23

$\mathcal{H}(b)$ spaces generated by a nonextreme symbol b

As we have already said, many properties of $\mathcal{H}(b)$ depend on whether b is or is not an extreme point of the closed unit ball of H^∞. Recall that, by the de Leeuw–Rudin theorem (Theorem 6.7), b is a nonextreme point of the closed unit ball of H^∞ if and only if $\log(1-|b|^2) \in L^1(\mathbb{T})$, i.e.

$$\int_{\mathbb{T}} \log(1-|b|^2)\, dm > -\infty. \tag{23.1}$$

In this chapter, we study some specific properties of the space $\mathcal{H}(b)$ when b is a nonextreme point. Roughly speaking, when b is a nonextreme point, the space $\mathcal{H}(b)$ looks like the Hardy space H^2.

In this situation, an important property is the existence of an outer function a such that $a(0) > 0$ and which satisfies $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. This function a is introduced in Section 23.1 and we will see that $\mathcal{H}(\bar{b}) \doteq \mathcal{M}(\bar{a})$. In Section 23.2, we characterize the inclusion $\mathcal{M}(u) \subset \mathcal{H}(b)$ where $u \in H^\infty$. An important object in the nonextreme case is the associated function f^+ introduced in Section 23.3. This function, which is defined via the equation $T_{\bar{b}} f = T_{\bar{a}} f^+$, enables us to give a useful formula for the scalar product in $\mathcal{H}(b)$. We also show, in Section 23.3, that $b \in \mathcal{H}(b)$ and we compute its norm. It turns out that the analytic polynomials belong to and are dense in $\mathcal{H}(b)$. This is the content of Section 23.4. Then, in Section 23.5, we give a formula for $\|X_b f\|_b$, $f \in \mathcal{H}(b)$, and we compute the defect operator D_{X_b}. Recall that, in Section 19.2, we gave a geometric representation of $\mathcal{H}(b)$ space based on the abstract functional embedding. In Section 23.6, we obtain another representation, which corresponds to the Sz.-Nagy–Foiaş model for the contraction X_b. In Section 23.7, we characterize $\mathcal{H}(b)$ spaces when b is a nonextreme point. The analog for the extreme case will be done in Section 25.8. In Section 23.8, we exhibit some new inhabitants of $\mathcal{H}(b)$. In the last section, we finally show that the $\mathcal{H}(b)$ space can be viewed as the domain of the adjoint of an unbounded Toeplitz operator with symbol in the Smirnov class.

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23.1 The pair (a, b)

If b satisfies the condition (23.1), then we define a to be the unique outer function whose modulus on $\mathbb{T}$ is $(1-|b|^2)^{1/2}$ and is positive at the origin. Hence, on the open unit disk, a is given by the formula

$$a(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \; \log(1 - |b(\zeta)|^2)^{1/2} \, dm(\zeta) \right) \qquad (z \in \mathbb{D}). \quad (23.2)$$

Clearly, $a \in H^\infty$ with $\|a\|_\infty \leq 1$ and

$$|a|^2 + |b|^2 = 1 \qquad \text{(a.e. on } \mathbb{T}). \quad (23.3)$$

Whenever we use the pair (a, b), we mean that they are related as described above. We sometimes say that a is the Pythagorean mate associated with b.

Theorem 23.1 *For each pair (a, b), we have*

$$\frac{a}{1-b} \in H^2.$$

Proof By Corollary 4.26, $1/(1-b)$ is an outer function in H^p for each $0 < p < 1$. Since a is an outer function in H^∞, then $a/(1-b)$ is also an outer function in H^p for each $0 < p < 1$. But, by (13.50) and (23.3),

$$\frac{|a|^2}{|1-b|^2} = \frac{1-|b|^2}{|1-b|^2} \in L^1(\mathbb{T}),$$

or equivalently $a/(1-b) \in L^2(\mathbb{T})$. Hence, Corollary 4.28 ensures that $a/(1-b) \in H^2$. □

Theorem 23.2 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$\mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}).$$

Moreover,

$$\mathcal{M}(a) \hookrightarrow \mathcal{M}(\bar{a}) \hookrightarrow \mathcal{H}(b),$$

i.e. both inclusions are contractive. In particular, $\mathcal{M}(a)$ is contractively contained in $\mathcal{H}(b)$.

Proof The relation $\mathcal{M}(a) \hookrightarrow \mathcal{M}(\bar{a})$ follows from Theorem 17.17. Using Theorem 12.4 and (23.3), we see that

$$T_{\bar{a}} T_a = T_{|a|^2} = T_{1-|b|^2} = I - T_{\bar{b}} T_b.$$

Hence, Corollary 16.8 implies that $\mathcal{M}(\bar{a}) = \mathcal{M}(T_{\bar{a}}) = \mathcal{M}((I - T_{\bar{b}} T_b)^{1/2}) = \mathcal{H}(\bar{b})$. The contractive inclusion $\mathcal{H}(\bar{b}) \hookrightarrow \mathcal{H}(b)$ is contained in Theorem 17.9. □

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Theorem 23.2 ensures that $\mathcal{M}(\bar{a})$ embeds contractively in $\mathcal{H}(b)$. The following result provides another contraction between these spaces.

Theorem 23.3 *Let b be a nonextreme point of the closed unit ball of H^∞. Then the operator T_b maps $\mathcal{M}(\bar{a})$ contractively into $\mathcal{H}(b)$.*

Proof According to Lemma 16.20, the operator T_b acts as a contraction from $\mathcal{H}(\bar{b})$ into $\mathcal{H}(b)$. The result follows since, by Theorem 23.2, we have $\mathcal{H}(\bar{b}) \doteq \mathcal{M}(\bar{a})$. □

According to Theorem 23.2, $\mathcal{H}(\bar{b}) \doteq \mathcal{M}(\bar{a})$, and thus, if $f \in \mathcal{H}(\bar{b})$, then there exists a unique $g \in H^2$ such that

$$f = T_{\bar{a}} g. \tag{23.4}$$

The uniqueness of g follows from the fact that $T_{\bar{a}}$ is injective; see Theorem 12.19(ii). In other words, $T_{\bar{a}}$ is an isometry from H^2 onto $\mathcal{M}(\bar{a})$. Therefore, if $f_1 = T_{\bar{a}} g_1$ and $f_2 = T_{\bar{a}} g_2$, with $g_1, g_2 \in H^2$, then

$$\langle f_1, f_2 \rangle_{\bar{b}} = \langle T_{\bar{a}} g_1, T_{\bar{a}} g_2 \rangle_{\mathcal{M}(\bar{a})} = \langle g_1, g_2 \rangle_2. \tag{23.5}$$

We recall that k_w denotes the Cauchy kernel.

