a &= 0 \\ \partial_a \xi_{\mathfrak{z}} &= \frac{1}{2}([\partial_a w_a, \xi_a] + [w_a, \partial_a \xi_a]) & \partial_b \xi_{\mathfrak{z}} &= \frac{1}{2}([\partial_b w_a, \xi_a] + [w_a, \partial_b \xi_a]) \\ &= 0 & &= \frac{1}{2}[\partial_b h_a, \xi_a] \end{aligned}$$

we get

$$0 = \alpha(\partial_a h_{\mathfrak{z}}) + \frac{\eta_2}{\tau_2} \partial_b h_a$$

and for the \mathfrak{z} -part we get

$$\begin{aligned} 0 &= -\partial_b \xi_{\mathfrak{z}} + \frac{1}{2}[\xi_a, \partial_b h_a] + \frac{1}{\tau_2}(\eta_1 \partial_a h_{\mathfrak{z}} + \eta_2 (\partial_b w)_{\mathfrak{z}}) \\ &= [\xi_a, \partial_b h_a] + \frac{1}{\tau_2}(\eta_1 \partial_a h_{\mathfrak{z}} + \eta_2 (\partial_b w)_{\mathfrak{z}}). \end{aligned}$$

LEMMA 7.5. We describe the kernel of D . If $D(\alpha, \eta, \xi) = 0$ then the vector $\xi_{\mathfrak{z}} - \frac{1}{2}[w, \xi_a]$ is arbitrary, $\eta_2 = 0$ and $\alpha = 0$. Moreover:

- (a) if $\partial_p h_{\mathfrak{z}} \neq 0$ then $\eta_1 = 0$ and $\xi \in \ker[\partial_b h_a, \cdot]$;
- (b) if $\partial_p h_{\mathfrak{z}} = 0$ then η_1 is arbitrary and ξ_a satisfies

$$[\xi_a, \partial_b h_a] + \eta_1 \partial_a h_{\mathfrak{z}} = 0.$$

Proof. To see $\eta_2 = 0$, recall from (5.5) that $\psi \partial_a h_{\mathfrak{z}} = \partial_b h_a$, so if

$$\alpha = r \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix} \in TW$$

then

$$r \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix} \begin{pmatrix} \partial_p h_{\mathfrak{z}} \\ \partial_p h_a \end{pmatrix} = -\frac{\eta_2}{\tau_2} r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \partial_p h_{\mathfrak{z}} \\ \partial_p h_a \end{pmatrix}.$$

Multiplying by Ψ_{θ}^{-1} on the left tells us that $\begin{pmatrix} \partial_p h_{\mathfrak{z}} \\ \partial_p h_a \end{pmatrix}$ is a real eigenvector of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ unless $r = \eta_2 = 0$. Since this matrix has only imaginary eigenvalues this is impossible.

The \mathfrak{z} -part of the equation now becomes

$$[\xi_a, \partial_b h_a] + \frac{\eta_1}{\tau_2} \partial_a h_{\mathfrak{z}} = 0.$$

Recall from Lemma 5.4 that $\partial_b h_{\mathfrak{z}} = [C_a, \partial_b h_a]$ so that $\partial_b h_{\mathfrak{z}} = 0$ (the t -direction is orthogonal to the commutator). Since $w_3 = h_3 + C_3$ and $\mathfrak{n}_3 \perp [\mathfrak{n}, \mathfrak{n}]$, we see that if $\partial_p h_{\mathfrak{z}} \neq 0$ then $\eta_1 = 0$. The rest is now clear by inspection. \square

By comparing with Corollary 5.14 and Lemma 5.15 we see that the kernel of D is equal to the tangent space of the moduli space, proving the cleanliness claimed in Theorem 7.1.

From the expected and actual dimension formulae for the moduli spaces we see that the moduli spaces of pseudoholomorphic tori are regular if and only if $\partial_p h_{\mathfrak{z}} \neq 0$. We will now write down sections of the obstruction bundles for each moduli space which is not regular. Since a fibre of the obstruction bundle is a space of $(0, 1)$ -forms, it suffices to specify the value of a section σ on the vector ∂_a .

