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ON QUASISIMILARITY FOR LOG-HYPONORMAL OPERATORS

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Abstract. In this paper we show that the normal parts of quasisimilar log-hyponormal operators are unitarily equivalent. A Fuglede-Putnam type theorem for log-hyponormal operators is proved. Also, it is shown that a log-hyponormal operator that is quasisimilar to an isometry is unitary and that a log-hyponormal spectral operator is normal.

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1. Introduction. Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional complex Hilbert spaces and let $L(\mathcal{H}, \mathcal{K})$ denote the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, we write $L(\mathcal{H})$ in place of $L(\mathcal{H}, \mathcal{K})$. Let $T \in L(\mathcal{H})$. T is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ for some $0 < p \leq 1$. It is known that if T is p-hyponormal and $0 < q < p$, then T is q-hyponormal, by the Lowner-Heinz's inequality [**11**, **14**]. If $p = 1$, T is said to be *hyponormal* and if $p = 1/2$, T is said to be *semi-hyponormal*. T is said to be *log-hyponormal* if T is invertible and satisfies the following inequality

$$\log(T^*T) \geq \log(TT^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal but that the converse is not true [**16**]. However it is very interesting that we may regard log-hyponormal operators as 0-hyponormal operators [**16**, **17**]. Let $T = U|T|$ be the polar decomposition of T. We usually define the Aluthge transform of T by $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$. Let $\widetilde{T} = V|\widetilde{T}|$ be the polar decomposition of $\widetilde{T}$, and define the second Aluthge transform of T by $\widehat{T} = |\widetilde{T}|^{1/2}V|\widetilde{T}|^{1/2}$. It is known that if T is log-hyponormal, then $\widetilde{T}$ is semi-hyponormal and $\widehat{T}$ is hyponormal [**1**, **16**, **21**]. An operator $X \in L(\mathcal{H}_2, \mathcal{H}_1)$ is called a *quasiaffinity* if X is injective and has dense range $R(X)$. For $T_1 \in L(\mathcal{H}_1)$ and $T_2 \in L(\mathcal{H}_2)$, if there exist quasiaffinities $X \in L(\mathcal{H}_2, \mathcal{H}_1)$ and

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$Y \in L(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_1X = XT_2$ and $YT_1 = T_2Y$, then we say that T_1 and T_2 are *quasisimilar*.

Xia [**20**] investigated properties of hyponormal and semi-hyponormal operators. Aluthge [**1**] introduced p-hyponormal operators and investigated their properties using Aluthge transforms. The idea of a log-hyponormal operator is due to Ando [**3**] and the first paper in which log-hyponormality appeared is [**9**]. See [**2**, **16**, **17**, **18**] for properties of log-hyponormal operators.

Jeon and Duggal [**13**] proved that the normal parts of quasisimilar p-hyponormal operators are unitarily equivalent, a p-hyponormal operator compactly quasisimilar to an isometry is unitary, and a p-hyponormal spectral operator is normal.

In this paper we prove that similar results hold for log-hyponormal operators; i.e., the normal parts of quasisimilar log-hyponormal operators are unitarily equivalent, Fuglede-Putnam's theorem holds for log-hyponormal operators, a log-hyponormal operator quasisimilar to an isometry is normal, and a log-hyponormal spectral operator is normal.

2. Normal parts of quasisimilar log-hyponormal operators.

THEOREM 1. *Let $\mathcal{M}$ be an invariant subspace of a log-hyponormal operator $T \in L(\mathcal{H})$ and $T|_{\mathcal{M}}$ the restriction of T to $\mathcal{M}$. If $T|_{\mathcal{M}}$ is invertible, then $T|_{\mathcal{M}}$ is log-hyponormal.*

