HOMOTOPY AND ISOTOPY PROPERTIES OF TOPOLOGICAL SPACES

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1. Introduction. The most important notion in topology is that of a *homeomorphism* $f: X \to Y$ from a topological space X onto a topological space Y. If a homeomorphism $f: X \to Y$ exists, then the topological spaces X and Y are said to be *homeomorphic* (or *topologically equivalent*), in symbols,

$$X \equiv Y.$$

The relation $\equiv$ among topological spaces is obviously reflexive, symmetric, and transitive; hence it is an equivalence relation. For an arbitrary family F of topological spaces, this equivalence relation $\equiv$ divides F into disjoint equivalence classes called the *topology types* of the family F. Then, the main problem in topology is the topological classification problem formulated as follows.

The topological classification problem: Given a family F of topological spaces, find an effective enumeration of the topology types of the family F and exhibit a representative space in each of these topology types.

A number of special cases of this problem were solved long ago. For example, the family of Euclidean spaces is classified by their dimensions and the family of closed surfaces is classified by means of orientability and Euler characteristic. However, the problem is far from being solved; in fact, the topological classification of the family of three-dimensional compact manifolds still remains an outstanding unsolved problem.

To overcome the difficulty of the topological classification problem, topologists introduced weaker equivalence relations, namely, the homotopy and isotopy equivalences, which would give rise to larger but fewer classes of spaces than the topology types.

A continuous map $f: X \to Y$ is said to be a *homotopy equivalence* provided that there exists a continuous map $g: Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity maps on X and Y respectively. Two topological spaces X and Y are said to be *homotopically equivalent* (in symbol, $X \simeq Y$) if there exists a homotopy equivalence $f: X \to Y$.

It is easily verified that the relation $\simeq$ among topological spaces is reflexive, symmetric, and transitive; hence it is an equivalence relation. For any given family F of topological spaces, this equivalence relation $\simeq$ divides F into disjoint equivalence classes called the *homotopy types* of the family. Analogous to

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the topological classification problem, one can formulate the *homotopy classification* problem in the obvious fashion.

To introduce the notion of isotopy equivalence, let us first recall the definition of an imbedding. A continuous map $f: X \to Y$ is said to be an *imbedding* provided that f is a homeomorphism of X onto the subspace $f(X)$ of Y.

A homotopy $h_t: X \to Y$, $(t \in I)$, is said to be an *isotopy* if, for each $t \in I$, h_t is an imbedding. Two imbeddings $f, g: X \to Y$ are said to be *isotopic* if there exists an isotopy $h_t: X \to Y$, $(t \in I)$, such that $h_0 = f$ and $h_1 = g$.

An imbedding $f: X \to Y$ is said to be an *isotopy equivalence* if there exists an imbedding $g: Y \to X$ such that the composite imbeddings $g \circ f$ and $f \circ g$ are isotopic to the identity imbeddings on X and Y respectively. Two topological spaces X and Y are said to be *isotopically equivalent* (in symbol, $X \cong Y$) if there exists an isotopy equivalence $f: X \to Y$.

The relation $\cong$ among topological spaces is obviously an equivalence relation. For any given family F of topological spaces, this equivalence relation $\cong$ divides F into disjoint equivalence classes called the *isotopy types* of the family. One can formulate the *isotopy classification problem* in the obvious fashion.

By the definitions given above, it is clear that every homeomorphism is an isotopy equivalence and that every isotopy equivalence is a homotopy equivalence.

Examples in the sequel will show that the converses are not always true. Hence, for any given family F of topological spaces, every topology type of F is contained in some isotopy type of F, and every isotopy type of F is contained in some homotopy type of F. Consequently, the topological classification problem can break into three steps as follows:

Step 1. *Homotopy classification.* Determine effectively all of the homotopy types of the family F.

Step 2. *Isotopy classification.* For each homotopy type α of the family F, determine effectively all of the isotopy types of the family α.

Step 3. *Topological classification.* For each isotopy type β of the family F, determine effectively all of the topology types of the family β and exhibit a representative space in each of the topology types.

