Appendix C
Canonical Manifold of Mappings

This appendix sketches the construction of a canonical manifold of mappings structure for smooth mappings between (finite-dimensional) manifolds (for details see Amiri et al., 2020, Appendix A). Before we begin, let us reconsider for a moment the locally convex space $C^\infty(M,E)$ (M a compact manifold and E a locally convex space; see §2.1). The topology and vector space structure allow us to compare two smooth maps f and g by measuring their difference $f-g$ on compact sets. As a general manifold N lacks an addition, we cannot mimic this construction (though the topology still makes sense!). On first sight it might be tempting to think that one could use the charts of N to construct charts for $C^\infty(M,N)$. However, if N does not admit an atlas with only one chart, there will be smooth mappings whose image is not contained in one chart. Thus the charts of N turn out to be not very useful. Instead one needs to find a replacement of the vector space addition to construct a way in which 'charts vary smoothly' over N.

C.1 Local Additions

In this section we first define a replacement for vector additions which allow us to construct a canonical manifold structure on spaces of mappings.

C.1 Definition Let M be a smooth manifold. A *local addition on M* is a smooth map

$$\Sigma\colon TM \supseteq U \to M,$$

defined on an open neighbourhood $U \Subset TM$ of the *zero-section* of the tangent bundle $0_M := \{0_p \in T_pM \mid p \in M\}$ such that $\Sigma(0_p) = p$ for all $p \in M$,

$$U' := \{(\pi_M(v), \Sigma(v)) \mid v \in U\}$$

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is open in $M \times M$ (where $\pi_M : TM \to M$ is the bundle projection) and the mapping $\theta := (\pi_M, \Sigma) : U \to U'$ is a C^∞-diffeomorphism.

C.2 Let G be a Lie group. Then G admits a local addition. To see this, let $\varphi : U \to V$ be a chart of G with $\mathbf{1} \in U$ and $\varphi(\mathbf{1}) = 0$. Then $\tilde{U} := T_0\varphi^{-1}(V)$ is open in $T_{\mathbf{1}}G$ and we define $\alpha_{\mathbf{1}} : \tilde{U} \to U,\ v \mapsto \varphi^{-1}(T_{\mathbf{1}}\varphi(v))$. Note that $\alpha_{\mathbf{1}}$ is a diffeomorphism. Now the tangent bundle of G is trivial, that is, $TG \cong G \times T_{\mathbf{1}}G$, Lemma 3.12, and we obtain an open set $\Omega := \bigcup_{g \in G} T\lambda_g(\tilde{U}) \cong G \times \tilde{U} \Subset G \times T_{\mathbf{1}}G$. Define the smooth map

$$\Sigma : \Omega \to G, \quad v \mapsto \pi_G(v) \cdot \alpha_{\mathbf{1}}(T\lambda_{\pi_G(v)^{-1}}(v)).$$

We note that Ω is a neighbourhood of the zero section and $\Sigma(0_g) = g \cdot \alpha_{\mathbf{1}}(0_{\mathbf{1}}) = g \cdot \mathbf{1} = g$. Finally, $(\pi_G, \Sigma) : \Omega \to G \times G$ is a diffeomorphism (onto an open set) with inverse $(\pi_G, \Sigma)^{-1}(g,h) = T\lambda_g(\alpha_{\mathbf{1}}^{-1}(g^{-1}h))$.

C.3 Remark If (M,g) is a strong Riemannian manifold (see Chapter 4), then the Riemannian exponential map $\exp_g$ of g induces a local addition on M (Michor, 1980, Lemma 10.1).

We leave the proof of the following statement as Exercise C.1.1.

C.4 Lemma (Schmeding and Wockel, 2015, Lemma 7.5; or Michor, 1980, 10.11) *Let M be a manifold which admits a local addition Σ. Then TM admits a local addition (it is $T\Sigma \circ \kappa$, where κ is the flip of the double tangent bundle).*

With the help of a local addition we can pull to vector fields over a given function.