Theorem 23.4 *Let (a, b) be a pair. Then*

$$k_w \in \mathcal{H}(\bar{b}) \qquad (w \in \mathbb{D})$$

and, for every function $f \in \mathcal{H}(\bar{b})$, we have

$$\langle f, k_w \rangle_{\bar{b}} = \frac{g(w)}{a(w)},$$

where $g \in H^2$ is related to f via (23.4). *Moreover, we have*

$$\|k_w\|_{\bar{b}} = \frac{1}{|a(w)|\,(1 - |w|^2)^{1/2}}. \tag{23.6}$$

Proof According to (12.7), we have $T_{\bar{a}} k_w = \overline{a(w)} k_w$. Since a is outer, then $a(w) \neq 0$ and we can write the last identity as

$$k_w = T_{\bar{a}}\left(\frac{k_w}{\overline{a(w)}} \right). \tag{23.7}$$

This representation shows that $k_w \in \mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b})$ and the function corresponding to k_w via (23.4) is equal to $k_w/\overline{a(w)}$. Therefore, for each $f \in \mathcal{H}(\bar{b})$, by (23.5), we have

$$\langle f, k_w \rangle_{\bar{b}} = a(w)^{-1} \langle g, k_w \rangle_2 = a(w)^{-1} g(w).$$

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In particular, if we take $f = k_w$, we obtain

$$\|k_w\|_{\bar{b}}^2 = a(w)^{-1} k_w(w)/\overline{a(w)} = |a(w)|^{-2}(1 - |w|^2)^{-1}.$$

Remember, as we established in (4.19), that $k_w(w) = 1/(1 - |w|^2)$. □

Recall that, in Section 17.5, we studied the question of inclusion of different $\mathcal{H}(\bar{b})$ spaces. In the case when b is nonextreme, we can state the condition (17.12) in terms of the associated function a.

Corollary 23.5 *Let (a_1, b_1) and (a_2, b_2) be two pairs. Then the following are equivalent:*

(i) $\mathcal{H}(\bar{b}_2) \subset \mathcal{H}(\bar{b}_1)$;
(ii) $a_2/a_1 \in H^\infty$.

Proof (i) $\Longrightarrow$ (ii) By Theorem 17.12, there is a constant $c > 0$ such that

$$1 - |b_2(\zeta)|^2 \le c(1 - |b_1(\zeta)|^2) \qquad \text{(a.e. on } \mathbb{T}).$$

Hence,

$$|a_2|^2 \le c\,|a_1|^2 \qquad \text{(a.e. on } \mathbb{T}).$$

This means that $a_2/a_1 \in L^\infty(\mathbb{T})$. But, since a_1 is outer, the function a_2/a_1 in fact belongs to H^∞.

(ii) $\Longrightarrow$ (i) Assume that $a_2 = a_1 g$, with some function $g \in H^\infty$. Then we have $T_{\bar{a}_2} = T_{\bar{a}_1} T_{\bar{g}}$, which trivially implies that $\mathcal{M}(\bar{a}_2) \subset \mathcal{M}(\bar{a}_1)$. The conclusion follows now from Theorem 23.2, because we have $\mathcal{H}(\bar{b}_k) = \mathcal{M}(\bar{a}_k)$, $k = 1, 2$. □

Exercises

Exercise 23.1.1 Let (a, b) be a pair. Show that

$$|a(\lambda)|^2 + |b(\lambda)|^2 \le 1 \qquad (\lambda \in \mathbb{D}).$$

Moreover, if b is not constant, the inequality is strict.
Hint: (First method) Note that $|a|^2 + |b|^2$ is harmonic and apply the maximum principle for harmonic functions.

(Second method) By Theorem 12.10, we know that, for any $\varphi \in H^\infty$, we have $T_\varphi T_{\bar{\varphi}} \le T_{\bar{\varphi}} T_\varphi$. Apply this inequality to get $\|T_{\bar{a}} k_\lambda\|_2^2 + \|T_{\bar{b}} k_\lambda\|_2^2 \le \|k_\lambda\|_2^2$.

Exercise 23.1.2 Let b be a nonextreme point of the closed unit ball of H^∞, and let a be the associated outer function. Show that $a/b \in H^\infty$ if and only if $\|b\|_\infty < 1$.

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23.2 Inclusion of $\mathcal{M}(u)$ into $\mathcal{H}(b)$

Theorem 23.2 reveals that $\mathcal{M}(a)$ is a linear manifold in $\mathcal{H}(b)$. Generally speaking, it is important to distinguish a submanifold of $\mathcal{H}(b)$ that is of the form $\mathcal{M}(u)$ for a certain bounded analytic function u. The following result is a characterization of this type.

Theorem 23.6 *Let (a, b) be a pair, and let u be a function in H^∞. Then the following are equivalent:*

(i) $u/a \in H^\infty$;
(ii) $\mathcal{M}(u) \subset \mathcal{M}(a)$;
(iii) $\mathcal{M}(u) \subset \mathcal{H}(b)$.

Proof (i) $\Longleftrightarrow$ (ii) This is already contained in Theorem 17.1.

(ii) $\Longrightarrow$ (iii) This follows from Theorem 23.2.

(iii) $\Longrightarrow$ (i) According to Lemma 16.6, there is a constant $c > 0$ such that

$$\|f\|_b \le c\,\|f\|_{\mathcal{M}(u)}, \tag{23.8}$$

for every function $f \in \mathcal{M}(u)$. Now applying Theorem 16.7 gives

$$T_u T_{\bar{u}} \le c^2 (I - T_b T_{\bar{b}}). \tag{23.9}$$

Applying (23.9) to k_w, $w \in \mathbb{D}$, gives

$$\|T_{\bar{u}} k_w\|_2^2 \le c(\|k_w\|_2^2 - \|T_{\bar{b}} k_w\|_2^2).$$

But, by (12.7), $T_{\bar{u}} k_w = \overline{u(w)} k_w$ and $T_{\bar{b}} k_w = \overline{b(w)} k_w$, and thus we obtain

$$|u(w)|^2 \le c(1 - |b(w)|^2) \qquad (w \in \mathbb{D}).$$

In particular, we deduce from this inequality that

$$|u(\zeta)|^2 \le c(1 - |b(\zeta)|^2) \qquad (\text{a.e. } \zeta \in \mathbb{T}).$$

By definition, we have $|a|^2 = 1 - |b|^2$ almost everywhere on $\mathbb{T}$ and thus we get

$$|u(\zeta)|^2 \le c|a(\zeta)|^2 \qquad (\text{a.e. } \zeta \in \mathbb{T}).$$

Hence, u/a belongs to $L^\infty(\mathbb{T})$. But, since a is outer, Corollary 4.28 ensures that u/a belongs to H^∞. □

Considering the set-theoretic inclusion, Theorem 23.6 also reveals that among spaces $\mathcal{M}(u)$, $u \in H^\infty$, that fulfill $\mathcal{M}(u) \subset \mathcal{H}(b)$, the space $\mathcal{M}(a)$ is the largest one.