LEMMA 7.6. *The section $\sigma(\partial_a) = \mathbf{n}_3$ is a nowhere-vanishing section of the obstruction bundle over moduli spaces $h_1 \neq 0, \partial_p h_3 = 0$.*

Proof. Since $\partial_p h_3 = 0$ and $\partial_q h_3 - \tau_1 \partial_p h_3 = 0$ we see that in this case $\partial_q h_3 = 0$ as well. If (α, η, ξ) is an infinitesimal variation then we show that

$$\int D_{(\psi, \tau, w)} \bar{\partial}(\alpha, \eta, \xi)(\partial_a) \cdot \sigma(\partial_a) \, \text{dvol} = 0$$

by examining the contributions from the three parts separately.

First, since $\alpha(\partial_a h_3) \in \mathfrak{a}$, it is obviously orthogonal to \mathbf{n}_3 so this term vanishes. Next, the integrand contribution from η is

$$\frac{1}{\tau_2} \left(\eta_1 \partial_a h_3 + \eta_2 \left(\partial_b h_a + \partial_b h_3 + \frac{1}{2} [\partial_b h_a, C_a] \right) \right) \cdot \mathbf{n}_3.$$

Since $\partial_p h_3 = \partial_q h_3 = 0$ this vanishes. Finally, the contribution from ξ is

$$\int \left(\psi \partial_a \xi - \partial_b \xi - \frac{1}{2} \psi [w_a, \partial_a \xi_a] + \frac{1}{2} ([\xi_a, \partial_b h_a] + [w_a, \partial_b \xi_a]) \right) \cdot \mathbf{n}_3 \, \text{dvol}.$$

The first two terms vanish by integrating by parts (using Γ -equivariance of ξ_3 , see §7.2) since \mathbf{n}_3 is constant. The other three vanish because \mathbf{n}_3 is orthogonal to $\psi[\mathfrak{n}, \mathfrak{n}]$ and to $[\mathfrak{n}, \mathfrak{n}]$. \square

This completes the proof of Theorem 7.1.

7.4 Orientations

To determine the orientations on our moduli spaces we need to write down a homotopy from the linearised $\bar{\partial}$ -operator to its complex-linear part $D^{\mathbf{C}}$. By this we mean the part which is complex-linear in ξ . Define

$$S(\xi) = \frac{1}{2} ([\xi, \partial_b w] + [w, \partial_b \xi] - \psi[\xi, \partial_a w] - \psi[w, \partial_a \xi])$$

and set

$$\begin{aligned} D^\epsilon(\alpha, \eta, \xi)(\partial_a) &= \psi \partial_a \xi - \partial_b \xi + \alpha \left(\partial_a w - \frac{1}{2} [w, \partial_a w] \right) + \frac{1}{\tau_2} (\eta_1 \partial_a w + \eta_2 \partial_b w) \\ &\quad + S(\xi) - \frac{\epsilon}{2} (S(\xi) - \psi S(\psi \xi)). \end{aligned} \tag{7.7}$$

This is D when $\epsilon = 0$ and $D^{\mathbf{C}}$ when $\epsilon = 1$. We obtain (after cross-differentiating)

$$\begin{aligned} \Delta \xi &= \psi [\partial_{[a} \xi, \partial_{b]} w] + \frac{1}{2} [w, \Delta \xi] \\ &\quad - \frac{\epsilon}{2} (\psi [\partial_{[a} \xi, \partial_{b]} w] + [\psi \partial_{[a} \xi, \partial_{b]} w]) + \frac{\epsilon}{4} (\psi [w, \psi \Delta \xi] - [w, \Delta \xi]). \end{aligned}$$

This gives the \mathfrak{a} and \mathfrak{z} parts

$$\Delta \xi_a - \left(1 - \frac{\epsilon}{2} \right) \psi [\partial_{[a} \xi_a, \partial_{b]} w_a] = \frac{\epsilon}{4} \psi [w_a, \psi \Delta \xi_3] \tag{7.8}$$

$$\Delta \xi_3 + \frac{\epsilon}{2} [\psi \partial_{[a} \xi_3, \partial_{b]} w_a] = \frac{2 - \epsilon}{4} [w, \Delta \xi_a]. \tag{7.9}$$

If we define $Q = \xi_3 - \frac{1}{2}[w, \xi_a]$ then

$$\begin{aligned} \partial Q &= \partial \xi_3 - \frac{1}{2}[\partial w, \xi_a] - \frac{1}{2}[w, \partial \xi_a] \\ \Delta Q &= \Delta \xi_3 - [\partial_a w_a, \partial_a \xi_a] - [\partial_b w_a, \partial_b \xi_a] - \frac{1}{2}[w_a, \Delta \xi_a]. \end{aligned}$$