C.5 Let $f \in C^\infty(K,M)$ and assume that M admits a local addition $\Sigma : TM \supseteq \Omega \to M$. If $g \in C^\infty(K,M)$ such that

$$(f,g)(K) \subseteq (\pi_M, \Sigma)(\Omega) \Subset M \times M,$$

define the mapping

$$\Phi_f(g) := (\pi_M, \Sigma)^{-1} \circ (f,g) \in C^\infty(K,TM).$$

A quick computation shows that $\pi_M \circ \Phi_f(g) = f$. The idea is that $\Phi_f(g)$ yields at every $x \in K$ a vector in $T_{f(x)}M$ which gets mapped by the local addition to $g(x)$. Thus the mapping $\Phi_f(g)$ measures the difference between f and g (similar to $f - g$ in the vector space case). Note that Φ_f takes its values in the subspace

$$C^\infty_f(K,TM) := \{h \in C^\infty(K,TM) \mid \pi_M \circ h = f\}$$

of mappings over the given map f.

To make mappings of the form Φ_f into charts for $C^\infty(K,M)$, we first need to study spaces of the form $C^\infty_f(K,TM)$. In particular, we need these spaces to be locally convex spaces and will showcase this in the next section. The idea will be to relate these sets to spaces of sections of certain vector bundles. Note, however, that the space of functions $C^\infty_f(K,TM)$ depends heavily on the function f. In general, these spaces will *not* be isomorphic if we change the base function f.

Exercise

C.1.1 Let M be a manifold, TM its tangent bundle and $T^2M = T(TM)$ the double tangent bundle. Assume that M admits a local addition $\Sigma\colon TM \supseteq U \to M$.

(a) Show that there is a bundle isomorphism $\kappa\colon T^2M \to T^2M$ over the identity such that in local coordinates we have (up to identification) the identity $\kappa(x,v,u,w) = (x,u,v,w)$. We call κ the *(canonical) flip* of the double tangent bundle.

(b) Prove that $T\Sigma \circ \kappa$ is a local addition for TM. Explain then why $T\Sigma$ is not a local addition.

C.2 Vector Bundles and Their Sections

Recall that a *vector bundle* is a pair of manifolds E (*total space*) and M (*base (space)*) together with a surjective submersion $p\colon E \to M$ such that for every $x \in M$ the *fibre* $E_x := p^{-1}(x) \subseteq E$ is a real vector space and there is $x \in U_x \Subset M$ and a diffeomorphism $\kappa_x\colon U_x \times F_x \to p^{-1}(U_x)$, a *bundle trivialisation* such that

- $p \circ \kappa_x(y,v) = y$, for all $y \in U_x$;
- for each $y \in U_x$, the map $\kappa_x(y,\cdot)\colon F_x \to p^{-1}(y)$ is a vector space isomorphism.

Finally, for two vector bundles $p_i\colon E_i \to M_i, i = 1,2$, a pair of smooth maps $F\colon E_1 \to E_2$ and $f\colon M_1 \to M_2$ is called *vector bundle morphism* if $p_2 \circ F = f \circ p_1$ and $F|_{p_1^{-1}(x)}^{p_2^{-1}(f(x))}$ is linear for every $x \in M_1$ (we also say that F is a bundle morphism over f).

C.6 (Sections of vector bundles) For a vector bundle $p\colon E \to M$, a smooth map $f \in C^\infty(M,E)$ is called a *smooth section* if $p \circ f = \mathrm{id}_M$. We denote the *set of all sections* by

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$$\Gamma(E) := \{f \in C^\infty(M,E) \mid p \circ f = \mathrm{id}_M\}.$$

Since every fibre of a vector bundle is a vector space, pointwise addition and scalar multiplication of sections makes sense and $\Gamma(E)$ becomes a vector space. This space can be topologised as a locally convex space as follows.

C.7 Let $p\colon E \to M$ be a vector bundle. We pick a family of bundle trivialisations $\kappa_i\colon p^{-1}(U_i) \to U_i \times F_i, i \in I$ which cover E (i.e. the U_i cover M). If we denote by $\mathrm{pr}_{F_i}\colon U_\varphi \times F_i \to F_i$ the canonical projection, then the *principal part* of the section X in the trivialisation κ_i is defined as

$$X_{\kappa_i} := \mathrm{pr}_{F_i} \circ \kappa_i \circ X|_{U_i} \in C^\infty(U_i, F_i).$$

Note that this entails that with respect to the bundle trivialisation we have

$$\kappa_i \circ X = (\mathrm{id}_{U_i}, X_{\kappa_i}), \quad \text{for all } X \in \Gamma(E).$$

We declare a topology on $\Gamma(E)$ as the initial topology with respect to the map

$$\mathcal{I}_E\colon \Gamma(E) \to \prod_{i\in I} C^\infty(U_i, F_i), \quad X \mapsto (X_{\kappa_i})_{i\in I},$$

where the factors on the right-hand side carry the compact open C^∞-topology. Now by definition of the vector space operations we have $\mathcal{I}_E(X + rU) = \mathcal{I}_E(X) + r\mathcal{I}_E(U)$. Note that since the right-hand side is a locally convex space by Proposition 2.4, this also shows that $\Gamma(E)$ is a locally convex space. We shall see that this structure does not depend on the choice of trivialisations in Exercise C.2.1.