Exercise

Exercise 23.2.1 Let (a, b) be a pair, and let u be a function in H^∞. Show that the following are equivalent.

(i) $u/a \in H^\infty$ and $\|u/a\|_\infty \le 1$.
(ii) $\mathcal{M}(u) \hookrightarrow \mathcal{M}(a)$.
(iii) $\mathcal{M}(u) \hookrightarrow \mathcal{H}(b)$.

Hint: See the proof of Theorem 23.6.

23.3 The element f^+

Let $f \in \mathcal{H}(b)$. Thus, using Theorems 17.8 and 23.2, we know that $T_{\bar{b}} f \in \mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$. Theorem 12.19(ii) says that $T_{\bar{a}}$ is injective. Therefore, there is a unique element of H^2, henceforth denoted by f^+, such that

$$T_{\bar{b}} f = T_{\bar{a}} f^+. \tag{23.10}$$

It is also useful to mention that, if a function $f \in H^2$ satisfies $T_{\bar{b}} f = T_{\bar{a}} g$, for some function $g \in H^2$, then it follows from Theorems 17.8 and 23.2 that f surely belongs to $\mathcal{H}(b)$ and $g = f^+$. The element f^+ is a useful tool in studying the properties of $f \in \mathcal{H}(b)$. In this section, we study some elementary properties of f^+.

Looking at the definition in (23.10), it is no wonder that this operation is invariant under a Toeplitz operator with a conjugate-analytic symbol.

Lemma 23.7 *Let b be a nonextreme point of the closed unit ball of H^∞, let $f \in \mathcal{H}(b)$ and let $\varphi \in H^\infty$. Then*

$$(T_{\bar{\varphi}} f)^+ = T_{\bar{\varphi}} f^+.$$

Proof We know from Theorem 18.13 that $\mathcal{H}(b)$ is invariant under $T_{\bar{\varphi}}$. Consequently, we have $T_{\bar{\varphi}} f \in \mathcal{H}(b)$. Then, according to Theorem 12.4,

$$T_{\bar{b}} T_{\bar{\varphi}} f = T_{\bar{\varphi}} T_{\bar{b}} f = T_{\bar{\varphi}} T_{\bar{a}} f^+ = T_{\bar{a}} T_{\bar{\varphi}} f^+.$$

Hence, remembering the uniqueness of $(T_{\bar{\varphi}} f)^+$, the identity $T_{\bar{b}}(T_{\bar{\varphi}} f) = T_{\bar{a}}(T_{\bar{\varphi}} f^+)$ means that $(T_{\bar{\varphi}} f)^+ = T_{\bar{\varphi}} f^+$. □

Theorem 23.8 *Let $f_1, f_2 \in \mathcal{H}(b)$. Then we have*

$$\langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle f_1^+, f_2^+ \rangle_2.$$

In particular, for each $f \in \mathcal{H}(b)$,

$$\|f\|_b^2 = \|f\|_2^2 + \|f^+\|_2^2.$$

Proof Using Theorem 17.8, we can write

$$\begin{aligned}\langle f_1, f_2\rangle_b &= \langle f_1, f_2\rangle_2 + \langle T_{\bar{b}} f_1, T_{\bar{b}} f_2\rangle_{\bar{b}}\\ &= \langle f_1, f_2\rangle_2 + \langle T_{\bar{a}} f_1^+, T_{\bar{a}} f_2^+\rangle_{\bar{b}}.\end{aligned}$$

Since $\mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$, we have

$$\langle T_{\bar{a}} f_1^+, T_{\bar{a}} f_2^+\rangle_{\bar{b}} = \langle T_{\bar{a}} f_1^+, T_{\bar{a}} f_2^+\rangle_{\mathcal{M}(\bar{a})}.$$

Since, according to Theorem 12.19(ii), $T_{\bar{a}}$ is injective, it follows that

$$\langle T_{\bar{a}} f_1^+, T_{\bar{a}} f_2^+\rangle_{\mathcal{M}(\bar{a})} = \langle f_1^+, f_2^+\rangle_2,$$

and this implies

$$\langle f_1, f_2\rangle_b = \langle f_1, f_2\rangle_2 + \langle f_1^+, f_2^+\rangle_2. \qquad \square$$

Theorem 23.8 is very useful in computing the norm of elements of $\mathcal{H}(b)$. Two such computations are discussed below.

Corollary 23.9 *Let b be a nonextreme point of the closed unit ball of H^∞. Then $b \in \mathcal{H}(b)$, with*

$$b^+ = \frac{1}{\overline{a(0)}} - a,$$

and, moreover, we have

$$\begin{aligned}\|b\|_b^2 &= |a(0)|^{-2} - 1\\ \|S^*b\|_b^2 &= 1 - |b(0)|^2 - |a(0)|^2.\end{aligned}$$

Proof According to Theorems 18.1 and 23.2, we have $b \in \mathcal{H}(b)$ if and only if $T_{\bar{b}} b \in \mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$. But

$$T_{\bar{b}} b = P_+|b|^2 = P_+(1 - |a|^2) = 1 - T_{\bar{a}} a,$$

and we can write $1 = P_+(\bar{a}/\overline{a(0)}) = T_{\bar{a}}(1/\overline{a(0)})$. Therefore, we obtain

$$T_{\bar{b}} b = T_{\bar{a}}\left(\frac{1}{\overline{a(0)}} - a\right) \in \mathcal{M}(\bar{a}).$$

This fact ensures that $b \in \mathcal{H}(b)$. Moreover, the last identity also reveals that

$$b^+ = \frac{1}{\overline{a(0)}} - a. \qquad (23.11)$$

A simple calculation shows that

$$\|b^+\|_2^2 = \|a\|_2^2 + \frac{1}{|a(0)|^2} - 2.$$

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Hence, by Theorem 23.8 and the fact that $\|a\|_2^2 + \|b\|_2^2 = 1$, we obtain

$$\begin{aligned}\|b\|_b^2 &= \|b\|_2^2 + \|b^+\|_2^2\\ &= \|b\|_2^2 + \|a\|_2^2 + \frac{1}{|a(0)|^2} - 2\\ &= \frac{1}{|a(0)|^2} - 1.\end{aligned}$$

By Lemma 23.7 and (23.11), we see that

$$(S^*b)^+ = -S^*a. \tag{23.12}$$

According to Theorem 23.8 and (8.16), we thus have

$$\begin{aligned}\|S^*b\|_b^2 &= \|S^*b\|_2^2 + \|S^*a\|_2^2\\ &= \|b\|_2^2 + \|a\|_2^2 - |b(0)|^2 - |a(0)|^2\\ &= 1 - |b(0)|^2 - |a(0)|^2.\end{aligned}$$

This completes the proof. □

By Theorem 23.2, we know that $\mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}) \subset \mathcal{H}(b)$. The following result reveals that, in a sense, $\mathcal{M}(\bar{a})$ is a large subset of $\mathcal{H}(b)$. In the extreme case, this is far from being true. For example, if b is inner, then $\mathcal{H}(\bar{b}) = \{0\}$.