The left-hand side of (7.8) is bounded. The right-hand side can be rewritten as

$$\frac{\epsilon}{4} \psi \left[w_a, \psi \left(\Delta Q + [\partial_a w_a, \partial_a \xi_a] + [\partial_b w_a, \partial_b \xi_a] + \frac{1}{2}[w_a, \Delta \xi_a] \right) \right],$$

which is a sum of terms which are linear or quadratic in a and b ; in particular, it is unbounded unless the coefficients vanish and hence the whole right-hand side is zero. Equation (7.8) therefore reduces to a linear elliptic equation which (up to the factor of $1 - \epsilon/2$) we have dealt with before. In particular, we deduce that ξ_a is constant. Equation (7.9) now reduces to a linear elliptic equation which we have dealt with before and we deduce that Q is constant.

Returning to the original (7.7), and bearing in mind that ξ_a and $\xi_3 - \frac{1}{2}[w, \xi_a]$ are constant, we have \mathfrak{a} - and \mathfrak{z} -components

$$\begin{aligned} 0 &= \alpha(\partial_a h_3) + \frac{\eta_2}{\tau_2} \partial_b h_a + \frac{\epsilon}{4} \psi \left[\psi \left(\xi_3 - \frac{1}{2}[w, \xi_a] \right), \partial_b w \right] \\ 0 &= \left(1 - \frac{\epsilon}{4} \right) [\xi_a, \partial_b w] + \frac{1}{\tau_2} (\eta_1 \partial_a h_3 + \eta_2 (\partial_b w)_3). \end{aligned}$$

LEMMA 7.10. *When $\partial_p h_3 \neq 0$, the space of solutions to this equation is three-dimensional. Explicitly, if we write $\alpha = r \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}$, the solutions are*

$$\xi_a = 0, \quad Q = \partial_a h_3, \quad r = 0, \quad \eta = 0, \tag{S1}$$

$$\xi_a = \partial_b h_a, \quad Q = 0, \quad r = 0, \quad \eta = 0 \tag{S2}$$

and

$$\begin{aligned} \xi_a &= -\frac{\partial_p h_4 C_{[2} \partial_b h_1]}{2(1 - \epsilon/4) |\partial_b h_a|^2} \begin{pmatrix} -\partial_b h_2 \\ \partial_b h_1 \end{pmatrix}, \quad r = \partial_p h_3 \\ Q &= \frac{4}{\epsilon} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \partial_q h_4 \\ -\partial_q h_3 \end{pmatrix}, \quad \eta = i \partial_p h_4 \tau_2. \end{aligned} \tag{S3}$$

Proof. The second solution is obvious; the first follows from (5.5) and (7.8); the third follows from (5.6) and (7.9) and the fact that

$$[-\partial_b h_2 \mathbf{n}_1 + \partial_b h_1 \mathbf{n}_2, \partial_b h_1 \mathbf{n}_1 + \partial_b h_2 \mathbf{n}_2] = |\partial_b h_a|^2 \mathbf{n}_4. \quad \square$$

The canonical orientation is now given by picking the oriented basis (S1), (S2) (which are related by ψ , thanks to (5.5)) and (S1) in that order. When we enumerate the tori we will be able to assume after an $SL(2, \mathbf{Z})$ -transformation that $\partial_p h_3 > 0$ so (S1) is positively oriented relative to the base W . We rescale (S1) by ϵ and let ϵ tend to zero. Solution (S1) becomes

$$\xi_a = 0, \quad Q = 4 \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \partial_q h_4 \\ -\partial_q h_3 \end{pmatrix}, \quad r = 0, \quad \eta = 0.$$

Note that by (5.5),

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \partial_q h_4 \\ -\partial_q h_3 \end{pmatrix} = \begin{pmatrix} \partial_p h_4 \\ -\partial_p h_3 \end{pmatrix}.$$

By Corollary 5.14 and Lemma 5.12 we know that the unparametrised moduli space $\mathcal{M}(W, A)$, consisting of curves He^{C_0+D} , admits a reparametrisation action which one can use to ensure that the (S1) and (S2) components of D vanish so that

$$D = \lambda \begin{pmatrix} 0 \\ 0 \\ \partial_p h_4 \\ -\partial_p h_3 \end{pmatrix}.$$

The moduli space $\mathcal{M}_{1,1}(W, A)$ therefore consists of triples $(He^{C_0+D(\lambda)}, p, q)$ with $(p, q) \in T^2$. We have shown that the orientation on the moduli space is precisely the one given by the three-form $d\lambda \wedge dp \wedge dq$.