Now to model spaces of smooth mappings, we need a certain type of bundle constructed from a map and a vector bundle. While we study this situation we recall two ways to construct new vector bundles from given vector bundles.

C.8 (Pullback bundles and their sections) Let M and K be smooth manifolds. If $p\colon E \to M$ is a smooth vector bundle over M and $f\colon K \to M$ is a smooth map, then

$$f^*(E) := \bigcup_{x\in K} \{x\} \times E_{f(x)}$$

is a split submanifold of $K \times E$ (as it locally looks like $\mathrm{graph}(f) \times E_x$ inside $K\times M\times E_x$ around points in $\{x\}\times E_x$). We endow $f^*(E)$ with this submanifold structure. Together with the natural vector space structure on $\{x\} \times E_{f(x)} \cong E_{f(x)}$ and the map $p_f\colon f^*(E) \to K$, $(x,y) \mapsto x$, we obtain a vector bundle $f^*(E)$ over K, the so-called *pullback of E along f*. For each local trivialisation $\theta = (p|_{E|_U}, \theta_2)\colon E|_U \to U \times F$ of E and $W := f^{-1}(U)$, the map

$$f^*(E)|_W \to W \times F, \quad (x,y) \mapsto (x, \theta_2(y))$$

is a local trivialisation of $f^*(E)$. We endow

$$C_f^\infty(K,E) := \{\tau \in C^\infty(K,E) \mid p \circ \tau = f\}$$

with the topology induced by $C^\infty(K,E)$. With pointwise operations, $C_f^\infty(K,E)$ is a vector space and the map

$$\Psi\colon \Gamma(f^*(E)) \to C_f^\infty(K,E),\ \ \sigma \mapsto \mathrm{pr}_2 \circ \sigma$$

is a bijection with inverse $\tau \mapsto (\mathrm{id}_K, \tau)$. As $(\mathrm{pr}_2)_*\colon C^\infty(K, K\times E) \to C^\infty(K,E)$ is a continuous map and also $\tau \mapsto (\mathrm{id}_K,\tau) \in C^\infty(K,K) \times C^\infty(K,E) \cong C^\infty(K, K\times E)$ is continuous, we deduce that $C_f^\infty(K,E)$ is a locally convex topological vector space and Ψ is an isomorphism of topological vector spaces.

C.9 (Whitney sum of bundles) Let $p_i\colon E_i \to M, i = 1,2$ be vector bundles over the same base M. Then we can form the *direct product* of these vector bundles, which is the vector bundle

$$p_1 \times p_2\colon E_1 \times E_2 \to M \times M$$

over the product manifold $M\times M$. Consider the diagonal map $\Delta\colon M \to M\times M$, $m \mapsto (m,m)$. Then the *Whitney sum* of p_1 and p_2 is defined as the bundle

$$p_1 \oplus p_2\colon E_1 \oplus E_2 := \Delta^*(E_1 \times E_2) \to M.$$

Note that by construction we have as fibres $(E_1 \oplus E_2)_x = (E_1)_x \times (E_2)_x$ and if κ_i is a bundle trivialisation of E_i over a common open set $U \Subset M$, then the restriction of $\kappa_1 \times \kappa_2$ to $(p_1 \oplus p_2)^{-1}(U)$ is a bundle trivialisation of the Whitney sum.

We just mention that there is a version of the exponential law for spaces of sections; see Amiri et al. (2020, Appendix A).

Exercises

C.2.1 Show that the mapping $\mathcal{I}_E\colon \Gamma(E) \to \prod_{i\in I} C^\infty(U_i, F)$ from C.7 has a closed image. Then prove that the topology does not depend on the choice of local trivialisations, that is, if $\mathcal{B} := \{\nu_j\}_{j\in J}$ is another family of trivialisations covering E, then the topologies induced by $\mathcal{I}_E$ and $\mathcal{I}_{E,\mathcal{B}}$ coincide.
Hint: If $\mathcal{A},\mathcal{B}$ are families of trivialisations, then $\mathcal{A} \cup \mathcal{B}$ is also such a family. Thus we may assume without loss of generality that $\mathcal{A} \subseteq \mathcal{B}$

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and it suffices to prove that the topology induced by $\mathcal{B}$ cannot be finer than the one induced by $\mathcal{A}$. To prove this, adapt Lemma 2.5.