Proof. Put $T = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ and $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Since

$$\log X = \lim_{p \downarrow 0} \frac{X^p - 1}{p}$$

for an arbitrary positive invertible operator X, we have

$$\begin{aligned} P(\log T^*T)P &= \lim_{p\downarrow 0} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{(T^*T)^p - 1}{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\leq \lim_{p\downarrow 0} \frac{1}{p} \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T^*T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)^p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ &= \lim_{p\downarrow 0} \begin{pmatrix} \frac{(A^*A)^p - 1}{p} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \log A^*A & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

by Hansen's inequality [**10**], and

$$\begin{aligned} P(\log TT^*)P &= \lim_{p\downarrow 0} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{(TT^*)^p - 1}{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\geq \lim_{p\downarrow 0} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{\left(T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T^* \right)^p - 1}{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \lim_{p\downarrow 0} \begin{pmatrix} \frac{(AA^*)^p - 1}{p} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \log AA^* & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

by Löwner-Heinz's inequality [**11**, **14**]. Since T is log-hyponormal,

$$\begin{pmatrix} \log AA^* & 0 \\ 0 & 0 \end{pmatrix} \leq P(\log TT^*)P$$
$$\leq P(\log T^*T)P \leq \begin{pmatrix} \log A^*A & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $A = T|_{\mathcal{M}}$ is also log-hyponormal. □

THEOREM 2. *Let $T \in L(\mathcal{H})$ be log-hyponormal. Then $T = T_1 \oplus T_2$ on the space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where T_1 is normal and T_2 is pure and log-hyponormal; i.e., T_2 has no invariant subspace $\mathcal{M}$ such that $T_2|_{\mathcal{M}}$ is normal.*

Proof. It is easy to prove that $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where T_1 is normal and T_2 is pure, by [**18**, Lemma 12]. Since $\sigma(T_2) \subset \sigma(T)$, T_2 is invertible and log-hyponormal, by Theorem 1. □

The next lemma was proved for dominant operators in [**15**, Theorem 1] and for p-hyponormal operators in [**13**]. Recall that an operator $T \in B(\mathcal{H})$ is said to be *dominant* if for any $\lambda \in \mathbb{C}$ there exists an $M_\lambda \geq 0$ such that $\|(T-\lambda)^*x\| \leq M_\lambda \|(T-\lambda)x\|$, for all $x \in \mathcal{H}$. If M_λ is a constant, then T is said to be *M-hyponormal*.

LEMMA 3. *Let $T_1 \in L(\mathcal{H}_1)$ be a log-hyponormal operator and let $T_2 \in L(\mathcal{H}_2)$ be a normal operator. If there exists an operator $X \in L(\mathcal{H}_2, \mathcal{H}_1)$ with dense range such that $T_1X = XT_2$, then T_1 is normal.*

Proof. First, we decompose T_1 into its normal and pure parts by $T_1 = T_{11} \oplus T_{12}$ with respect to a decomposition $\mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12}$. Let $T_{12} = U_{12}|T_{12}|$ be the polar decomposition of T_{12} and $\widetilde{T}_{12} = |T_{12}|^{1/2}U_{12}|T_{12}|^{1/2}$. Let $\widetilde{T}_{12} = V_{12}|\widetilde{T}_{12}|$ be the polar decomposition of $\widetilde{T}_{12}$ and $\widehat{T}_{12} = |\widetilde{T}_{12}|^{1/2}V_{12}|\widetilde{T}_{12}|^{1/2}$. Since T_{11} is normal, we have that $\widetilde{T}_1 = T_{11} \oplus \widetilde{T}_{12}$ and $\widehat{T}_1 = T_{11} \oplus \widehat{T}_{12}$. Let $W = |\widetilde{T}_{12}|^{1/2}|T_{12}|^{1/2}$. Since $N(|T_{12}|) = N(T_{12}) = \{0\}$, by Theorem 2, $|T_{12}|^{\frac{1}{2}}$ is a quasiaffinity. Hence $\widehat{T}_{12}$ is injective and W is a quasiaffinity such that $\widehat{T}_{12}W = WT_{12}$. Let $Y = I_{H_{11}} \oplus W$. Then $\widehat{T}_1$ is hyponormal and Y is a quasiaffinity such that $\widehat{T}_1 Y = YT_1$. Thus we have that $\widehat{T}_1(YX) = (YX)T_2$ and YX has dense range. Hence $\widehat{T}_1$ is normal, by [**15**, Theorem 1], and so T_1 is normal by [**16**, Theorem 7]. □

The following lemma is due to Williams [**19**, Lemma 1.1].

