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On the Connectedness of Moduli Spaces of Flat Connections over Compact Surfaces

Nan-Kuo Ho and Chiu-Chu Melissa Liu

Abstract. We study the connectedness of the moduli space of gauge equivalence classes of flat G-connections on a compact orientable surface or a compact nonorientable surface for a class of compact connected Lie groups. This class includes all the compact, connected, simply connected Lie groups, and some non-semisimple classical groups.

1 Introduction

Given a compact Lie group G and a compact surface Σ, let $\mathcal{M}(\Sigma, G)$ denote the moduli space of gauge equivalence classes of flat G-connections on Σ. We know that $\mathcal{M}(\Sigma, G)$ can be identified with $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, where G acts on $\mathrm{Hom}(\pi_1(\Sigma), G)$ by conjugate action of G on itself (see *e.g.,* [G]). It is known that if G is compact, connected, simply connected (in particular, G is semisimple), and Σ is orientable, then $\mathcal{M}(\Sigma, G)$ is nonempty and connected (see *e.g.,* [Li, AMM]). Thus it is natural to ask about the connectedness of $\mathcal{M}(\Sigma, G)$ for nonorientable Σ. From classification of compact surfaces, all nonorientable compact surfaces are homeomorphic to the connected sum of the real projective planes $\mathbb{RP}^2$.

Recall that we have the following structure theorem of compact connected Lie groups [K, Theorem 4.29]:

Theorem 1 *Let G be a compact connected Lie group with center $Z(G)$, and let S be the identity component of $Z(G)$. Let $\mathfrak{g}$ be the Lie algebra of G, and let G_{ss} be the analytic subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Then G_{ss} has finite center, S and G_{ss} are closed subgroups, and G is the commuting product, $G = G_{ss}S$.*

In Theorem 1, G_{ss} is semisimple, and the map $G_{ss} \times S \to G = G_{ss}S$ given by $(\bar{g}, s) \mapsto \bar{g}s$ is a finite cover which is also a group homomorphism. In particular, if G_{ss} is simply connected, we have the following result (the part about orientable surfaces is well-known):

Theorem 2 *Let G be a compact connected Lie group with center $Z(G)$, and let S be the identity component of $Z(G)$. Let $\mathfrak{g}$ be the Lie algebra of g, and let G_{ss} be the analytic subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Suppose that G_{ss} is simply connected. Then $\mathcal{M}(\Sigma, G)$ is nonempty and connected if Σ is a compact orientable surface, and $\mathcal{M}(\Sigma, G)$ is nonempty and has $2^{\dim S}$ connected components if Σ is homeomorphic to k copies of $\mathbb{RP}^2$, where $k \neq 1, 2, 4$.*

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For example, $U(n)$ and $Spin^{\mathbb{C}}(n)$ satisfy the hypothesis of Theorem 2. We have

$$U(n)_{ss} = SU(n),\ Spin^{\mathbb{C}}(n)_{ss} = Spin(n),\ Z(U(n)) \cong Z(Spin^{\mathbb{C}}(n)) \cong U(1).$$

When $\dim S = 0$, the hypothesis of Theorem 2 is equivalent to the condition that G is a compact, simply connected, connected Lie group. So we have the following special case:

Corollary 3 *Let G be a compact, connected, simply connected Lie group. Then $\mathcal{M}(\Sigma, G)$ is connected if Σ is a compact orientable surface or is homeomorphic to k copies of $\mathbb{RP}^2$, where $k \neq 1, 2, 4$.*

Corollary 3 is also a special case of the following result in [HL] (the part about orientable surfaces is well-known, see [Li]):

Theorem 4 *Let G be a compact, connected, semisimple Lie group. Let Σ be a compact orientable surface which is not homeomorphic to a sphere. Then there is a bijection*

$$\pi_0(\mathcal{M}(\Sigma, G)) \to H^2(\Sigma, \pi_1(G)) \cong \pi_1(G).$$

Let Σ be homeomorphic to k copies of $\mathbb{RP}^2$, where $k \neq 1, 2, 4$. Then there is a bijection

$$\pi_0(\mathcal{M}(\Sigma, G)) \to H^2(\Sigma, \pi_1(G)) \cong \pi_1(G)/2\pi_1(G),$$

where $2\pi_1(G)$ denote the subgroup $\{a^2 \mid a \in \pi_1(G)\}$ of the finite abelian group $\pi_1(G)$.