In order to carry out the three steps of the topological classification problem for a given family F of topological spaces, one must make use of the various properties of spaces which are preserved by homotopy equivalences, isotopy equivalences, and homeomorphisms respectively. These properties are called the homotopy properties, the isotopy properties, and the topological properties respectively. It follows that every homotopy property is an isotopy property and that every isotopy property is a topological property. Examples in the sequel will show that the converses of these implications do not always hold.

The main purpose of the present paper is to give general tests for homotopy and isotopy properties in terms of hereditary and weakly hereditary properties with the elementary properties in general topology as illustrations. These will be given in §§ 2 and 3. In the final section of the paper, we will describe a general method of constructing new homotopy and isotopy properties out of old ones as a striking and profound synthesis of various isolated known results.

2. Homotopy properties. A property P of topological spaces is called a *homotopy property* provided that it is preserved by all homotopy equivalences. Precisely, P is a homotopy property if and only if, for an arbitrary homotopy equivalence $f: X \to Y$, that X has P implies that Y also has P. If a homotopy property P is given in the form of a number, a set, a group, or some other similar object, P is said to be a *homotopy invariant.*

Some of the elementary properties in general topology are homotopy properties. As examples, one can easily prove the following assertions.

PROPOSITION 2.1. *Contractibility is a homotopy property of topological spaces.*

PROPOSITION 2.2. *The cardinal number of components of a topological space X is a homotopy invariant.*

COROLLARY 2.3. *Connectedness is a homotopy property of topological spaces.*

PROPOSITION 2.4. *The cardinal number of path-components of a topological space X is a homotopy invariant.*

COROLLARY 2.5. *Pathwise connectedness is a homotopy property of topological spaces.*

Nevertheless, most properties studied in general topology are not homotopy properties. To demonstrate this fact, let us first introduce the notion of weakly hereditary properties.

A property P of topological spaces is said to be *hereditary* if each subspace of a topological space with P also has P; it is said to be *weakly hereditary* if every closed subspace of a topological space with P also has P. For examples, the following properties of a topological space X are weakly hereditary:

(A) X is a T_1-space, that is, every point in X forms a closed set of X.

(B) X is a Hausdorff space.

(C) X is a regular space.

(D) X is a completely regular space.

(E) X is a discrete space, that is, every set in X is open.

(F) X is an indiscrete space, that is, the only open sets in X are the empty set $\square$ and the set X itself.

(G) X is a metrizable space.

(H) The first axiom of countability is satisfied in X, that is, the neighbourhoods of any point in X have a countable basis.

(I) The second axiom of countability is satisfied in X, that is, the open sets of X have a countable basis.

(J) X can be imbedded in a given topological space Y.

(K) For a given integer $n \geqslant 0$, $\dim X \leqslant n$. Here, the *inductive dimension* $\dim X$ is defined as follows: $\dim X = -1$ if X is empty, and $\dim X \leqslant n$ if for every point $p \in X$ and every open neighbourhood U of p there exists an open neighbourhood $V \subset U$ of p such that $\dim \partial V \leqslant n - 1$, where ∂V denotes the boundary $\bar{V} \setminus V$ of V in X (**2**, p. 153).

(L) X is a normal space.

(M) X is a compact space.

(N) X is a Lindelof space, that is, every open covering of X has a countable subcovering.

(O) X is a paracompact space.

(P) X is a locally compact space.

(Q) For a given integer $n \geqslant 0$, $\operatorname{Dim} X \leqslant n$. Here, the *covering dimension* $\operatorname{Dim} X$ is defined as follows: $\operatorname{Dim} X \leqslant n$ if every finite open covering of X has a refinement of order $\leqslant n$ (**2**, p. 153).

The first eleven properties (A)–(K) listed above are also hereditary.