Corollary 23.10 *Let b be a nonextreme point of the closed unit ball of H^∞. Then, relative to the topology of $\mathcal{H}(b)$, the space $\mathcal{H}(\bar{b})$ is a dense submanifold of $\mathcal{H}(b)$.*

Proof By Theorem 23.2, $\mathcal{M}(\bar{a}) \overset{=}{\hookrightarrow} \mathcal{H}(\bar{b}) \hookrightarrow \mathcal{H}(b)$. Let $f \in \mathcal{H}(b)$ and assume that, relative to the inner product of $\mathcal{H}(b)$, f is orthogonal to $\mathcal{M}(\bar{a})$. Thus, in particular, we have

$$\langle f, T_{\bar{a}} S^{*n} f\rangle_b = 0 \tag{23.13}$$

for all $n \geq 0$. Using Theorem 12.4, we can write

$$T_{\bar{a}} S^{*n} f = T_{\bar{a}} T_{\bar{z}^n} f = T_{\bar{a}\bar{z}^n} f.$$

Again, since $z^n a(z) \in H^\infty$, by Lemma 23.7,

$$(T_{\bar{a}} S^{*n} f)^+ = T_{\bar{a}\bar{z}^n} f^+.$$

Therefore, according to Lemma 4.8 and Theorem 23.8, we have

$$\begin{aligned}\langle f, T_{\bar{a}} S^{*n} f\rangle_b &= \langle f, T_{\bar{a}\bar{z}^n} f\rangle_2 + \langle f^+, T_{\bar{a}\bar{z}^n} f^+\rangle_2\\ &= \langle T_{az^n} f, f\rangle_2 + \langle T_{az^n} f^+, f^+\rangle_2\\ &= \langle az^n f, f\rangle_2 + \langle az^n f^+, f^+\rangle_2\\ &= \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})[|f(e^{i\theta})|^2 + |f^+(e^{i\theta})|^2]\, e^{in\theta}\, d\theta\\ &= \hat{\varphi}(-n),\end{aligned}$$

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where φ denotes the L^1 function defined by $\varphi = (|f|^2 + |f^+|^2)a$ (the function φ belongs to $L^1(\mathbb{T})$ since it is the product of the H^∞ function a and the L^1 function $(|f|^2 + |f^+|^2)$). Thus, (23.13) and the previous computation imply that $\hat{\varphi}(n) = 0$ for all $n \leq 0$. This precisely means that $\varphi \in H_0^1$. Since a is an outer function and $|f|^2 + |f^+|^2 \in L^1(\mathbb{T})$, we deduce from Corollary 4.28 that $|f|^2 + |f^+|^2 \in H_0^1$. Since this function is real-valued, (4.12) implies that $|f|^2 + |f^+|^2 \equiv 0$. In particular, $f \equiv 0$. Therefore, $\mathcal{M}(\bar{a})$ is dense in $\mathcal{H}(b)$. □

Recall that, if $0 < r < 1$, then, by definition, a_r is the unique outer function whose modulus on $\mathbb{T}$ is $(1 - r^2|b|^2)^{1/2}$ and $a_r(0) > 0$. In other words, (a_r, rb) is a pair. Note that, on $\mathbb{T}$, we have

$$|a|^2 = 1 - |b|^2 \leq 1 - r^2|b|^2 = |a_r|^2,$$

which implies that $a/a_r \in L^\infty(\mathbb{T})$. Then, according to Corollary 4.28, the function a/a_r belongs to H^∞ and we have

$$\left\| \frac{a}{a_r} \right\|_\infty \leq 1. \tag{23.14}$$

A similar argument shows that a_r^{-1} belongs to H^∞.

Given a function f in $\mathcal{H}(b)$, the next result gives a method to find the associated function f^+. To give the motivation for the following result, note that, if incidentally $bf/a \in L^2(\mathbb{T})$, then

$$f^+ = P_+(\bar{b}f/\bar{a}). \tag{23.15}$$

Indeed, we have

$$T_{\bar{a}} P_+(\bar{b}f/\bar{a}) = P_+(\bar{a}P_+(\bar{b}f/\bar{a})) = P_+(\bar{a}\bar{b}f/\bar{a}) = T_{\bar{b}}f,$$

which, by uniqueness of f^+, gives the formula (23.15). However, if bf/a does not belong to $L^2(\mathbb{T})$, we appeal to a limiting process to get a similar result.

Theorem 23.11 *Let* $f \in \mathcal{H}(b)$. *Then*

$$\lim_{r \to 1} \|T_{\bar{b}/\bar{a}_r} f - f^+\|_2 = 0.$$

Proof Since $a_r^{-1} \in H^\infty$, multiplying both sides of $T_{\bar{b}} f = T_{\bar{a}} f^+$ by $T_{1/\bar{a}_r}$ gives

$$T_{\bar{b}/\bar{a}_r} f = T_{\bar{a}/\bar{a}_r} f^+.$$

Hence, by (23.14), we have

$$\|T_{\bar{b}/\bar{a}_r} f\|_2 = \|T_{\bar{a}/\bar{a}_r} f^+\|_2 \leq \left\| \frac{a}{a_r} \right\|_\infty \|f^+\|_2 \leq \|f^+\|_2 \tag{23.16}$$

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for all $r \in (0,1)$. Let us now prove that a/a_r tends to 1, as $r \longrightarrow 1$, in the weak-star topology of H^∞. According to Theorem 4.16, this is equivalent to saying that

$$\sup_{0\le r<1} \left\| \frac{a}{a_r} \right\|_\infty < +\infty$$

and

$$\lim_{r\to 1} \frac{a(z)}{a_r(z)} = 1 \qquad (z \in \mathbb{D}).$$

The first fact follows immediately from (23.14). To verify the second fact, recall that

$$a_r(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |a_r(\zeta)| \, dm(\zeta) \right),$$

and then an application of the dominated convergence theorem gives the result. Consequently, for every $\phi \in L^1(\mathbb{T})$, we have

$$\lim_{r\to 1} \int_{\mathbb{T}} \frac{\bar{a}}{\bar{a}_r} \phi \, dm = \int_{\mathbb{T}} \phi \, dm.$$

Now, let $u, v \in H^2$. Since $u\bar{v} \in L^1(\mathbb{T})$, the last identity gives

$$\begin{aligned} \lim_{r\to 1} \langle T_{\bar{a}/\bar{a}_r} u, v \rangle_2 &= \lim_{r\to 1} \langle \bar{a} u / \bar{a}_r, v \rangle_2 \\ &= \lim_{r\to 1} \int_{\mathbb{T}} \frac{\bar{a}}{\bar{a}_r} u\bar{v} \, dm = \int_{\mathbb{T}} u\bar{v} \, dm = \langle u, v \rangle_2. \end{aligned}$$