C.2.2 Verify that the pullback bundle in C.8 forms a split submanifold of $K \times E$.
Hint: Construct submanifold charts by hand as for the graph of a function.

C.2.3 Show that for two vector bundles $p_i\colon E_i \to M$ there is a canonical isomorphism of locally convex spaces $\Gamma(E_1) \times \Gamma(E_2) \cong \Gamma(E_1 \oplus E_2)$.

C.3 Construction of the Manifold Structure

General Assumption We let K be a compact manifold and M be a smooth manifold which admits a local addition $\Sigma\colon TM \supseteq U \to M$ such that

- Σ is *normalised*, that is, $T_{0_p}(\Sigma|_{T_pM}) = \mathrm{id}_{T_pM}$ for all $p \in M$ (a manifold with local addition has a normalised local addition; see Amiri et al., 2020, Lemma A.14);
- $\theta := (\pi_M, \Sigma)\colon U \to U'$ is a diffeomorphism.

C.10 (Manifold structure on $C^\infty(K,M)$) For $f \in C^\infty(K,M)$, the locally convex space of C^∞-sections of $f^*(TM)$ can be identified with

$$C^\infty_f(K,E) = \{\tau \in C^\infty(K,TM) \mid \pi_M \circ \tau = f\},$$

with the topology induced by $C^\infty(K,TM)$. Use notation as in Definition C.1. Then $O_f := C^\infty_f(K,E) \cap C^\infty(K,U)$ is an open subset of $C^\infty_f(K,E)$,

$$O'_f := \{g \in C^\infty(K,M) \mid (f,g)(K) \subseteq U'\}$$

is an open subset[1] of $C^\infty(K,M)$ and the map $\phi_f\colon O_f \to O'_f$, $\tau \mapsto \Sigma \circ \tau$ is a homeomorphism with inverse $g \mapsto \theta^{-1} \circ (f,g)$. By the preceding, if also $h \in C^\infty(K,M)$, then $\phi_h^{-1} \circ \phi_f$ has an open domain; $\phi_h^{-1} \circ \phi_f$ is smooth there, since (by the exponential law, Theorem 2.12), we only need to observe that the map

$$(\tau,x) \mapsto (\phi_h^{-1} \circ \phi_f)(\tau)(x) = \theta^{-1}(h(x), \Sigma(\tau(x))) \tag{C.1}$$

is C^∞. Hence $C^\infty(K,M)$ has a smooth manifold structure such that each of the maps ϕ_f^{-1} is a local chart.

We prove that the manifold structure on $C^\infty(K,M)$ is canonical and thus by Lemma 2.16(b) the construction C.10 is independent of the choice of local addition.

[1] The proof is similar to Exercise B.2.4.

C.11 Lemma *The manifold structure on* $C^\infty(K,M)$ *constructed in* C.10 *is canonical.*

Proof We first show that the evaluation map $\mathrm{ev}\colon C^\infty(K,M)\times K\to M$ is C^∞. It suffices to show that $\mathrm{ev}(\phi_f(\sigma),x)$ is C^∞ in $(\sigma,x)\in O_f\times K$ for all $f\in C^\infty(K,M)$. But

$$\mathrm{ev}(\phi_f(\sigma),x)=\Sigma(\sigma(x))=\Sigma(\mathrm{ev}(\sigma,x)),$$

where $\mathrm{ev}\colon C^\infty_f(K,E)\times K\to f^*(TM)$, $(\sigma,x)\mapsto\sigma(x)$ is C^∞ (see Alzaareer and Schmeding, 2015, Proposition 3.20). Now let $h\colon L\to C^\infty(K,M)$ be a map, where L is a manifold. If h is C^∞, then $h^\wedge=\mathrm{ev}\circ(h\times\mathrm{id}_K)$ is C^∞. If, conversely, $h^\wedge$ is C^∞, then h is continuous as a map to $C(K,M)$ with the compact open topology (see Proposition B.12) and $h(x)=h^\wedge(x,\cdot)\in C^\infty(K,M)$ for each $x\in L$. Given $x\in L$, let $f:=h(x)$. Then