LEMMA 4. **[19]** *Let $N_i \in L(\mathcal{H}_i)$ be normal for each $i = 1, 2$. If $X \in L(\mathcal{H}_2, \mathcal{H}_1)$ and $Y \in L(\mathcal{H}_1, \mathcal{H}_2)$ are injective such that $N_1X = XN_2$ and $YN_1 = N_2Y$, then N_1 and N_2 are unitarily equivalent.*

Conway [**4**] proved that the normal parts of quasisimilar subnormal operators are unitarily equivalent and gave an example showing that the pure parts of quasisimilar subnormal operators need not be quasisimilar. This result was generalized to classes of dominant operators in [**15**] and p-hyponormal operators in [**13**], respectively. We prove that these results hold for log-hyponormal operators.

THEOREM 5. *For each $i = 1, 2$, let $T_i \in L(\mathcal{H}_i)$ be log-hyponormal operators and let $T_i = N_i \oplus V_i$ on $\mathcal{H}_i = \mathcal{H}_{i1} \oplus \mathcal{H}_{i2}$, where N_i and V_i are the normal and pure parts, respectively, of T_i. If T_1 and T_2 are quasisimilar, then N_1 and N_2 are unitarily equivalent* *and there exist* $X_* \in L(\mathcal{H}_{22}, \mathcal{H}_{12})$ *and* $Y_* \in L(\mathcal{H}_{12}, \mathcal{H}_{22})$ *having dense ranges such that* $V_1 X_* = X_* V_2$ *and* $Y_* V_1 = V_2 Y_*$.

Proof. By hypothesis there exist quasiaffinities $X \in L(\mathcal{H}_2, \mathcal{H}_1)$ and $Y \in L(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_1 X = X T_2$ and $Y T_1 = T_2 Y$. Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$$

with respect to $\mathcal{H}_2 = \mathcal{H}_{21} \oplus \mathcal{H}_{22}$ and $\mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12}$, respectively. A simple matrix calculation shows that

$$V_1 X_3 = X_3 N_2 \text{ and } V_2 Y_3 = Y_3 N_1.$$

We claim that $X_3 = Y_3 = 0$. Let $\mathcal{M} = \overline{R(X_3)}$. Then $\mathcal{M}$ is a non-trivial invariant subspace of V_1. Since $V_1^* X_3 = X_3 N_2^*$, by [**18**, Theorem 3], $\mathcal{M}$ is an invariant subspace of V_1^*. Hence $\mathcal{M}$ reduces V_1, $\sigma(V_1|_{\mathcal{M}}) \subset \sigma(V_1)$ and $V_1|_{\mathcal{M}}$ is invertible. Let $V_1' = V_1|_{\mathcal{M}}$ and define an operator $X_3' : \mathcal{H}_{21} \to \mathcal{M}$ by $X_3' x = X_3 x$, for each $x \in \mathcal{H}_{21}$. Then V_1' is log-hyponormal, by Theorem 1, so that X_3' has dense range and satisfies $V_1' X_3' = X_3' N_2$. Hence V_1' is normal, by Lemma 3. Since V_1 is pure, this implies that $\mathcal{M} = \{0\}$ and $X_3 = 0$. Similarly, we have $Y_3 = 0$. Hence X_1 and Y_1 are injective.

Since $N_1 X_1 = X_1 N_2$ and $Y_1 N_1 = N_2 Y_1$, N_1 and N_2 are unitarily equivalent, by Lemma 4. Also, X_4 and Y_4 have dense ranges. Hence $V_1 X_4 = X_4 V_2$ and $Y_4 V_1 = V_2 Y_4$, so that the proof is complete. □

Corollary 6. *Let* $T_1 \in L(\mathcal{H}_1)$ *and* $T_2 \in L(\mathcal{H}_2)$ *be quasisimilar log-hyponormal operators. If* T_1 *is pure, then* T_2 *is also pure.*

Corollary 7. *Let* $T_1 \in L(\mathcal{H}_1)$ *be log-hyponormal and let* $T_2 \in L(\mathcal{H}_2)$ *be normal. If* T_1 *and* T_2 *are quasisimilar, then* T_1 *and* T_2 *are unitarily equivalent normal operators.*