This means that $T_{\bar{a}/\bar{a}_r} u$ is weakly convergent to u in H^2. Therefore, $T_{\bar{b}/\bar{a}_r} f = T_{\bar{a}/\bar{a}_r} f^+$ weakly converges to f^+ in H^2, as $r \longrightarrow 1$. But, according to (23.16), we have

$$\begin{aligned} \|T_{\bar{b}/\bar{a}_r} f - f^+\|_2^2 &= \|T_{\bar{b}/\bar{a}_r} f\|_2^2 + \|f\|_2^2 - 2\,\Re \langle T_{\bar{b}/\bar{a}_r} f, f^+ \rangle_2 \\ &\le 2\|f^+\|_2^2 - 2\,\Re \langle T_{\bar{b}/\bar{a}_r} f, f^+ \rangle_2. \end{aligned}$$

Hence, we get

$$\limsup_{r\to 1} \|T_{\bar{b}/\bar{a}_r} f - f^+\|_2^2 \le 2\|f^+\|_2^2 - 2 \lim_{r\to 1} \Re \langle T_{\bar{b}/\bar{a}_r} f, f^+ \rangle_2 = 0,$$

from which we deduce that $T_{\bar{b}/\bar{a}_r} f$ actually converges to f^+ in H^2 norm, as $r \longrightarrow 1$. □

Using this fact and Theorem 23.8, we can give another proof of formula (18.20) in the nonextreme case.

Theorem 23.12 *The map* $\mathfrak{G} : h \longmapsto h^+$ *is a partial isometry of* $\mathcal{H}(b)$ *onto* $\mathcal{H}(a)$, *and its kernel is* $\ker T_{\bar{b}} \cap \mathcal{H}(b)$.

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Proof Let $h \in \mathcal{H}(b)$. Note that $h^+ \in H^2$ and then $h^+ \in \mathcal{H}(a)$ if and only if $T_{\bar{a}} h^+ \in \mathcal{H}(\bar{a})$. By applying Theorem 23.2 to a (which is of course also a nonextreme point of the closed unit ball of H^∞), then $\mathcal{M}(\bar{b}) = \mathcal{H}(\bar{a})$ and we deduce that

$$T_{\bar{a}} h^+ = T_{\bar{b}} h \in \mathcal{H}(\bar{a}).$$

Hence $h^+ \in \mathcal{H}(a)$. Now, let $\varphi \in \mathcal{H}(a)$. Then $T_{\bar{a}}\varphi \in \mathcal{H}(\bar{a})$. Using Theorem 23.2 once more, there exists $h \in H^2$ such that $T_{\bar{a}}\varphi = T_{\bar{b}} h$. Since $T_{\bar{b}} h \in \mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b})$, we deduce that $h \in \mathcal{H}(b)$ and the last equation gives that $h^+ = \varphi$. That means that $\mathfrak{G}$ is a surjective map from $\mathcal{H}(b)$ onto $\mathcal{H}(a)$.

Let $h \in \mathcal{H}(b)$. Since $T_{\bar{a}}$ is one-to-one, we have

$$\begin{aligned} \mathfrak{G}(h) = 0 &\Longleftrightarrow h^+ = 0 \\ &\Longleftrightarrow T_{\bar{a}} h^+ = 0 \\ &\Longleftrightarrow T_{\bar{b}} h = 0 \\ &\Longleftrightarrow h \in \ker T_{\bar{b}}. \end{aligned}$$

Hence $\ker \mathfrak{G} = \ker T_{\bar{b}} \cap \mathcal{H}(b)$.

It remains to check that $\mathfrak{G}$ is a partial isometry. So let $h \in \mathcal{H}(b)$, $h \perp \ker T_{\bar{b}}$. On the one hand, we have

$$\|h\|_b^2 = \|h\|_2^2 + \|h^+\|_2^2,$$

and on the other,

$$\|h^+\|_a^2 = \|h^+\|_2^2 + \|T_{\bar{a}} h^+\|_{\bar{a}}^2 = \|h^+\|_2^2 + \|T_{\bar{b}} h\|_{\mathcal{M}(\bar{b})}^2.$$

Since $h \in \ker T_{\bar{b}}$, we have $\|T_{\bar{b}} h\|_{\mathcal{M}(\bar{b})}^2 = \|h\|_2^2$, which gives

$$\|h^+\|_a^2 = \|h^+\|_2^2 + \|h\|_2^2 = \|h\|_b^2.$$

In other words, $\mathfrak{G}$ is a partial isometry. □

Exercises

Exercise 23.3.1 Assume that b is not an extreme point of the closed unit ball of H^∞.

(i) Prove that

$$rT_{r\bar{b}/\bar{a}_r} b = \overline{a_r^{-1}(0)} - a_r.$$

(ii) Deduce that

$$\|b\|_b^2 = |a(0)|^{-2} - 1.$$

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(iii) Prove that, for $n \geq 1$, we have

$$rT_{r\bar{b}/\bar{a}_r} X^n b = -S^{*n} a_r.$$

(iv) Show that $T_{\bar{b}/\bar{a}_r} 1 = \overline{b(0)}/\overline{a_r(0)}$.

(v) Deduce that

$$\langle X^n b, 1 \rangle_b = \hat{b}(n) - b(0)\hat{a}(n)a(0)^{-1} \qquad (n \geq 1).$$

Hint: Use (iii) and (iv).

Exercise 23.3.2 Assume that b is not an extreme point of the closed unit ball of H^∞ and assume that b has a zero of order m at the origin. Show that

$$\langle X^n b, z^m \rangle_b = \hat{b}(n+m) - \hat{b}(m)\hat{a}(n)a(0)^{-1} \qquad (n \geq 1).$$

Hint: Use Exercise 23.3.1(iii) and Exercise 18.9.3(ii).

Exercise 23.3.3 Assume that b has a zero of order m (possibly 0) at the origin and assume that b is not an extreme point of the closed unit ball of H^∞. Show that

$$\langle X^n b, b \rangle_b = -\hat{a}(n)/a(0) \qquad (n \geq 1).$$

Hint: Use Exercise 18.9.1 with $f = X^n b$ and Exercise 23.3.2.

23.4 Analytic polynomials are dense in $\mathcal{H}(b)$

Theorem 17.4 tells us that the analytic polynomials are dense in $\mathcal{M}(\bar{a})$. Then Theorem 23.2 says that the latter linear manifold is dense and contractively contained in $\mathcal{H}(b)$. Hence, it is natural to deduce some result about the family of analytic polynomials in $\mathcal{H}(b)$.