Our proofs of Theorem 2 and Theorem 4 rely on the following result of Alekseev, Malkin, and Meinrenken:

Fact 5 [AMM, Theorem 7.2] *Let G be a compact, connected, simply connected Lie group. Let ℓ be a positive integer. Then the commutator map $\mu_G^\ell \colon G^{2\ell} \to G$ defined by*

$$\mu_G^\ell(a_1, b_1, \ldots, a_\ell, b_\ell) = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_\ell b_\ell a_\ell^{-1} b_\ell^{-1} \tag{1}$$

is surjective, and $(\mu_G^\ell)^{-1}(g)$ is connected for all $g \in G$.

The surjectivity in Fact 5 follows from Goto's commutator theorem [HM, Theorem 6.55].

The case of orientable surfaces is discussed in Section 2. The case of the connected sum of $2\ell+1$ copies of $\mathbb{RP}^2$, or equivalently, the connected sum of a Riemann surface of genus ℓ and $\mathbb{RP}^2$, is studied in Section 3. The case of the connected sum of $2\ell + 2$ copies of $\mathbb{RP}^2$, or equivalently, the connected sum of a Riemann surface of genus ℓ and a Klein bottle, is studied in Section 4.

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2 Σ Is a Riemann Surface with Genus ℓ

In this section, we give a proof of Theorem 2 for Riemann surfaces of genus $\ell \geq 1$. The genus zero case is trivial because the fundamental group is trivial.

Lemma 6 *Let G, G_{ss}, S be as in Theorem 2. Let ℓ be a positive integer. Then the image of the commutator map $\mu_G^\ell \colon G^{2\ell} \to G$ defined by (1) is G_{ss}, and $(\mu_G^\ell)^{-1}(\bar{g})$ is connected for all $\bar{g} \in G_{ss}$.*

Proof By Fact 5, $\mu_G^\ell(G_{ss}^{2\ell}) = G_{ss}$, so $\mu_G^\ell(G^{2\ell}) \supset G_{ss}$. We now show that $\mu_G^\ell(G^{2\ell}) \subset G_{ss}$. Given $(a_1, b_1, \ldots, a_\ell, b_\ell) \in G^{2\ell}$, there exist $\bar{a}_i, \bar{b}_i \in G_{ss}$, $s_i, t_i \in S$ such that

$$a_i = \bar{a}_i s_i, \; b_i = \bar{b}_i t_i$$

for $i = 1, \ldots, \ell$. We have

$$\mu_G^\ell(a_1, b_1, \ldots, a_\ell, b_\ell) = \mu_G^\ell(\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_\ell, \bar{b}_\ell) \in G_{ss}.$$

So $\mu_G^\ell(G^{2\ell}) \subset G_{ss}$.

By Fact 5, $(\mu_G^\ell)^{-1}(\bar{g}) \cap G_{ss}^{2\ell}$ is connected for all $\bar{g} \in G_{ss}$, so it suffices to show that for any $\bar{g} \in G_{ss}$ and $(a_1, b_1, \ldots, a_\ell, b_\ell) \in (\mu_G^\ell)^{-1}(\bar{g})$, there is a path $\gamma \colon [0,1] \to (\mu_G^\ell)^{-1}(\bar{g})$ such that $\gamma(0) = (a_1, b_1, \ldots, a_\ell, b_\ell)$ and $\gamma(1) \in (\mu_G^\ell)^{-1}(\bar{g}) \cap G_{ss}^{2\ell}$.