3. A Fuglede-Putnam type theorem for log-hyponormal operators.

Theorem 8. *Let* $T_1 \in L(\mathcal{H}_1)$ *and* $T_2^* \in L(\mathcal{H}_2)$ *be log-hyponormal or* p*-hyponormal operators satisfying* $T_1 X = X T_2$, *for some operator* $X \in L(\mathcal{H}_2, \mathcal{H}_1)$. *Then* $T_1^* X = X T_2^*$, $\overline{R(X)}$ *reduces* T_1, $N(T)^{\perp}$ *reduces* T_2, *and* $T_1|_{\overline{R(X)}}$, $T_2|_{N(X)^{\perp}}$ *are unitarily equivalent normal operators.*

Proof. Duggal [**5**, Theorem7] proved the case in which T_1 and T_2^* are p-hyponormal. First we prove the case in which T_1 and T_2^* are log-hyponormal. Let $T_1 = U_1|T_1|$ and $T_2^* = U_2^*|T_2^*|$ be the polar decompositions of T_1 and T_2^*, respectively. Let $\widetilde{T}_1 = |T_1|^{1/2} U_1 |T_1|^{1/2}$, $\widetilde{T_2^*} = |T_2^*|^{1/2} U_2^* |T_2^*|^{1/2}$ and $W = |T_1|^{\frac{1}{2}} X |T_2^*|^{\frac{1}{2}}$. Since $T_1 X = X T_2$, we have $\widetilde{T}_1 W = W(\widetilde{T_2^*})^*$. Since $\widetilde{T}_1$ and $\widetilde{T_2^*}$ are semi-hyponormal, by [**16**], $\overline{R(W)}$ reduces $\widetilde{T}_1$ and $N(W)^{\perp}$ reduces $(\widetilde{T_2^*})^*$. Also $\widetilde{T}_1|_{\overline{R(W)}}$ and $(\widetilde{T_2^*})^*|_{N(W)^{\perp}}$ are unitarily equivalent normal operators, by [**5**, Theorem 7]. By [**18**, Lemma 3], T_1 and T_2^* are of the forms

$$T_1 = \widetilde{T}_1|_{\overline{R(W)}} \oplus S_1 = N_1 \oplus S_1 \text{ on } \overline{R(W)} \oplus R(W)^{\perp},$$
$$T_2^* = \widetilde{T_2^*}|_{N(W)^{\perp}} \oplus S_2^* = N_2^* \oplus S_2^* \text{ on } N(W)^{\perp} \oplus N(W),$$

where N_1 and N_2 are unitarily equivalent normal operators. Since T_1 and T_2^* are invertible, N_1, N_2, S_1 and S_2 are also invertible. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \text{ and } W = \begin{pmatrix} W_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to $\mathcal{H}_1 = \overline{R(W)} \oplus R(W)^{\perp}$ and $\mathcal{H}_2 = N(W)^{\perp} \oplus N(W)$, respectively. Then $W = |T_1|^{\frac{1}{2}} X |T_2^*|^{\frac{1}{2}}$ implies that

$$\begin{pmatrix} W_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |N_1|^{\frac{1}{2}} X_{11} |N_2^*|^{\frac{1}{2}} & |N_1|^{\frac{1}{2}} X_{12} |S_2^*|^{\frac{1}{2}} \\ |S_1|^{\frac{1}{2}} X_{21} |N_2^*|^{\frac{1}{2}} & |S_1|^{\frac{1}{2}} X_{22} |S_2^*|^{\frac{1}{2}} \end{pmatrix}. \tag{3.1}$$

Hence $X_{12} = 0$, $X_{21} = 0$, $X_{22} = 0$ and $X_{11} : N(W)^{\perp} \to \overline{R(X)}$ is a quasiaffinity. This implies that $\overline{R(X)} = \overline{R(X_{11})} = \overline{R(W)}$, $N(X) = N(W)$ and $T_1|_{\overline{R(X)}}$, $T_2|_{N(X)^{\perp}}$ are unitarily equivalent normal operators. Since $T_1 X = X T_2$, we have that $N_1 X_{11} = X_{11} N_1$. Thus $N_1^* X_{11} = X_{11} N_1^*$ and $T_1^* X = X T_2^*$.