8. Enumeration of tori in K

The aim of this section is to compute the Gromov–Witten invariant in a non-zero homology class $A = \sum A_{ij}E_{ij}$. By Lemma 4.14 we can transform this homology class by an automorphism φ of Γ to a class $\varphi_*A = [m, m, n, n]$ where $m = \gcd(A_{13}, A_{23})$ and $n = \gcd(A_{14}, A_{24})$. Equation (4.6) implies that

$$\varphi_* \text{GW}_{1,1}(W, A) = \text{GW}_{1,1}(W, \varphi_*A),$$

so without loss of generality we can assume that $A = [m, m, n, n]$ for the sake of computing its Gromov–Witten invariants.

For such a class, Theorem 7.1 tells us that:

- (a) if $m = 0$ then the Gromov–Witten invariant vanishes;
- (b) if $m \neq 0$ then the moduli space $\mathcal{M}_{1,1}(W, A)$ is regular.

We therefore consider only the case $m \neq 0$. By Lemma 5.3,

$$\mathcal{M}_{1,1}(W, A) = \mathcal{M}^{\text{ful}}(W, A) \times_{T^2} T^2,$$

which is a union over all fully reduced homomorphisms ρ with $[\rho] = A$ of

$$\mathcal{M}_\rho(W) \times_{T^2} T^2.$$

In terms of the Lie algebra homomorphism $h : \mathbf{R}^2 \rightarrow \mathfrak{n}$ (the logarithm of the unique homomorphic extension $H : \mathbf{R}^2 \rightarrow N$ of ρ), the fully reduced homomorphisms have the matrix form

$$\begin{pmatrix} \partial_p h_1 & \partial_q h_1 \\ \partial_p h_2 & \partial_q h_2 \\ \partial_p h_3 & \partial_q h_3 \\ \partial_p h_4 & \partial_q h_4 \end{pmatrix} = \begin{pmatrix} 0 & -\text{sgn}(m)d \\ 0 & -\text{sgn}(m)d \\ \frac{|m|}{d} & \partial_q h_3 \\ -\frac{n}{\text{sgn}(m)d} & \partial_q h_4 \end{pmatrix},$$

where d is a positive divisor of $\gcd(m, n)$, $\partial_q h_3 \in \mathbf{Z}$, $0 \leq \partial_q h_3 < |m|/d$ and $\partial_q h_4 + \frac{1}{2}\partial_q h_1\partial_q h_2 \in \mathbf{Z}$.

We can now use the concrete description of the moduli space given in Corollary 5.14 and its orientation as given in § 7.4 to describe the evaluation cycle. The moduli space consists of maps

$He^{C_0+D(\lambda)}$ where $H = \exp(h)$ and

$$D(\lambda) = \lambda \begin{pmatrix} 0 \\ 0 \\ \partial_p h_4 \\ -\partial_p h_3 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 \\ \frac{\partial_q h_4 - \tau_1 \partial_p h_4}{\partial_q h_1} \\ 0 \\ 0 \end{pmatrix}, \quad \lambda \in \left[0, \frac{1}{\gcd(\partial_p h_3, \partial_p h_4)} \right].$$

Note that h determines τ_1 and hence also C_0 . For a Lie algebra homomorphism h and a real number λ we denote by $u(\lambda, h) : T^2 \rightarrow K$ the curve represented by

$$He^{C_0+D(\lambda)} : \mathbf{R}^2 \rightarrow N.$$

LEMMA 8.1. *Let $k \in \mathbf{Z}$ and let $\rho : \mathbf{Z}^2 \rightarrow \Gamma$ be a reduced homomorphism with underlying Lie algebra homomorphism h . Consider ρ' , the modified homomorphism whose underlying Lie algebra map h' has the same derivatives as h except for*

$$\partial_q h'_4 = \partial_q h_4 + k.$$

Then the tori $u(\lambda, h)$ and $u(\lambda, h')$ are equal if and only if $k \in (\partial_q h_1)\mathbf{Z} = d\mathbf{Z}$.