A topological space X is said to be a *singleton space* if X consists of a single point. Obviously, every singleton X has all of the properties (A)–(Q). On the other hand, none of these properties prevails in all topological spaces. Hence we deduce, as a consequence of the following theorem, the fact that *none of these properties* (A)–(Q) *is a homotopy property.*

THEOREM 2.6. *Let P be a weakly hereditary topological property such that every singleton space has P and suppose that there exists a topological space X which does not have P. Then P is not a homotopy property.*

Proof. Let X be a topological space which does not have P. Consider the cone $C(X)$ over X which is the quotient space obtained by identifying the top $X \times 1$ of the cylinder $X \times I$ to a single point v, called the vertex of the cone $C(X)$. Then the space X may be identified with bottom $X \times 0$ of the cone $C(X)$ and hence X becomes a closed subspace of $C(X)$. Since P is a weakly hereditary property which X does not have, $C(X)$ cannot have P. On the other hand, it is well known that the inclusion map $i: v \subset C(X)$ is a homotopy equivalence. Since the singleton space v has P but $C(X)$ does not have P, P is not a homotopy property. This completes the proof of (2.6).

Although most of the properties studied in general topology are not homotopy properties as shown by the foregoing theorem, it is well known that almost all invariants studied in algebraic topology are homotopy invariants, namely, the homology groups, the homotopy groups, etc.

For topological spaces which are homotopically equivalent to CW-complexes, Postnikov, in his celebrated work (**3**), gave a complete system of homotopy invariants, now called the *Postnikov system* of the space. Any pair of these spaces are homotopically equivalent if and only if their Postnikov

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systems are isomorphic. Hence, the homotopy classification problem of these spaces has been solved by Postnikov at least theoretically although his process is too complicated to be practicable.

3. Isotopy properties. A property P of topological spaces is called an *isotopy property* provided that it is preserved by all isotopy equivalences. Precisely, P is an isotopy property if and only if, for an arbitrary isotopy equivalence $f: X \to Y$, that X has P implies that Y also has P. If an isotopy property P is given in the form of a number, a set, a group, or some other similar object, P is said to be an *isotopy invariant.*

Most of the elementary properties in general topology are isotopy properties. For example, the eleven properties (A)–(K) listed in § 2 are isotopy properties in immediate consequence of the following theorem.

THEOREM 3.1. *Every hereditary topological property of spaces is an isotopy property.*

Proof. Let P be any hereditary topological property of spaces. Assume that $f: X \to Y$ is an isotopy equivalence and that the space X has the property P. It suffices to prove that Y also has P.

By definition of an isotopy equivalence, there exists an imbedding $g: Y \to X$ such that the composed imbeddings $g \circ f$ and $f \circ g$ are isotopic to the identity imbeddings on X and Y respectively. The image $g(Y)$ is a subspace of X. Since P is hereditary, this implies that $g(Y)$ has the property P. As an imbedding, g is a homeomorphism of Y onto $g(Y)$. Since P is a topological property and $g(Y)$ has P, it follows that Y also has P. This completes the proof of (3.1).

THEOREM 3.2. *The inductive dimension* dim X *of a topological space* X *is an isotopy invariant.*

Proof. Let $f: X \to Y$ be any given isotopy equivalence and assume that

$$\dim X = m, \dim Y = n.$$

It suffices to prove that $m = n$.

Since $\dim X \leqslant m$ and $f: X \to Y$ is an isotopy equivalence, it follows from the fact that the property (K) of § 2 is an isotopy property that $\dim Y \leqslant m$. Hence, we obtain $n \leqslant m$. By considering any isotopy inverse $g: Y \to X$ of f, we can also prove that $m \leqslant n$. Hence $m = n$ and (3.2) is proved.

Not all topological properties of spaces are isotopy properties. Examples are given by the following propositions.

PROPOSITION 3.3. *Compactness is not an isotopy property of topological spaces.*

Proof. Let Y denote the closed unit interval $I = [0, 1]$ and X the open unit interval $(0, 1)$ which is the interior of Y. It is well known that Y is compact

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but X is non-compact. Hence, it suffices to prove that the inclusion $i: X \subset Y$ is an isotopy equivalence.

For this purpose, let $j: Y \to X$ denote the imbedding defined by

$$j(t) = \tfrac{1}{3}(t+1), \qquad (0 \leqslant t \leqslant 1).$$

It remains to prove that the composed imbeddings $j \circ i$ and $i \circ j$ are isotopic to the identity imbeddings on X and Y respectively.