Given $\bar{g} \in G_{ss}$ and $(a_1, b_1, \ldots, a_\ell, b_\ell) \in (\mu_G^\ell)^{-1}(\bar{g})$, there exist $\bar{a}_i, \bar{b}_i \in G_{ss}$, $s_i, t_i \in S$ such that

$$a_i = \bar{a}_i s_i, \; b_i = \bar{b}_i t_i$$

for $i = 1, \ldots, \ell$. Let $\mathfrak{g}$ and $\mathfrak{s}$ be the Lie algebras of G and S, respectively. Then $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$. Let $\exp \colon \mathfrak{g} \to G$ be the exponential map. There exist $X_i, Y_i \in \mathfrak{s}$ such that

$$\exp(X_i) = s_i, \; \exp(Y_i) = t_i.$$

for $i = 1, \ldots, \ell$. Define $\gamma \colon [0,1] \to G^{2\ell}$ by

$$\gamma(t) = \left(a_1 \exp(-tX_1), b_1 \exp(-tY_1), \ldots, a_l \exp(-tX_\ell), b_l \exp(-tY_\ell)\right).$$

Then the image of γ lies in $(\mu_G^\ell)^{-1}(\bar{g})$, and

$$\gamma(0) = (a_1, b_1, \ldots, a_\ell, b_\ell), \;\; \gamma(1) = (\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_\ell, \bar{b}_\ell) \in (\mu_G^\ell)^{-1}(\bar{g}) \cap G_{ss}^{2\ell}. \quad \blacksquare$$

Corollary 7 *Let G be as in Theorem 2. Let Σ be a Riemann surface of genus $\ell \geq 1$. Then $\mathcal{M}(\Sigma, G)$ is nonempty and connected.*

Proof Let μ_G^ℓ be the commutator map defined by (1), and let e be the identity element of G. Then $\mathrm{Hom}(\pi_1(\Sigma), G)$ can be identified with $(\mu_G^\ell)^{-1}(e)$, which is nonempty and connected by Lemma 6. So

$$\mathcal{M}(\Sigma, G) = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

is nonempty and connected. ■

3 Σ Is the Connected Sum of a Riemann Surface of Genus ℓ with One $\mathbb{RP}^2$

The following Proposition 8 is a well-known fact. We present an elementary proof for completeness. We use the notation in [Hu, Chapter III].

Proposition 8 *Let Φ be an irreducible root system of a Euclidean space E, and let W be the Weyl group of Φ. Then there exists $w \in W$ such that 1 is not an eigenvalue of the linear map $w\colon E \to E$.*

Proof The Coxeter element $w_c \in W$ has no eigenvalue equal to 1 [Hu, Section 3.16]. Here we will construct such an element (not necessarily w_c) case by case. From the classification of irreducible root systems (see *e.g.,* [Hu, Chapter III]), we have the following cases.

A_ℓ ($\ell \geq 1$): E is the ℓ-dimensional subspace of $\mathbb{R}^{\ell+1}$ orthogonal to the vector $\epsilon_1 + \cdots + \epsilon_{\ell+1}$, and

$$\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq \ell + 1, i \neq j\}.$$

The linear map $L\colon \mathbb{R}^{\ell+1} \to \mathbb{R}^{\ell+1}$ given by $L\epsilon_i = \epsilon_{i+1}$ for $1 \leq i \leq \ell$ and $L\epsilon_{\ell+1} = \epsilon_1$ restricts to a linear map $w\colon E \to E$ which is an element of W. The eigenvalues of w are $e^{2\pi i/(\ell+1)}$, $1 \leq i \leq \ell$.

B_ℓ ($\ell \geq 2$): $E = \mathbb{R}^\ell$, and

$$\Phi = \{\pm\epsilon_i \mid 1 \leq i \leq \ell\} \cup \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i, j \leq \ell, i \neq j\}.$$

The linear map $w\colon E \to E$ given by $v \mapsto -v$ is an element of W.