Proof. Under this change, C_0 changes to $C'_0 = C_0 + (k/\partial_q h_1)\mathbf{n}_2$. We have

$$\exp(h') \exp(C'_0 + D(\lambda)) = \exp(h + qk\mathbf{n}_4) \exp\left(\frac{k}{\partial_q h_1}\mathbf{n}_2\right) \exp(C_0 + D(\lambda))$$

and

$$\begin{aligned} \exp(h + qk\mathbf{n}_4) \exp\left(\frac{k}{\partial_q h_1}\mathbf{n}_2\right) &= \exp\left(h + qk\mathbf{n}_4 + \frac{k}{\partial_q h_1}\mathbf{n}_2 - \frac{1}{2}qk\mathbf{n}_4\right) \\ &= \exp\left(h + \frac{k}{\partial_q h_1}\mathbf{n}_2 + \frac{1}{2}qk\mathbf{n}_4\right) \\ &= \exp\left(\frac{k}{\partial_q h_1}\mathbf{n}_2\right) \exp(h) \end{aligned}$$

and this agrees with $\exp(h)$ modulo the right action of Γ if and only if $\partial_q h_1$ divides k . □

COROLLARY 8.2. *Let $A = [m, m, n, n] \in H_2(K; \mathbf{Z})$ be a homology class with $m \neq 0$. For each divisor d of $\gcd(m, n)$ there are $|m|/d$ values of $\partial_q h_3$ and d values of $\partial_q h_4$ giving distinct tori and hence $\mathcal{M}_{1,1}(W, A)$ has $|m|\sigma_0(\gcd(m, n))$ components. □*

We need to calculate the homology class of the evaluation cycle for each of these components. For simplicity, we first ignore the equivalence relation $(u, z) \sim (u \circ \varphi^{-1}, \varphi(z))$ for $\varphi \in \text{Aut}(u)$ mentioned in Corollary 5.14; this means we are passing to an $|\text{Aut}(u)|$ -sheeted cover of the moduli space which we write as $\mathcal{M}'_{1,1}(W, A)$. We will later divide out by the size of the automorphism group to compensate for this.

Using the coordinates $(\lambda, p, q) \mapsto (He^{C_0+D(\lambda)}, p + iq)$ on $\mathcal{M}'_{1,1}(W, A)$, the evaluation map sends (λ, p, q) to

$$\left(\begin{pmatrix} 1 & q\partial_q h_2 + \frac{\partial_q h_4 - \tau_1 \partial_p h_4}{\partial_q h_1} & R & 0 \\ 0 & 1 & q\partial_q h_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda\partial_p h_4 + p\partial_p h_3 + q\partial_q h_3 \end{pmatrix}, \psi \right) \in K \times W,$$

where ψ is the unique complex structure for which $He^{C_0+D(\lambda)}$ is (j_τ, J_ψ) -holomorphic,

$$R = -\lambda\partial_p h_3 + p\partial_p h_4 + q\partial_q h_4 + \frac{1}{2}q^2\partial_q h_1\partial_q h_2$$

and $\lambda \in [0, d/\gcd(m, n)]$. Since ψ is determined by the derivatives of the Lie algebra homomorphism h , see (5.8), it is constant over each component of the moduli space and the evaluation map can be thought of as a three-cycle in K . This cycle represents the three-dimensional homology class

$$-(\partial_q h_1 E_{134} + \partial_q h_2 E_{234}) \frac{|\partial_p h_3|^2}{\gcd(\partial_p h_3, \partial_p h_4)} = \operatorname{sgn}(m) \frac{m^2 + n^2}{\gcd(m, n)} (E_{134} + E_{234})$$

as we can see by integrating the forms e_{ijk} pulled back along the map $\exp(h + C + \frac{1}{2}[h, C])$.

As we remarked above, we are currently overcounting because we have not divided out by the equivalence relation $(u, z) \sim (u \circ \varphi^{-1}, \varphi(z))$ for $\varphi \in \operatorname{Aut}(u)$. By Lemma 6.1, if we write $k = \partial_q h_4 + \partial_q h_1 \partial_q h_2 / 2 \gcd(\partial_q h_1, \partial_q h_2)$ and $\ell = \partial_q h_3$ then the torus corresponding to the choice of d dividing $\gcd(m, n)$, $0 < k \leq d$ and $0 < \ell \leq |m|/d$ contributes

$$\frac{1}{\gcd(\gcd(m, n), mk + n\ell)}.$$

This gives an factor of

$$(\dagger) = \sum_{d|\gcd(m,n)} \sum_{k=1}^d \sum_{\ell=1}^{|m|/d} \frac{1}{\gcd(\gcd(m, n), (mk + n\ell)/d)}.$$