$$\psi_f\colon C(K,M)\to C(K,M)\times C(K,M)\cong C(K,M\times M),\quad g\mapsto(f,g)$$

is a continuous map and $W:=h^{-1}(O'_f)=(\phi_f\circ h)^{-1}(C(K,U'))$ is an open x-neighbourhood in L. Since

$$((\phi_f)^{-1}\circ h|_W)^\wedge(y,z)=(\theta^{-1}\circ(f,h(y)))(z)=\theta^{-1}(f(z),h^\wedge(y,z))$$

is C^∞, the map $\phi_f^{-1}\circ h|_W$ (and hence also $h|_W$) is C^∞ (apply Theorem 2.12 to the spaces $C^\infty(U_i,F)$ containing the principal parts of the sections). □

C.12 Let $\gamma\in C^\infty(K,M)$ and view $T_\gamma C^\infty(K,M)$ as a set of equivalence classes of smooth curves $c\colon\,]-\varepsilon,\varepsilon[\to C^\infty(K,M), c(0)=\gamma$. As the manifold structure is canonical, c is smooth if and only if $c^\wedge\colon\,]-\varepsilon,\varepsilon[\times K\to M$ is smooth. Hence for the canonical chart $\phi_\gamma\colon O_\gamma\to O'_\gamma\subseteq C^\infty(K,M)$, the map $T_0\phi_\gamma\colon C^\infty_\gamma(K,TM)\to\to T_\gamma C^\infty(K,M)$ is an isomorphism of TVS. For $x\in K$ denote by ε_x the point evaluation in x. Since Σ is normalised we obtain

$$\begin{aligned}T\varepsilon_xT\phi_f(0,\tau)&=T\varepsilon_x([t\mapsto\Sigma\circ(t\tau)])=[t\mapsto\Sigma(t\tau(x))]\\&=[t\mapsto\Sigma|_{T_{f(x)}M}(t\tau(x))]=T\Sigma|_{T_{f(x)}M}(\tau(x))=\tau(x).\end{aligned}$$

Summing up, this implies that for each fibre there is a linear bijection

$$\Phi_\gamma\colon T_\gamma C^\infty(K,M)\to C^\infty_\gamma(K,TM),\quad[c]\mapsto(k\mapsto[t\mapsto c^\wedge(t,k)]).\tag{C.2}$$

We will now sketch the proof that the fibre maps (C.2) induce a bundle isomorphism

$$\Phi_M: TC^\infty(K,M)\to C^\infty(K,TM),\quad T_\gamma C^\infty(K,M)\ni V\mapsto\Phi_\gamma(V)$$

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such that the following diagram commutes

$$\begin{array}{ccc} TC^\infty(K,M) & \xrightarrow{\Phi_M} & C^\infty(K,TM) \\ & {\scriptstyle \pi_{C^\infty(K,M)}} \searrow \quad \swarrow {\scriptstyle (\pi_M)_*} & \\ & C^\infty(K,M). & \end{array}$$

Sketch of Proof (See Amiri et al., 2020, Appendix A.) If $\lambda_p : T_pM \to TM$ is the inclusion and $\kappa : T^2M \to T^2M$ the canonical flip, then $\Theta : TM \oplus TM \to \pi_{TM}^{-1}(0_M) \subseteq T^2M, \Theta(v,w) = \kappa(T\lambda_{\pi(v)}(v,w))$ is a bundle isomorphism. Let $0 : M \to TM$ be the zero-section. Then Θ induces a diffeomorphism

$$\Theta_\gamma : O_\gamma \to O_{0\circ\gamma}, \quad \eta \mapsto \Theta \circ (0 \circ \gamma, \eta).$$

From the local addition Σ, we construct a local addition $\widetilde{\Sigma}$ on TM and consider the charts $\phi_{0\circ\gamma}$ on $C^\infty(K,TM)$. Then the sets $S_\gamma := T\phi_\gamma(O_\gamma \times C^\infty_\gamma(K,TM))$ form an open cover of $T(C^\infty(K,M))$ for $\gamma \in C^\infty(K,M)$. We deduce that the sets $\Phi_M(S_\gamma)$ form a cover of $C^\infty(K,TM)$ by sets which are open as $\Phi_M(S_\gamma) = (\phi_{0\circ\gamma} \circ \phi_\gamma)(O_\gamma \times C^\infty_\gamma(M,TM)) = \phi_{0\circ\gamma}(O_{0\circ\gamma})$. Hence we can check that Φ_M restricts to a C^∞-diffeomorphism on these open sets, that is,