Next we prove the case in which T_1 is p-hyponormal and T_2^* is log-hyponormal. In this case we have the equation (3.1), where N_2 and S_2 are invertible by the argument of the case above. Since N_1 and N_2 are unitarily equivalent, N_1 is invertible. It follows that $X_{12} = 0$, $|S_1|^{\frac{1}{2}} X_{21} = 0$, $|S_1|^{\frac{1}{2}} X_{22} = 0$ and $S_1 X_{21} = 0$, $S_1 X_{22} = 0$. Then $T_1 X = X T_2$ implies that

$$\begin{pmatrix} N_1 X_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11} N_2 & 0 \\ X_{21} N_2 & X_{22} S_2 \end{pmatrix}.$$

Hence $X_{21} = 0$, $X_{22} = 0$ and X_{11} is a quasiaffinity. The rest of the proof is similar to the case above.

Lastly we prove the case in which T_1 is log-hyponormal and T_2^* is p-hyponormal. Here we have the equation (3.1), where N_1 and S_1 are invertible. Since N_1 and N_2 are unitarily equivalent, N_2 is invertible. Hence $X_{21} = 0$, $X_{12} |S_2^*|^{\frac{1}{2}} = 0$, $X_{22} |S_2^*|^{\frac{1}{2}} = 0$ and $X_{12} S_2 = 0$, $X_{22} S_2 = 0$. Then $T_1 X = X T_2$ implies that

$$\begin{pmatrix} N_1 X_{11} & N_1 X_{12} \\ 0 & S_1 X_{22} \end{pmatrix} = \begin{pmatrix} X_{11} N_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $X_{12} = 0$, $X_{22} = 0$ and X_{11} is a quasiaffinity. The rest of the proof is similar to the case above. □

Remark. Let $T = U|T|$ be the polar decomposition of T. We define $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. Since $T^* = |T| U^* = U^* U |T| U^* = U^* |T^*|$, T^* has the polar decomposition $T^* = U^* |T^*|$. Hence $\widetilde{T^*} = |T^*|^{\frac{1}{2}} U^* |T^*|^{\frac{1}{2}} = U (\widetilde{T})^* U^*$ and $(\widetilde{T})^* = U^* \widetilde{T^*} U$. Thus $\widetilde{T^*}$ is log-hyponormal (resp., p-hyponormal) if and only if $(\widetilde{T})^*$ is log-hyponormal (resp, p-hyponormal).

A generalization to dominant operator is given by the following Corollary.

Corollary 9. *Let $T_1 \in L(\mathcal{H}_1)$ be dominant and $T_2^* \in L(\mathcal{H}_2)$ log-hyponormal. If $T_1 X = X T_2$, for some operator $X \in L(\mathcal{H}_2, \mathcal{H}_1)$, then $T_1^* X = X T_2^*$, $\overline{R(X)}$ reduces T_1, $N(T)^{\perp}$ reduces T_2, and $T_1|_{\overline{R(X)}}$, $T_2|_{N(X)^{\perp}}$ are unitarily equivalent normal operators.*

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Proof. Decompose T_1 and T_2^* into their normal and pure parts. Then we have

$$T_1 = N_1 \oplus S_1 \text{ on } \mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12},$$
$$T_2 = N_2 \oplus S_2 \text{ on } \mathcal{H}_2 = \mathcal{H}_{21} \oplus \mathcal{H}_{22},$$

and

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} : \mathcal{H}_2 = \mathcal{H}_{21} \oplus \mathcal{H}_{22} \to \mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12},$$

where N_1, N_2 are normal, S_1 is dominant and S_2^* is log-hyponormal. Then $T_1X = XT_2$ implies that

$$\begin{pmatrix} N_1X_{11} & N_1X_{12} \\ S_1X_{21} & S_1X_{22} \end{pmatrix} = \begin{pmatrix} X_{11}N_2 & X_{12}S_2 \\ X_{21}N_2 & X_{22}S_2 \end{pmatrix}.$$