LEMMA 8.3. *The following equality holds:*

$$\sum_{d|\gcd(m,n)} \sum_{k=1}^d \sum_{\ell=1}^{|m|/d} \frac{1}{\gcd(\gcd(m, n), (mk + n\ell)/d)} = \frac{|m|}{\gcd(m, n)^2} \sum_{d|\gcd(m,n)} d^2.$$

Before we prove this lemma we give the formula for the one-point Gromov–Witten invariant $\operatorname{GW}_{1,1}(W, A)$ when $A = [m, m, n, n]$:

$$\frac{m(m^2 + n^2)\sigma_2(\gcd(m, n))}{\gcd(m, n)^3} (E_{134} + E_{234}) \otimes [\star] \in H_3(K \times W; \mathbf{Z}).$$

We now ignore the $[\star]$ factor. Pushing this result forward using (4.6) allows us to compute the Gromov–Witten invariant $\operatorname{GW}_{1,1}(W, A)$ for $A = [ma, mb, na, nb]$ where $\gcd(a, b) = 1$:

$$\operatorname{GW}_{1,1}(W, A) = \frac{(m^2 + n^2)\sigma_2(\gcd(m, n))}{\gcd(m, n)^3} (maE_{134} + mbE_{234}). \tag{8.4}$$

This proves Theorem 1.2. □

Proof of Lemma 8.3. For convenience, define $\mu = \gcd(m, n)$, $\bar{m} = |m|/\mu$, $\bar{n} = n/\mu$ and $\lambda = \bar{m}k + \bar{n}\ell$. We have

$$\frac{1}{\gcd(\gcd(m, n), (mk + n\ell)/d)} = \frac{d/\mu}{\gcd(d, \operatorname{sgn}(m)\bar{m}k + \bar{n}\ell)}.$$

We convert the sum over k into a sum over λ :

$$\begin{aligned} \sum_{k=1}^d \frac{1}{\gcd(d, \operatorname{sgn}(m)\bar{m}k + \bar{n}\ell)} &= \sum_{\lambda=1}^d \frac{\#\{k : \operatorname{sgn}(m)\bar{m}k + \bar{n}\ell \equiv \lambda \pmod{d}\}}{\gcd(d, \lambda)} \\ &= \sum_{\lambda=1}^d \frac{\gcd(\bar{m}, d)}{\gcd(d, \lambda)} \operatorname{if}(\gcd(\bar{m}, d) \mid \lambda - \bar{n}\ell) \end{aligned}$$

where $\operatorname{if}(X)$ is the Boolean function taking the value 1 if X is true and 0 otherwise. To get this line we use the fact that a linear congruence $ax = y \pmod{d}$ has $\gcd(a, d)$ solutions modulo d if $\gcd(a, d) \mid y$ and none otherwise. Now perform the sum over ℓ :

$$\begin{aligned} \sum_{\ell=1}^{|m|/d} \operatorname{if}(\gcd(\bar{m}, d) \mid \lambda - \bar{n}\ell) &= \frac{|m|}{d \gcd(\bar{m}, d)} \\ &= \frac{\bar{m}}{\gcd(\bar{m}, d)} \frac{\mu}{d} \end{aligned}$$

since $\lambda - \bar{n}\ell \equiv 0 \pmod{\gcd(\bar{m}, d)}$ has a unique solution modulo $\gcd(\bar{m}, d)$, since $\gcd(\bar{m}, \bar{n}) = 1$, and hence $|m|/d \gcd(\bar{m}, d)$ solutions in $\{1, \dots, |m|/d\}$. Substituting this back into the full formula gives

$$\begin{aligned} (\dagger) &= \sum_{d|\mu} \sum_{\lambda=1}^d \frac{\gcd(\bar{m}, d)}{\gcd(d, \lambda)} \frac{d}{\mu} \frac{\bar{m}}{\gcd(\bar{m}, d)} \frac{\mu}{d} \\ &= \sum_{d|\mu} \sum_{\lambda=1}^d \frac{\bar{m}}{\gcd(d, \lambda)} \\ &= \frac{|m|}{\gcd(m, n)^2} \sum_{d|\gcd(m, n)} d^2 \end{aligned}$$

where in the last line we have used Cesàro’s formula

$$\sum_{d|n} \sum_{i=1}^d f(\gcd(i, d)) = \sum_{d|n} f\left(\frac{n}{d}\right) d,$$

valid for any arithmetic function f : this follows from [Dic52, p. 129] and the elementary properties of Dirichlet convolutions. \square

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