Theorem 23.13 *Let b be a nonextreme point of the closed unit ball of H^∞, and let $\mathcal{P}$ denote the linear manifold of analytic polynomials. Then the following hold.*

(i) $\mathcal{P} \subset \mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$.
(ii) *$\mathcal{P}$ is a dense manifold in $\mathcal{M}(\bar{a})$.*
(iii) *$\mathcal{P}$ is a dense manifold in $\mathcal{H}(b)$.*

Proof (i) The inclusion $\mathcal{P} \subset \mathcal{M}(\bar{a})$ was shown in Theorem 17.4, and $\mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$ was established in Theorem 23.2.

(ii) This is also from Theorem 17.4.

(iii) Let $f \in \mathcal{H}(b)$ and let $\varepsilon > 0$. According to Corollary 23.10, there exists $g \in \mathcal{M}(\bar{a})$ such that

$$\|f - g\|_b \leq \frac{\varepsilon}{2},$$

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and, appealing to part (ii), there is a $p \in \mathcal{P}$ such that

$$\|g - p\|_{\mathcal{M}(\bar{a})} \le \frac{\varepsilon}{2}.$$

But, by Theorem 23.2,

$$\|g - p\|_b \le \|g - p\|_{\mathcal{M}(\bar{a})}.$$

The three inequalities above imply that $\|f - p\|_b \le \varepsilon$. □

Let u_o be the inner part and b_o be the outer part of a function b in the closed unit ball of H^∞. Since $|b_o| = |b|$ a.e. on $\mathbb{T}$, if b is nonextreme, then b_o is also nonextreme. In particular, we will have, according to Theorems 23.13 and 18.7,

$$\mathcal{P} \subset \mathcal{H}(b_o) \subset \mathcal{H}(b).$$

Since $\mathcal{P}$ is dense in $\mathcal{H}(b)$, we immediately get that $\mathcal{H}(b_o)$ is also dense in $\mathcal{H}(b)$. The situation in the extreme case is dramatically different because we will see in Section 25.6 that $\mathcal{H}(b_o)$ is a closed subspace of $\mathcal{H}(b)$ and, if u_o is not a finite Blaschke product, the orthogonal complement of $\mathcal{H}(b_o)$ in $\mathcal{H}(b)$ is of infinite dimension.

23.5 A formula for $\|X_b f\|_b$

We recall that $\mathcal{H}(b)$ is invariant under the backward shift S^* and that the restriction of S^* to $\mathcal{H}(b)$ was denoted by X_b. In this section, we give a formula for $\|X_b f\|_b$.

Theorem 23.14 *Assume that b is a nonextreme point of the closed unit ball of H^∞. Then we have*

$$X_b^* X_b = I - k_0^b \otimes k_0^b - |a(0)|^2 b \otimes b.$$

Moreover, for every $f \in \mathcal{H}(b)$, we have

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2 - |a(0)|^2 |\langle f, b\rangle_b|^2. \tag{23.17}$$

Proof According to Corollary 18.23, we have

$$\begin{aligned} X_b^* X_b f &= SS^* f - \langle X_b f, S^* b\rangle_b b \\ &= f - f(0) - \langle X_b f, X_b b\rangle_b b \\ &= f - f(0) - \langle f, X_b^* X_b b\rangle_b b \end{aligned} \tag{23.18}$$

for every $f \in \mathcal{H}(b)$. By Corollary 23.9, $b \in \mathcal{H}(b)$, and thus by setting $f = b$ in (23.18), we obtain

$$X_b^* X_b b = b - b(0) - \langle b, X_b^* X_b b\rangle_b b = b - b(0) - \|X_b b\|_b^2\, b.$$

Using Corollary 23.9 again and the formula for $X_b b = S^* b$, we simplify the preceding identity to get

$$X_b^* X_b b = (|b(0)|^2 + |a(0)|^2) b - b(0).$$

Plugging the preceding expression for $X_b^* X_b b$ and the formula $f(0) = \langle f, k_0^b \rangle_b$ into (23.18) gives

$$\begin{aligned} X_b^* X_b f &= f - \langle f, k_0^b \rangle_b - (|b(0)|^2 + |a(0)|^2) \langle f, b \rangle_b b + \overline{b(0)} \langle f, 1 \rangle_b \\ &= f - \langle f, k_0^b \rangle_b - |a(0)|^2 \langle f, b \rangle_b b + \overline{b(0)} (\langle f, 1 \rangle_b - b(0) \langle f, b \rangle_b) b \\ &= f - \langle f, k_0^b \rangle_b - |a(0)|^2 \langle f, b \rangle_b b + \overline{b(0)} \langle f, 1 - \overline{b(0)} b \rangle_b b \\ &= f - |a(0)|^2 \langle f, b \rangle_b b - \langle f, k_0^b \rangle_b k_0^b \\ &= (I - k_0^b \otimes k_0^b - |a(0)|^2 b \otimes b) f. \end{aligned}$$

Using this formula for $X_b^* X_b$, we can write

$$\begin{aligned} \|X_b f\|_b^2 &= \langle X_b f, X_b f \rangle_b \\ &= \langle X_b^* X_b f, f \rangle_b \\ &= \big\langle f - \langle f, k_0^b \rangle_b k_0^b - |a(0)|^2 \langle f, b \rangle_b b,\ f \big\rangle_b \\ &= \|f\|_b^2 - |\langle f, k_0^b \rangle_b|^2 - |a(0)|^2 |\langle f, b \rangle_b|^2 \\ &= \|f\|_b^2 - |f(0)|^2 - |a(0)|^2 |\langle f, b \rangle_b|^2. \end{aligned}$$

This completes the proof. □

We recall that, in Corollary 18.27, we proved that the defect operator $D_{X_b^*} = (I - X_b X_b^*)^{1/2}$ has rank one, its range is spanned by $S^* b$ and its nonzero eigenvalue equals $\|S^* b\|_b$. The analogous result for D_{X_b} depends on whether b is an extreme or nonextreme point of the closed unit ball of H^∞.