C_ℓ ($\ell \geq 3$): $E = \mathbb{R}^\ell$, and

$$\Phi = \{\pm 2\epsilon_i \mid 1 \leq i \leq \ell\} \cup \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i, j \leq \ell, i \neq j\}.$$

The linear map $w\colon E \to E$ given by $v \mapsto -v$ is an element of W.

D_ℓ (ℓ is even, $\ell \geq 4$): $E = \mathbb{R}^\ell$, and

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i, j \leq \ell, i \neq j\}.$$

If ℓ is even, the linear map $w\colon E \to E$ given by $v \mapsto -v$ is an element of W. If l is odd, the linear map $w\colon E \to E$ given by $\epsilon_1 \mapsto \epsilon_2$, $\epsilon_2 \mapsto -\epsilon_1$, and $\epsilon_i \mapsto -\epsilon_i$ for $i \geq 3$ is an element of W, and the eigenvalues of w are $i, -i, -1$.

E_ℓ ($\ell = 6, 8$): $E = \mathbb{R}^\ell$, and

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i, j \leq \ell\} \cup \left\{ \frac{1}{2} \sum_{i=1}^{\ell} (-1)^{k(i)} \epsilon_i \mid k(i) = 0, 1, \sum_{i=1}^{\ell} k(i) \text{ is even} \right\}.$$

The linear map $w\colon E \to E$ given by $v \mapsto -v$ is an element of W.

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E_7: $E = \mathbb{R}^7$, and

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i, j \le 7\} \cup \Big\{ \frac{1}{2}\sum_{i=1}^{7}(-1)^{k(i)}\epsilon_i \mid k(i) = 0, 1, \sum_{i=1}^{7} k(i) \text{ is odd} \Big\}.$$

The linear map $w\colon E \to E$ given by $\epsilon_1 \mapsto \epsilon_2$, $\epsilon_2 \mapsto -\epsilon_1$, and $\epsilon_i \mapsto -\epsilon_i$ for $i \ge 3$ is an element of W, and the eigenvalues of w are $i, -i, -1$.

F_4: $E = \mathbb{R}^4$, and

$$\Phi = \{\pm\epsilon_i \mid 1 \le i \le 4\} \cup \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i, j \le 4, i \ne j\} \cup \{\pm\tfrac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}.$$

The linear map $w\colon E \to E$ given by $v \mapsto -v$ is an element of W.

G_2: E is the subspace of $\mathbb{R}^3$ orthogonal to $\epsilon_1 + \epsilon_2 + \epsilon_3$, and

$$\begin{aligned}\Phi = \{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_2 - \epsilon_3), \pm(\epsilon_3 - \epsilon_1), \pm(2\epsilon_1 - \epsilon_2 - \epsilon_3),\\ \pm(2\epsilon_2 - \epsilon_3 - \epsilon_1), \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2)\}.\end{aligned}$$

The linear map $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\epsilon_1 = \epsilon_2$, $L\epsilon_2 = \epsilon_3$, $L\epsilon_3 = \epsilon_1$ restricts to a linear map $w\colon E \to E$ which is an element of W. The eigenvalues of w are $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. ■

The root system of a semisimple Lie algebra can be decomposed into irreducible root systems, so Proposition 8 implies:

Corollary 9 *Let G be a compact, connected, simply connected Lie group. Let $\mathfrak{t}$ be the Lie algebra of the maximal torus T of G. Then there exists an element w in the Weyl group of G such that 1 is not an eigenvalue of the linear map $w\colon \mathfrak{t} \to \mathfrak{t}$.*

Theorem 10 *Let G, G_{ss}, S be as in Theorem 2. Let Σ be a compact surface whose topological type is the connected sum of a Riemann surface of genus $\ell \ge 1$ and $\mathbb{RP}^2$. Then $\mathcal{M}(\Sigma, G)$ is nonempty and has $2^{\dim S}$ connected components.*

Proof The space $\mathrm{Hom}(\pi_1(\Sigma), G)$ can be identified with

$$X = \{(a_1, b_1, \ldots, a_\ell, b_\ell, c) \in G^{2\ell+1} \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_\ell b_\ell a_\ell^{-1} b_\ell^{-1} c^2 = e\},$$

where e is the identity element of G. So

$$\mathcal{M}(\Sigma, G) = X/G,$$

where G acts on $G^{2\ell+1}$ by diagonal conjugation. Note that the action of G preserves X.