$$\Phi_M \circ T\phi_\gamma = \phi_{0\circ\gamma} \circ \Theta_\gamma$$

for each $\gamma \in C^\infty(K,M)$ (as all other mappings in the formula are smooth diffeomorphisms). Now we can rewrite $\Phi_M(T\phi_\gamma(\sigma,\tau))$ as

$$\begin{aligned}([t \mapsto \Sigma(\sigma(x) + t\tau(x))])_{x\in K} &= ([t \mapsto (\Sigma \circ \lambda_{\gamma(x)})(\sigma(x) + t\tau(x))])_{x\in K} \\ &= (T(\Sigma \circ \lambda_{\gamma(x)})(\sigma(x),\tau(x)))_{x\in K} = (\Sigma_{TM}((\kappa \circ T\lambda_{\gamma(x)})(\sigma(x),\tau(x))))_{x\in K} \\ &= ((\Sigma_{TM} \circ \Theta_\gamma)(\sigma,\tau)(x))_{x\in K} = (\phi_{0\circ\gamma} \circ \Theta_\gamma)(\sigma,\tau).\end{aligned}$$

Thus Φ_M is a C^∞-diffeomorphism.

C.13 (Smooth maps into the Whitney sum over strong Riemannian manifolds) By Lemma C.4, TM admits a local addition whose product with itself yields a local addition on the product manifold. Thus $C^\infty(K,M), C^\infty(K,TM)$, $C^\infty(K,TM\times TM)$ are canonical manifolds. Taking the Whitney sum of $(\pi_M)_*$: $C^\infty(K,TM)\to C^\infty(K,M)$ (C.12) with itself, we obtain the bundle $C^\infty(K,TM)\oplus C^\infty(K,TM)$. Our aim is to identify it with the bundle $C^\infty(K,TM \oplus TM)$. Observe that $C^\infty(K,TM) \oplus C^\infty(K,TM)$ is a split submanifold of $C^\infty(K,M) \times C^\infty(K,TM)^2$. Now the factors of the product are canonical manifolds. Thus Lemma 2.16 yields a diffeomorphism

$$C^\infty(K,M) \times C^\infty(K,TM)^2 \cong C^\infty(K,M \times (TM)^2),$$

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which takes the split submanifold $C^\infty(K,TM) \oplus C^\infty(K,TM)$ to $C^\infty(K,TM \oplus TM)$. As diffeomorphisms preserve split submanifolds, see Exercise 4.4.2, $C^\infty(K,TM \oplus TM)$ must be a split submanifold of $C^\infty(K, M \times (TM)^2)$. Finally, Lemma 2.16(c) shows that $C^\infty(K,TM \oplus TM)$ is a canonical manifold diffeomorphic to $TC^\infty(K,M) \oplus TC^\infty(K,M)$.

C.14 Remark By uniqueness of canonical manifolds, $C^\infty(K,TM \oplus TM)$ from C.13 coincides with the manifold structure we could have obtained via a local addition on $TM \oplus TM$. The same proof works if we only assume that $C^\infty(K,M)$ and $C^\infty(K,TM)$ are canonical manifolds (without assuming that M has a local addition).

Exercise

C.3.1 Work out the missing details in the sketch of the proof in C.12.

C.3.2 Prove that a manifold with a local addition admits a normalised local addition.

C.4 Manifolds of Curves and the Energy of a Curve

In this appendix we consider the manifold structure on spaces of curves on a compact interval. The reason for this is that we defined for a (weak) Riemannian manifold (M,g) the energy $\mathrm{En}\colon C^\infty([0,1],M) \to \mathbb{R}$ of a curve and would like to differentiate En to find geodesics. Hence a manifold structure on the space of curves $C^\infty([0,1],M)$ is needed. Many details of the construction will be left to the reader as Exercise C.4.1. Moreover, we will not systematically introduce tangent bundles for manifolds with boundary such as $[0,1]$ (see e.g. Michor, 1980). Thus the compact open C^∞-topology needs to be defined without recourse to tangent bundles.