Define an isotopy $k_t: Y \to Y$, $(t \in I)$, by taking

$$k_t(y) = \tfrac{1}{3}(t + 3y - 2ty)$$

for each $t \in I$ and each $y \in Y = I$. Since $k_t(X) \subset X$ for each $t \in I$, k_t also defines an isotopy $h_t: X \to X$, $(t \in I)$.

Since h_0 and k_0 are the identity maps on X and Y respectively and since $h_1 = j \circ i$ and $k_1 = i \circ j$, it follows that $j \circ i$ and $i \circ j$ are isotopic the identity imbeddings. This completes the proof of (3.3).

Since the open interval $Y = (0, 1)$ is homeomorphic to the real line R, we have also proved the following corollary.

COROLLARY 3.4. *The unit interval* $I = [0, 1]$ *and the real line* R *are isotopically equivalent.*

Since the product of an arbitrary family of isotopy equivalences is clearly also an isotopy equivalence, we have the following generalization of (3.4).

COROLLARY 3.5. *For any cardinal number* α, *the topological powers* I^α *and* R^α *are isotopically equivalent.*

In particular, if α is a finite integer $n \to 0$, the n-cube I^n and the Euclidean n-space R^n are isotopically equivalent.

On the other hand, if α is infinite, R^α is not locally compact while I^α is compact and hence locally compact. This proves the following proposition.

PROPOSITION 3.6. *Local compactness is not an isotopy property of topological spaces.*

4. Homotopy functors and isotopy functors. By a *covariant homotopy functor*, we mean an operator ϕ which assigns to each topological space X a topological space $\phi(X)$ and to each continuous map $f: X \to Y$ a continuous map

$$\phi(f): \phi(X) \to \phi(Y)$$

satisfying the following three conditions:

(HF1) ϕ preserves identity, that is, if f is the identity map so is $\phi(f)$.

(HF2) ϕ preserves composition, that is, if $f: X \to Y$ and $g: Y \to Z$ are continuous maps then we have

$$\phi(g \circ f) = \phi(g) \circ \phi(f).$$

(HF3) ϕ preserves homotopy, that is, if the family $h_t: X \to Y$, $(t \in I)$, of continuous maps is a homotopy, so is the family

$$\phi(h_t): \phi(X) \to \phi(Y), \qquad (t \in I).$$

If, in the preceding definition of a homotopy functor ϕ, we have

$$\phi(f): \phi(Y) \to \phi(X), \phi(g \circ f) = \phi(f) \circ \phi(g),$$

then the operator ϕ is called a *contravariant homotopy functor*.

Similarly, by a *covariant isotopy functor*, we mean an operator which assigns to each topological space X a topological space $\psi(X)$ and to each imbedding $f: X \to Y$ an imbedding

$$\psi(f): \psi(X) \to \psi(Y)$$

satisfying the following three conditions:

(IF1) ψ preserves identity, that is, if f is the identity imbedding so is $\psi(f)$.

(IF2) ψ preserves composition, that is, if $f: X \to Y$ and $g: Y \to Z$ are imbeddings then we have

$$\psi(g \circ f) = \psi(g) \circ \psi(f).$$

(IF3) ψ preserves isotopy, that is, if the family $k_t: X \to Y$, $(t \in I)$, of imbeddings is an isotopy, so is the family

$$\psi(k_t): \psi(X) \to \psi(Y), \qquad (t \in I).$$

One can define *contravariant isotopy functors* by reversing the direction of the imbeddings $\psi(f)$ and obvious modifications in (IF2) and (IF3).