Corollary 23.15 *Let b be a nonextreme point of the closed unit ball of H^∞. The operator $D_{X_b}^2 = I - X_b^* X_b$ has rank two. It has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1 - |b(0)|^2 - |a(0)|^2$. Moreover, if $e_1 = 1$ and $e_2 = -b(0) k_0^b + |a(0)|^2 b$, then*

$$\ker(D_{X_b}^2 - \lambda_1 I) = \mathbb{C} e_1 \quad \textit{and} \quad \ker(D_{X_b}^2 - \lambda_2 I) = \mathbb{C} e_2.$$

Proof Using Theorem 23.17, we have

$$D_{X_b}^2 = k_0^b \otimes k_0^b + |a(0)|^2 b \otimes b.$$

Since b and k_0^b are linearly independent, $D_{X_b}^2$ has rank two, and it is sufficient to study its restriction to the two-dimensional space $\mathbb{C} k_0^b \oplus \mathbb{C} b$. Relative to the basis (k_0^b, b), this restriction has the following matrix:

$$A = \begin{pmatrix} \|k_0^b\|_b^2 & \langle b, k_0^b \rangle_b \\ |a(0)|^2 \langle k_0^b, b \rangle_b & |a(0)|^2 \|b\|_b^2 \end{pmatrix}.$$

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According to (18.8), Theorem 18.11 and Corollary 23.9, we have

$$\|k_0^b\|_b^2 = 1 - |b(0)|^2, \quad \langle b, k_0^b\rangle_b = b(0) \quad \text{and} \quad |a(0)|^2\|b\|_b^2 = 1 - |a(0)|^2.$$

Hence,

$$A = \begin{pmatrix} 1 - |b(0)|^2 & b(0) \\ \overline{b(0)}|a(0)|^2 & 1 - |a(0)|^2 \end{pmatrix}.$$

It is now easy to compute the eigenvalue and eigenvectors of this matrix. The characteristic polynomial is given by

$$\det(A - \lambda I) = \lambda^2 - \lambda(2 - |a(0)|^2 - |b(0)|^2) + 1 - |a(0)|^2 - |b(0)|^2.$$

As already noted, we have $1 - |a(0)|^2 - |b(0)|^2 > 0$. Hence, there are two real roots, which are 1 and $1 - |a(0)|^2 - |b(0)|^2$. Therefore, $\lambda_1 = 1$ and $\lambda_2 = 1 - |a(0)|^2 - |b(0)|^2$ are the two eigenvalues. To compute the eigenvectors, we need to solve linear systems. Let $u = \alpha k_0^b + \beta b$, $\alpha, \beta \in \mathbb{C}$. Then $u \in \ker(D_{X_b}^2 - \lambda_1 I)$ if and only if

$$\begin{pmatrix} 1 - |b(0)|^2 & b(0) \\ \overline{b(0)}|a(0)|^2 & 1 - |a(0)|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

This equivalent to

$$\begin{cases} \alpha|b(0)|^2 = \beta b(0), \\ \alpha\overline{b(0)}|a(0)|^2 = \beta|a(0)|^2. \end{cases}$$

Since $a(0) \neq 0$, this equivalent to $\beta = \alpha\overline{b(0)}$ and we get that $u \in \ker(D_{X_b}^2 - \lambda_1 I)$ if and only if $u = \alpha k_0^b + \alpha\overline{b(0)}b = \alpha$. This proves that

$$\ker(D_{X_b}^2 - \lambda_1) = \mathbb{C}1.$$

Similarly, $u \in \ker(D_{X_b}^2 - \lambda_2 I)$ if and only if

$$\begin{pmatrix} 1 - |b(0)|^2 & b(0) \\ \overline{b(0)}|a(0)|^2 & 1 - |a(0)|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda_2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

which is equivalent to

$$\begin{cases} \beta b(0) = \alpha(\lambda_2 - 1 + |b(0)|^2), \\ \alpha\overline{b(0)}|a(0)|^2 = \beta(\lambda_2 - 1 + |a(0)|^2). \end{cases}$$

Using the fact that $\lambda_2 = 1 - |a(0)|^2 - |b(0)|^2$, we see that the system is equivalent to $\alpha = -\beta b(0)/|a(0)|^2$. Hence, $u \in \ker(D_{X_b}^2 - \lambda_1 I)$ if and only if

$$u = -\beta\frac{b(0)}{|a(0)|^2}k_0^b + \beta b = \frac{\beta}{|a(0)|^2}(-b(0)k_0^b + |a(0)|^2 b),$$

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which gives

$$\ker(D^2_{X_b} - \lambda_2 I) = \mathbb{C}(-b(0)k_0^b + |a(0)|^2 b). \qquad \square$$

We are now ready to explicitly determine the defect operator D_{X_b}.

Corollary 23.16 *Let b be a nonextreme point of the closed unit ball of H^∞. Then the following hold.*

(i) *The operator D_{X_b} has rank two and it has two eigenvalues $\mu_1 = 1$ and $\mu_2 = (1 - |b(0)|^2 - |a(0)|^2)^{1/2}$.*

(ii) *If $e_1 = 1$ and $e_2 = -b(0)k_0^b + |a(0)|^2 b$, then we have*

$$\ker(D_{X_b} - \mu_1 I) = \mathbb{C}e_1 \quad \textit{and} \quad \ker(D_{X_b} - \mu_2 I) = \mathbb{C}e_2.$$

(iii) *We have*

$$D_{X_b} = \frac{1}{|a(0)|^2 + |b(0)|^2}\left(|a(0)|^2 e_1 \otimes e_1 + \frac{1}{\mu_2}\, e_2 \otimes e_2\right).$$

Proof Parts (i) and (ii) follow immediately from Corollary 23.15 and the fact that $\mu_\ell = \sqrt{\lambda_\ell}$, $\ell = 1, 2$.

To prove (iii), note that $\langle e_1, e_2\rangle_b = 0$ since they correspond to eigenvectors associated with different eigenvalues of a self-adjoint operator. With respect to the orthogonal basis (e_1, e_2), the operator D_{X_b} can then be written as

$$D_{X_b} = \frac{1}{\|e_1\|_b^2}\, e_1 \otimes e_1 + \frac{\mu_2}{\|e_2\|_b^2}\, e_2 \otimes e_2.$$

It remains to compute $\|e_1\|_b$ and $\|e_2\|_b$. First, note that $e_1^+ = \overline{b(0)}/\overline{a(0)}$, which gives, using Theorem 23.8,

$$\|e_1\|_b^2 = \|e_1\|_2^2 + \|e_1^+\|_2^2 = 1 + \frac{|b(0)|^2}{|a(0)|^2} = \frac{|a(0)|^2 + |b(0)|^2}{|a(0)|^2}.$$

On the other hand, using Corollary 23.9, we have

$$\begin{aligned}
\|e_2\|_b^2 &= |b(0)|^2\|k_0^b\|_b^2 + |a(0)|^4\|b\|_b^2 - 2|a(0)|^2\, \Re(b(0)\langle k_0^b, b\rangle_b)\\
&= |b(0)|^2(1 - |b(0)|^2) + |a(0)|^4\left(\frac{1}{|a(0)|^2} - 1\right) - 2|a(0)|^2|b(0)|^2\\
&= (1 - |b(0)|^2 - |a(0)|^2)(|b(0)|^2 + |a(0)|^2)\\
&= \mu_2^2(|b(0)|^2 + |a(0)|^2).
\end{aligned}$$

Finally, we get

$$D_{X_b} = \frac{|a(0)|^2}{|a(0)|^2 + |b(0)|^2}\, e_1 \otimes e_1 + \frac{1}{\mu_2(|b(0)|^2 + |a(0)|^2)}\, e_2 \otimes e_2. \qquad \square$$