C.15 Let M be a (possibly infinite-dimensional) manifold. Let $c\colon [0,1] \to M$ be a smooth curve and $K \subseteq [0,1]$ compact such that $c(K) \subseteq U$, where (U,φ) is a chart of M. If $\varphi(U) \Subset E$ for the locally convex space E, we pick a seminorm $\|\cdot\|$ on E and define a C^k-neighbourhood $N^k(c,K,(U,\varphi),\|\cdot\|,\varepsilon)$ as the set

$$\left\{ g \in C^\infty([0,1],M) \,\middle|\, g(K) \subseteq U, \sup_{0 \le \ell \le k} \sup_{x \in K} \left\| \frac{\mathrm{d}^\ell}{\mathrm{d}x^k}(\varphi \circ g - \varphi \circ c)(x) \right\| < \varepsilon \right\}.$$

Then the family $N^k(c,K,(U,\varphi),\|\cdot\|,\varepsilon)$, where $c \in C^\infty([0,1],M)$, $K \subseteq [0,1]$ compact, $\|\cdot\|$ is a continuous seminorm of E and $\varepsilon > 0$ forms the base of a topology on $C^\infty([0,1],M)$ called the *compact-open C^∞-topology*. If $M = E$ is

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a locally convex space, then $C^\infty([0,1],E)$ is also a locally convex space[2] and the exponential law, Theorem 2.12, carries over to $C^\infty([0,1],O)$, $O \Subset E$.

C.16 Proposition *Let (M,Σ) be a manifold with local addition Σ and topologise the space $C^\infty([0,1],M)$ with the compact-open C^∞-topology. Then we have that $C^\infty([0,1],M)$ is a canonical manifold and $TC^\infty([0,1],M) \cong C^\infty([0,1],TM)$.*

C.17 Lemma *Let (M,g) be a (weak) Riemannian manifold such that M and TM admit local additions. Then the* energy

$$\mathrm{En}\colon C^\infty([0,1],M) \to \mathbb{R}, \quad c \mapsto \frac{1}{2}\int_0^1 g_{c(t)}(\dot{c}(t),\dot{c}(t))\mathrm{d}t$$

is smooth. We can express its derivative in a local chart (U,φ) (suppressing most identifications) as

$$d\mathrm{En}(c;h) = \int_0^1 \frac{1}{2} d_1 g_U(c,c'(t),c'(t);h) - d_1 g(c(t),h(t),c'(t);c'(t)) - g_U(c(t),h(t),c''(t))\mathrm{d}t,$$

where we view g as a map of three arguments, $g_U(c,a,b)$ and $h \in T_c C^\infty([0,1],M) \cong \{g \in C^\infty([0,1],TM) \mid \pi \circ g = c\}$ with $h(0) = 0_{c(0)}$ and $h(1) = 0_{c(1)}$.

Proof Since M and TM admit local additions, Exercise C.4.4 implies that both $C^\infty([0,1],M)$ and $C^\infty([0,1],TM)$ are canonical manifolds. Applying the exponential law, $C^\infty([0,1],M) \to C^\infty([0,1],TM)$, $c \mapsto \dot{c} = (t \mapsto T_t c(1))$ is smooth if and only if $C^\infty([0,1],M) \times [0,1] \to TM, (c,t) \mapsto T_t c(1)$ is smooth. We check this locally in a neighbourhood of a c: Pick a chart (U,φ) of M and $[a,b] \subseteq [0,1]$ with $c([a,b]) \subseteq U$. As the topology on $C^\infty([0,1],M)$ is finer than the compact-open topology, there exists a whole neighbourhood of curves g with $g([a,b]) \subseteq U$. Cover $[0,1]$ by compact intervals which c maps into a chart domain and work locally. To keep the notation simple, we will assume that $c([0,1]) \subseteq U$ or in other words, assume without loss of generality that $M \Subset E$ for some locally convex vector space E. Thus we need to prove that $C^\infty([0,1],M) \times [0,1] \to M \times E$, $(c,t) \mapsto (c(t),c'(t)) = (\mathrm{ev}(c,t),\mathrm{ev}(c',t))$ is smooth. The evaluation map is smooth on canonical manifolds and the mapping $C^\infty([0,1],M) \to C^\infty([0,1],E)$, $c \mapsto c'$, is the restriction of the continuous linear map $C^\infty([0,1],E) \to C^\infty([0,1],E)$, $c \mapsto c'$, to the open subset $C^\infty([0,1],M) \Subset C^\infty([0,1],E)$ (here we exploit the compact open C^∞-topology). We conclude that the mapping $C^\infty([0,1],M) \to C^\infty([0,1],TM)$,

[2] For $M = \mathbb{R}$, we have described this structure already in Example 1.6.