Let G_{ss}, S be as in Theorem 2. Then G_{ss} is a normal subgroup in G. Define

$$\check{S} = G/G_{ss} \cong S/(G_{ss} \cap S),$$

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where the isomorphism is induced by the inclusion $S \hookrightarrow G$. Note that $G_{ss} \cap S \subset Z(G_{ss})$ is a finite abelian group, so $S \to \check{S}$ is a finite cover, and $\check{S}$ is a compact torus with $\dim \check{S} = \dim S$. Let $\pi\colon G \to \check{S} = G/G_{ss}$ be the natural projection, and let $P\colon X \to G$ be defined by

$$(a_1, b_1, \dots, a_\ell, b_\ell, c) \mapsto c.$$

For $(a_1, b_1, \dots, a_\ell, b_\ell, c) \in X$, we have

$$c^{-2} = \mu_G(a_1, b_1, \dots, a_\ell, b_\ell)$$

which is an element of G_{ss} by Lemma 2. So

$$\pi(c) \in K \equiv \{k \in \check{S} \mid k^2 = \check{e}\} \cong (\mathbb{Z}/2\mathbb{Z})^{\dim S}$$

where $\check{e}$ is the identity element of $\check{S}$. Thus $\pi \circ P$ gives a continuous map

$$\tilde{o}\colon X \to K.$$

Note that $\tilde{o}$ factors through the quotient X/G, so we have

$$o\colon X/G \to K,$$

and $\tilde{o} = o \circ p$, where p is the projection $X \to X/G$. Let $X_k = \tilde{o}^{-1}(k)$ for $k \in K$. We will show that X_k is nonempty and connected for each $k \in K$, which implies $o^{-1}(k)$ is nonempty and connected for each $k \in K$. This will complete the proof since K is a group of order $2^{\dim S}$.

Let $k \in K \subset \check{S}$. We fix $\tilde{k} \in S$ such that $\pi(\tilde{k}) = k$. By Lemma 6, $P^{-1}(c)$ is nonempty and connected for all $c \in G$ such that $\pi(c) = k$. So X_k is nonempty. To prove that X_k is connected, it suffices to show that for any $c \in G$ such that $\pi(c) = k$, there is a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) \in P^{-1}(\tilde{k})$ and $\gamma(1) \in P^{-1}(c)$.

Since $\pi(c) = \pi(\tilde{k})$, we have $c\tilde{k}^{-1} \in G_{ss}$. There exists $g \in G_{ss}$ such that $g^{-1}c\tilde{k}^{-1}g \in T_{ss}$, where T_{ss} is the maximal torus of G_{ss}. Let $\mathfrak{g}_{ss}$ and $\mathfrak{t}_{ss}$ denote the Lie algebras of G_{ss} and T_{ss}, respectively, and let $\exp\colon \mathfrak{g}_{ss} \to G_{ss}$ be the exponential map. Then

$$g^{-1}cg\tilde{k}^{-1} = \exp(\xi)$$

for some $\xi \in \mathfrak{t}_{ss}$. By Corollary 9, there exists w in the Weyl group W of G_{ss} and $\xi' \in \mathfrak{t}_{ss}$ such that $w\cdot\xi' - \xi' = \xi$. Recall that $W = N(T_{ss})/T_{ss}$, where $N(T_{ss})$ is the normalizer of T_{ss} in G_{ss}, so $w = aT_{ss} \in N(T_{ss})/T_{ss}$ for some $a \in G_{ss}$. We have

$$a\exp(t\xi')a^{-1}\exp(-t\xi') = \exp(t\xi)$$

for any $t \in \mathbb{R}$.