Examples of homotopy and isotopy functors:

Example 1. *Topological powers*. Let n be any positive integer. Define an operator ϕ as follows. For each topological space X, let $\phi(X)$ denote the topological nth power X^n, that is, the topological product of n copies of the space X. For each continuous map $f: X \to Y$, let $\phi(f)$ stand for the nth power $f^n: X^n \to Y^n$ of f defined by

$$f^n(x_1, \ldots, x_n) = (fx_1, \ldots, fx_n).$$

Then the conditions (HF1)–(HF3) can easily be verified and hence ϕ is a covariant homotopy functor. Furthermore, if $f: X \to Y$ is an imbedding, $\phi(f) = f^n$ is clearly also an imbedding. Hence, the restriction ψ of ϕ on spaces and imbeddings is a covariant isotopy functor.

More generally, let G be a subgroup of the symmetric group S of the integers $1, \ldots, n$, that is, S is the group of all permutations of the n integers $1, \ldots, n$. Then G operates on the topological power X^n by permuting the factors of X^n. Let $\phi_G(X)$ denote the orbit space X^n/G. Since the operators in G obviously commute with the continuous maps $f^n: X^n \to Y^n$, each f^n induces a continuous map $\phi_G(f): \phi_G(X) \to \phi_G(Y)$. It follows that ϕ_G is a covariant homotopy

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functor and its restriction ψ_G on spaces and imbeddings is a covariant isotopy functor.

Example 2. *Residual functors.* Let n be an integer greater than 1. Define an operator ψ as follows. For each topological space X, let $\psi(X)$ denote the residual space $X^n \setminus d(X)$ obtained by deleting the diagonal $d(X)$ from the nth power X^n. If $f: X \to Y$ is an imbedding, the nth power f^n carries $\psi(X)$ into $\psi(Y)$ and hence defines an imbedding $\psi(f): \psi(X) \to \psi(Y)$. The conditions (IF1)–(IF3) are obviously satisfied and hence ψ is a covariant isotopy functor. This isotopy functor ψ is called the nth *residnal functor* and is denoted by R_n.

Let G be a subgroup of the symmetric group of n integers $1, \ldots, n$. Then G also operates on the residual space $\psi(X)$. Let $\psi_G(X)$ denote the orbit space $\psi(X)/G$. Then $\psi(f)$ induces an imbedding $\psi_G(f): \psi_G(X) \to \psi_G(Y)$ for each imbedding $f: X \to Y$. Thus, ψ_G is also a covariant isotopy functor.

Example 3. *Mapping spaces.* Let T be a given Hausdorff space. Define an operator ϕ as follows. For each topological space X, let $\phi(X)$ stand for the space Map (T, X) of all continuous maps from T into X with the compact-open topology. For each continuous map $f: X \to Y$, let

$$\phi(f): \mathrm{Map}(T, X) \to \mathrm{Map}(T, Y)$$

denote the function defined taking

$$[\phi(f)](\xi) = f \circ \xi$$

for each $\xi: T \to X$ in Map(T, X). One can verify that $\phi(f)$ is a continuous map and that the conditions (HF1)–(HF3) are satisfied. Hence ϕ is a covariant homotopy functor. Furthermore, if $f: X \to Y$ is an imbedding, so is $\phi(f)$. This implies that the restriction ψ of ϕ on spaces and imbeddings is a covariant isotopy functor.

Example 4. *Enveloping functors.* Let n be any positive integer greater than 1. Define an operator ψ as follows. For each topological space X, consider as in Example 2 the nth power X^n and identify X with the diagonal $d(X)$ in X^n. Then, $\psi(X)$ stands for the subspace of Map(I, X^α) consisting of the continuous paths $\sigma: I \to X^n$ such that $\sigma(t) \in X$ if and only if $t = 0$. For each imbedding $f: X \to Y$, it follows from the preceding examples that the imbedding $f^n: X^n \to Y^n$ induces an imbedding of Map(I, X^n) into Map (I, Y^n) which carries $\psi(X)$ into $\psi(Y)$ and hence defines an imbedding

$$\psi(f): \psi(X) \to \psi(Y).$$

One can easily verify that the conditions (IF1)–(IF3) are satisfied and hence ψ is a covariant isotopy functor. This isotopy functor ψ is called the nth *enveloping functor* and is denoted by E_n. For the remaining case $n = 1$, we may define $E_1(X)$ to be the subspace of Map(I, X) consisting of the continuous paths $\sigma: I \to X$ such that $\sigma(t) = \sigma(0)$ if and only if $t = 0$.