$c \mapsto \dot{c}$ is smooth. Since Exercise C.4.6 identifies the Whitney sums, we deduce from $(\dot{c}, \dot{c}) \in TM \oplus TM$ that $C^\infty([0,1],M) \to C^\infty([0,1],\mathbb{R})$, $c \mapsto g_*(\dot{c},\dot{c})$ is smooth (as pushforwards are smooth on canonical manifolds). However, $\frac{1}{2}\int_0^1 \colon C^\infty([0,1],\mathbb{R}) \to \mathbb{R}$ is continuous linear, whence En can be written as a composition of smooth mappings and is thus smooth.

To compute the derivative of the energy, we work in a local chart (U,φ) of M (though we will only label g and suppress the other identifications). Then the metric g becomes a map of three arguments g_U which is bilinear in the last two. Recall that the vector component of $\dot{c}$ is c'. Now by choice of h, there is a smooth curve $q\colon]-\varepsilon,\varepsilon[\to C^\infty([0,1],M)$ such that $\frac{\partial}{\partial s}\big|_{s=0} q(s) = h$ and $q(t)(0) = c(0)$ and $q(t)(1) = c(1)$ for all t (a smooth variation; see also Definition 7.1 for the meaning of the partial derivative). Then we compute, with the help of the exponential law (Exercise C.4.3),

$$\begin{aligned}
d\mathrm{En}(c;h) &= \frac{d}{ds}\bigg|_{s=0} \mathrm{En}(q(s)) \\
&= \frac{1}{2}\int_0^1 \frac{d}{ds}\bigg|_{s=0} g_U\left(q(s)(t), \frac{d}{dt}q(s)(t), \frac{d}{dt}q(s)(t)\right) \mathrm{d}t \\
&= \int_0^1 \frac{1}{2} d_1 g_U(c(t),c'(t),c'(t);h(t)) + g_U\left(c(t), \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}q(s)(t), c'(t)\right) \mathrm{d}t \\
&= \int_0^1 \frac{1}{2} d_1 g_U(c(t),c'(t),c'(t);h(t)) - d_1 g_U\left(c(t), \frac{d}{ds}\bigg|_{s=0} q(s)(t), c'(t); c'(t)\right) \\
&\qquad - g_U\left(c(t), \frac{d}{ds}\bigg|_{s=0} q(s)(t), c''(t)\right) \mathrm{d}t \\
&= \int_0^1 \frac{1}{2} d_1 g_U(c(t),c'(t),c'(t);h(t)) - d_1 g_U(c(t),h(t),c'(t);c'(t)) \\
&\qquad - g_U(c(t),h(t),c''(t))\mathrm{d}t.
\end{aligned}$$

In passing from the second to the third lines we used integration by parts together with the fact that $\frac{d}{ds}\big|_{s=0} q(s,t)$ vanishes at the endpoints of the interval. □

Almost all of the terms in the formula for the derivative of the energy in Lemma C.17 can be globalised to the Riemannian manifold. Derivation exploits of course that we work locally, and the second derivative of c needs to be taken (from the perspective of the Riemannian manifold M) in the fibre over $c(t)$. This already hints at the connection of this formula to the covariant derivative (which, however, was not yet needed).

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Exercises

C.4.1 Prove that the neighbourhoods defined in C.15 form the base of a topology.

C.4.2 Let E be a locally convex space. Show that the compact-open C^∞-topology turns $C^\infty([0,1],E)$ into a locally convex space. Show then that for $M = \mathbb{R}$ this topology coincides with the compact-open C^∞-topology from Example 1.6.

C.4.3 Establish a variant of the exponential law, Theorem 2.12, for manifolds of smooth mappings on $[0,1]$ (with values in open sets of locally convex spaces).

C.4.4 Generalise C.10 to prove Proposition C.16. Then proceed to show that $C^\infty([0,1],M)$ is a canonical manifold.

C.4.5 Follow the argument in C.12 to prove that $TC^\infty([0,1],M)$ can naturally be identified with $C^\infty([0,1],TM)$.

C.4.6 Adapt the argument in C.13 to establish an isomorphism

$$C^\infty([0,1],TM \oplus TM) \cong C^\infty([0,1],TM) \oplus C^\infty([0,1],TM)$$

if M and TM admit local additions.

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