The group G_{ss} is connected, so we may choose a path $\bar{g}\colon [0,1] \to G_{ss}$ such that $\bar{g}(0) = e$ and $\bar{g}(1) = g$. Define $\gamma\colon [0,1] \to G^{2\ell+1}$ by

$$\gamma(t) = (a(t), b(t), e, \dots, e, c(t)),$$

where

$$a(t) = \bar{g}(t) a \bar{g}(t)^{-1},$$
$$b(t) = \bar{g}(t) \exp(-2t\xi') \bar{g}(t)^{-1},$$
$$c(t) = \tilde{k} \bar{g}(t) \exp(t\xi) \bar{g}(t)^{-1}.$$

Then the image of γ lies in X_k, $\gamma(0) = (a, e, e, \dots, e, \tilde{k}) \in P^{-1}(\tilde{k})$, and

$$\gamma(1) = (gag^{-1}, g \exp(-2\xi') g^{-1}, e, \dots, e, c) \in P^{-1}(c). \qquad \blacksquare$$

4 Σ Is the Connected Sum of a Riemann Surface of Genus $\ell \geq 2$ with a Klein Bottle

Theorem 11 *Let G, G_{ss}, S be as in Theorem 2. Let Σ be a compact surface whose topological type is the connected sum of a Riemann surface of genus $\ell \geq 2$ and a Klein bottle. Then $\mathcal{M}(\Sigma, G)$ is nonempty and has $2^{\dim S}$ connected components.*

Proof The space $\mathrm{Hom}(\pi_1(\Sigma), G)$ can be identified with

$$X = \{(a_1, b_1, \dots, a_\ell, b_\ell, c_1, c_2) \in G^{2\ell+2} \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_\ell b_\ell a_\ell^{-1} b_\ell^{-1} c_1^2 c_2^2 = e\},$$

where e is the identity element of G. So

$$\mathcal{M}(\Sigma, G) = X/G,$$

where G acts on $G^{2\ell+2}$ by diagonal conjugation. Note that the action of G preserves X.

Let $\pi\colon G \to \check{S} = G/G_{ss}$ be defined as in the proof of Theorem 10, and let $P\colon X \to G^2$ defined by

$$(a_1, b_1, \dots, a_\ell, b_\ell, c_1, c_2) \mapsto (c_1, c_2).$$

For $(a_1, b_1, \dots, a_\ell, b_\ell, c_1, c_2) \in X$, we have

$$c_2^{-2} c_1^{-2} = \mu_G(a_1, b_1, \dots, a_\ell, b_\ell)$$

which is an element of G_{ss} by Lemma 6, so

$$(\pi(c_1)\pi(c_2))^2 = \pi(c_1)^2 \pi(c_2)^2 = \pi(c_1^2 c_2^2) = \check{e},$$

where $\check{e}$ is the the identity element of $\check{S}$. Define $\tilde{o}\colon X \to K$ by

$$(a_1, b_1, \dots, a_\ell, b_\ell, c_1, c_2) \mapsto \pi(c_1)\pi(c_2)$$

where K is defined as in the proof Theorem 10. Note that $\tilde{o}$ factors through the quotient X/G, so we have

$$o\colon X/G \to K,$$

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and $\tilde{o} = o \circ p$, where p is the projection $X \to X/G$. Let $X_k = \tilde{o}^{-1}(k)$ for $k \in K$. We will show that X_k is nonempty and connected for each $k \in K$, which will complete the proof.

Given $k \in K$, we fix $\tilde{k} \in S$ such that $\pi(\tilde{k}) = k$. By Lemma 6, $P^{-1}(c_1, c_2)$ is nonempty and connected for all $(c_1, c_2) \in G^2$ such that $\pi(c_1)\pi(c_2) = k$. So X_k is nonempty. To prove that X_k is connected, it suffices to show that for any $(c_1, c_2) \in G^2$ such that $\pi(c_1)\pi(c_2) = k$, there is a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) \in P^{-1}(e, \tilde{k})$ and $\gamma(1) \in P^{-1}(c_1, c_2)$.