Let $S_2^* = U_2^*|S_2^*|$ be the polar decomposition of S_2^* and $\widetilde{S_2^*} = |S_2^*|^{\frac{1}{2}}U_2^*|S_2^*|^{\frac{1}{2}}$. Let $\widetilde{S_2^*} = V_2^*|\widetilde{S_2^*}|$ be the polar decomposition of $\widetilde{S_2^*}$ and $\widehat{S_2^*} = |\widetilde{S_2^*}|^{\frac{1}{2}}V_2^*|\widetilde{S_2^*}|^{\frac{1}{2}}$. Applying [**6**, Corollary 1] to

$$S_1X_{21} = X_{21}N_2$$

and

$$S_1X_{22}|S_2^*|^{\frac{1}{2}}|\widetilde{S_2^*}|^{\frac{1}{2}} = X_{22}|S_2^*|^{\frac{1}{2}}|\widetilde{S_2^*}|^{\frac{1}{2}}(\widehat{S_2^*})^*$$

together with Theorem 8 to

$$N_1X_{12} = X_{12}S_2,$$

we have $X_{21} = 0$, $X_{22} = 0$, and $X_{12} = 0$. The rest of the proof is similar to the proof of Theorem 8. □

In Theorem 8 above if X is a quasiaffinity then $\overline{R(X)} = \mathcal{H}_1$ and $N(X)^{\perp} = \mathcal{H}_2$. Hence T_1 and T_2 are unitarily equivalent normal operators. Thus we can obtain an improvement to Corollary 7 as follows.

Corollary 10. *Let $T_1 \in L(\mathcal{H}_1)$ be a log-hyponormal operator and let $T_2 \in L(\mathcal{H}_2)$ be a normal operator. If there exists a quasiaffinity $X \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that $T_1X = XT_2$, then T_1 and T_2 are unitarily equivalent normal operators.*

Problem. Is it possible to replace the normality of T_2 in Corollary 10 with an isometry or a spectral operator?

4. Log-hyponormal operators quasisimilar to isometries or spectral operators.

Theorem 11. *Let $T_1 \in L(\mathcal{H}_1)$ be log-hyponormal and let $T_2 \in L(\mathcal{H}_2)$ be an isometry. If T_1 and T_2 is quasisimilar, then T_1 and T_2 are unitarily equivalent unitary operators.*

Proof. There exist quasiaffinities X and Y such that $T_1X = XT_2$ and $YT_1 = T_2Y$. Since T_1 is invertible and $YT_1 = T_2Y$, T_2 has dense range. Hence T_2 is unitary. Thus T_1 and T_2 are unitarily equivalent unitary operators by Corollary 10. □

Recall that a spectral operator (in the sense of Dunford) is an operator with a strongly countably additive resolution of the identity defined on the Borel sets of the complex plane. If T is spectral, then it has the canonical decomposition $T = S + Q$, where S and Q are its scalar and its radical parts, respectively. For a thorough discussion of spectral operators see [**7**]. Hoover [**12**] studied quasisimilar spectral operators. It was proved that M-hyponormal spectral operators and p-hyponormal spectral operators are normal in [**8**] and [**13**], respectively. We conclude with the result that log-hyponormal spectral operators are normal.

THEOREM 12. *Let $T_1 \in L(\mathcal{H}_1)$ be a log-hyponormal operator and let $T_2 \in L(\mathcal{H}_2)$ be a spectral operator. If there exists a quasiaffinity $X \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that $T_1 X = X T_2$, then T_1 is normal, T_2 is a scalar operator, and T_2 is similar to T_1.*

Proof. There exists a quasiaffinity Y such that $\widetilde{T}_1 Y = Y T_1$, by the same arguments as in the proof of Lemma 3, and so we have that $\widetilde{T}_1(YX) = (YX)T_2$. Hence YX is also a quasiaffinity and so $\widetilde{T}_1$ is normal by [**8**, Corollary 4]. If follows that $T_1 (= \widetilde{T}_1)$ is normal by [**16**]. Thus T_2 is a scalar-type spectral operator and similar to T_1 by [**9**, Corollary 4]. □

COROLLARY 13. *If $T \in L(\mathcal{H})$ is a log-hyponormal spectral operator, then T is normal.*

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