For each subgroup G of the symmetric group of n integers $1, \ldots, n$, similar modifications may be made as in Examples 1 and 2.

The usefulness of these functors can be seen from the following two theorems.

THEOREM 4.1. *If ϕ is a homotopy functor, then every homotopy property of $\phi(X)$ induces a homotopy property of X.*

Proof. Let P be an arbitrary homotopy property. Assume that $f: X \to Y$ is a homotopy equivalence and that $\phi(X)$ has P. We have to prove that $\phi(Y)$ must also have P. For this purpose, it suffices to show that $\phi(f)$ is also a homotopy equivalence.

Let $g: Y \to X$ be a continuous map such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity maps on X and Y respectively. Then there exist homotopies $h_t: X \to X$ and $k_t: Y \to Y$, $(t \in I)$, such that $h_0 = g \circ f$, $k_0 = f \circ g$, and h_1, k_1 are identity maps. By (HF3), $\phi(h_t)$ and $\phi(k_t)$ are homotopies. By (HF2), $\phi(h_0)$ and $\phi(k_0)$ are the two compositions of $\phi(f)$ and $\phi(g)$. By (HF1), $\phi(h_1)$ and $\phi(k_1)$ are the identity maps on $\phi(X)$ and $\phi(Y)$ respectively. Hence $\phi(f)$ is a homotopy equivalence. This completes the proof of (4.1).

For example, let us take P to be the pathwise connectedness. For each homotopy functor ϕ, we may define a new homotopy property which might be called the *ϕ-pathwise connectedness* as follows. A topological space X is said to be *ϕ-pathwise connected* provided that $\phi(X)$ is pathwise connected. By (4.1), we know that ϕ-pathwise connectedness is a homotopy property of topological spaces. In particular, if ϕ is the homotopy functor constructed in Example 3 with $T = S^1$ the unit 1-sphere, then one can easily see that a topological space X is ϕ-pathwise connected if and only if it is simply connected. Thus, this gives us the well-known fact that simple connectedness is a homotopy property of topological spaces.

Analogously, we have the following

THEOREM 4.2. *If ψ is an isotopy functor, then every isotopy property of $\psi(X)$ induces an isotopy property of X; in particular, every homotopy property of $\psi(X)$ induces an isotopy property of X.*

The proof of (4.2) is similar to that of (4.1) and hence omitted.

COROLLARY 4.3. *If ψ is an isotopy functor, then all homotopy invariants of $\psi(X)$, such as the homology groups of $\psi(X)$, are isotopy invariants of X.*

By suitable choices of the isotopy functors ψ, (4.3) provides many new isotopy invariants of topological spaces which enable us to solve the problems in isotopy theory. For example, let us consider a family of topological spaces

$$W_q^p, \qquad (p \geqslant 0, q \geqslant 0),$$

where W_q^p denotes the linear graph obtained by attaching p small triangles at each end of a line-segment ab and joining the two ends of ab by q broken lines ac_kb, $k = 1, 2, \ldots, q$. Let

$$r = 2p + q.$$

Since the Euler characteristic of W_q^p is

$$\chi(W_q^p) = 1 - r,$$

it follows that the homotopy classification problem of this family of spaces $\{W_q^p: p \geqslant 0,\ q \geqslant 0\}$ is solved by the homotopy invariant $r = 2p + q$. Precisely, W_q^p and W_t^s are homotopically equivalent if and only if

$$2p + q = 2s + t.$$

For the isotopy classification of the spaces W_q^p with the same $r = 2p + q$, let us use the second residual functor R_2. In **(1)**, it has been computed that the two-dimensional homology group of $R_2(W_q^p)$ is a free abelian group of rank $2p^2$ and the one-dimensional homology group of $R_2(W_q^p)$ is a free abelian group of rank $6p^2 + 4pq + q^2 + 2p + q - 1$ for all W_q^p with $2p + q > 0$. This solves the isotopy classification problem for the spaces W_q^p.

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