Given $(c_1, c_2) \in G^2$ such that $\pi(c_1)\pi(c_2) = k$, let $k_1 = \pi(c_1)$ and $k_2 = \pi(c_2)$. Choose $s_1 \in S$ such that $\pi(s_1) = k_1$ and let $s_2 = s_1^{-1}\tilde{k}$. Then $c_1 s_1^{-1}, c_2 s_2^{-1} \in G_{ss}$, and $s_1 s_2 = \tilde{k}$. Let $T_{ss}, \mathfrak{t}_{ss}, \mathfrak{g}_{ss}$ be as in the proof of Theorem 10. There exist $g_1, g_2 \in G_{ss}$, $\xi_1, \xi_2 \in \mathfrak{t}_{ss}$ such that

$$c_1 s_1^{-1} = g_1 \exp(\xi_1) g_1^{-1},\ c_2 s_2^{-1} = g_2 \exp(\xi_2) g_1^{-1}$$

There exists $X \in \mathfrak{s}$ such that $s_1 = \exp(X)$. Then $s_2 = \exp(-X)\tilde{k}$.

By Corollary 9, there exists w in the Weyl group W of G_{ss} and $\xi_1', \xi_2' \in \mathfrak{t}_{ss}$ such that $w \cdot \xi_i' - \xi_i' = \xi_i$, $i = 1, 2$. Recall that $W = N(T_{ss})/T_{ss}$, where $N(T_{ss})$ is the normalizer of T_{ss} in G_{ss}, so $w = aT_{ss} \in N(T_{ss})/T_{ss}$ for some $a \in G_{ss}$. We have

$$a \exp(t\xi_i') a^{-1} \exp(-t\xi_i') = \exp(t\xi_i)$$

where $i = 1, 2$.

The group G_{ss} is connected, so we may choose a path $\bar{g}_i\colon [0,1] \to G_{ss}$ such that $\bar{g}_i(0) = e$ and $\bar{g}(1) = g_i$ for $i = 1, 2$. Define $\gamma\colon [0,1] \to G^{2\ell+1}$ by

$$\gamma(t) = (a_1(t), b_1(t), a_2(t), b_2(t), e, \dots, e, c_1(t), c_2(t)),$$

where

$$\begin{aligned}
a_1(t) &= \bar{g}_2(t) a \bar{g}_2(t)^{-1},\\
b_1(t) &= \bar{g}_2(t) \exp(-2t\xi_2') \bar{g}_2(t)^{-1}\\
a_2(t) &= \bar{g}_1(t) a \bar{g}_1(t)^{-1},\\
b_2(t) &= \bar{g}_1(t) \exp(-2t\xi_1') \bar{g}_1(t)^{-1}\\
c_1(t) &= \bar{g}_1(t) \exp(t\xi_1) \bar{g}_1(t)^{-1} \exp(tX)\\
c_2(t) &= \bar{g}_2(t) \exp(t\xi_2) \bar{g}_2(t)^{-1} \exp(-tX)\tilde{k}
\end{aligned}$$

Then the image of γ lies in X_k, and

$$\begin{aligned}
\gamma(0) &= (a, e, a, e, e, \dots, e, e, \tilde{k}) \in P^{-1}(e, \tilde{k}),\\
\gamma(1) &= \big(g_2 a g_2^{-1}, g_2 \exp(-2\xi_2') g_2^{-1}, g_1 a g_1^{-1}, g_1 \exp(-2\xi_1') g_1^{-1}, e,\\
&\qquad \dots, e, c_1, c_2\big) \in P^{-1}(c_1, c_2).
\end{aligned}$$

■

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Department of Mathematics
National Cheng-Kung University
Taiwan
e-mail: nankuo@mail.ncku.edu.tw

Department of Mathematics
Harvard University
Cambridge, MA 02138
USA
e-mail: ccliu@math.harvard.edu