23.6 Another representation of $\mathcal{H}(b)$

In Section 19.2, we saw a representation of the $\mathcal{H}(b)$ space based on an abstract functional embedding. In the nonextreme case, we can also give a slightly different representation. Let b be a nonextreme point of the closed unit ball of H^∞ and let a be the outer function defined by (23.2). Denote $\mathbb{H}_b = L^2 \oplus L^2$ along with

$$\begin{array}{rccc} \pi: & L^2 & \longrightarrow & \mathbb{H}_b \\ & f & \longmapsto & bf \oplus (-af), \end{array}$$

and

$$\begin{array}{rccc} \pi_*: & L^2 & \longrightarrow & \mathbb{H}_b \\ & g & \longmapsto & g \oplus 0. \end{array}$$

Theorem 23.17 *The linear mapping $\Pi = (\pi, \pi_*) : L^2 \oplus L^2 \longrightarrow \mathbb{H}_b$ is an abstract functional embedding (AFE).*

Proof For any $f \in L^2$, we have

$$\begin{aligned} \|bf \oplus (-af)\|_{\mathbb{H}_b}^2 &= \|bf\|_2^2 + \|af\|_2^2 \\ &= \int_{\mathbb{T}} (|b|^2 + |a|^2)|f|^2 \, dm \\ &= \|f\|_2^2, \end{aligned}$$

the last equality following from the fact that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Thus π is an isometry. The map π_* is also clearly an isometry and one can easily check that

$$\pi_*^*(h_1 \oplus h_2) = h_1, \qquad h_1 \oplus h_2 \in L^2 \oplus L^2. \tag{23.19}$$

Now let $f \in H^2$ and $g \in H^2_-$. We have

$$\langle \pi f, \pi_* g\rangle_{\mathbb{H}_b} = \langle bf \oplus (-af), \ g \oplus 0\rangle_{\mathbb{H}_b} = \langle bf, g\rangle_2 = 0,$$

because $bf \in H^2$ and $g \in H^2_-$. That proves that $\pi H^2 \perp \pi_* H^2_-$. By (23.19), we also clearly have

$$\pi_*^* \pi f = \pi_*^*(bf \oplus (-af)) = bf.$$

Thus $\pi_*^*\pi$ is the multiplication operator by b and, in particular, it commutes with the shift operator and maps H^2 into H^2.

Finally, note that $\operatorname{Clos}(aL^2)$ is a reducing invariant subspace for the multiplication operator by z on L^2. Hence, it follows from Theorem 8.29 that there exists a measurable set $E \subset \mathbb{T}$ such that $\operatorname{Clos}(aL^2) = \chi_E L^2$. Since $a \in \chi_E L^2$, a should vanish a.e. on $\mathbb{T} \setminus E$ and then necessarily $m(\mathbb{T} \setminus E) = 0$. That implies that $\operatorname{Clos}(aL^2) = L^2$ and then the range of Π is dense in $\mathbb{H}_b$. □

Let $\mathbb{K}_b$ be the subspace defined by (19.4), and let $\mathbb{K}'_b$ and $\mathbb{K}''_b$ the subspaces defined by (19.7) and (19.6). It will be useful to have the following more explicit transcriptions.

Lemma 23.18 *Let* b *be a nonextreme point of the closed unit ball of* H^∞*. We have:*

(i) $\mathbb{K}_b = (H^2 \oplus L^2) \ominus \{bf \oplus (-af) : f \in L^2\}$;
(ii) $\mathbb{K}''_b = 0 \oplus H^2_-$;
(iii) $\mathbb{K}'_b = (H^2 \oplus H^2) \ominus \{bf \oplus (-af) : f \in H^2\}$.

Proof (i) Recall that

$$\mathbb{K}_b = \mathbb{H}_b \ominus (\pi(H^2) \oplus \pi_*(H^2_-)).$$

First note that

$$\{bf \oplus (-af) : f \in H^2\} = \pi(H^2),$$

and since π is an isometry, this space is a closed subspace of $H^2 \oplus L^2$. Now let $\varphi \oplus \psi \in L^2 \oplus L^2$. Then $\varphi \oplus \psi \in \mathbb{K}_b$ if and only if

$$\varphi \oplus \psi \perp \{bf \oplus (-af) : f \in H^2\}$$

and

$$\varphi \oplus \psi \perp \pi_*(H^2_-).$$

The second condition gives that, for any $h \in H^2_-$, we have

$$0 = \langle \varphi \oplus \psi,\ \pi_*(h)\rangle_{\mathbb{H}_b} = \langle \varphi \oplus \psi,\ h \oplus 0\rangle_{\mathbb{H}_b} = \langle \varphi, h\rangle_2.$$

This condition is thus equivalent to $\varphi \in H^2$. Thus, we get that

$$\mathbb{K}_b = \{\varphi \oplus \psi : \varphi \in H^2,\ \psi \in L^2 \text{ and } \varphi \oplus \psi \perp bf \oplus (-af),\ f \in H^2\}.$$

(ii) According to Lemma 19.5, we have

$$\mathbb{K}''_b = \mathbb{K}_b \cap (\pi_*(H^2))^\perp.$$

Then it is clear that $0 \oplus H^2_- \subset \mathbb{K}''_b$. Conversely, if $\varphi \oplus \psi \in \mathbb{K}''_b$, using (i), we first have $\varphi \in H^2$ and

$$\varphi \oplus \psi \perp bf \oplus (-af) \qquad (\forall\, f \in H^2). \tag{23.20}$$

On the other hand, since $\varphi \oplus \psi \perp \pi_*(H^2)$, that gives $\varphi \oplus \psi \perp f \oplus 0$, for any $f \in H^2$. Hence, $\langle \varphi, f\rangle_2 = 0$, $f \in H^2$, which implies that $\varphi \perp H^2$. But, since φ also belongs to H^2, we get that $\varphi = 0$. Now, if we use (23.20), we obtain

$$\langle \psi, af\rangle_2 = 0 \qquad (f \in H^2).$$

Since a is outer, aH^2 is dense in H^2. Hence, $\psi \perp H^2$. We thus obtain that $\varphi \oplus \psi \in 0 \oplus H^2_-$.

(iii) Recall that $\mathbb{K}_b' = \mathbb{K}_b \ominus \mathbb{K}_b''$. Hence, $\varphi \oplus \psi \in \mathbb{K}_b'$ if and only if $\varphi \in H^2$, $\varphi \oplus \psi \perp bf \oplus (-af)$, $f \in H^2$ and $\varphi \oplus \psi \perp 0 \oplus g$, $g \in H^2_-$. The last condition is equivalent to $\psi \perp H^2_-$, which means that $\psi \in H^2$ and that gives the desired description of $\mathbb{K}_b'$. □

According to Theorem 19.8, we know that the map

$$Q_b = \pi^*_{*|\mathbb{K}_b'} : \mathbb{K}_b' \longrightarrow \mathcal{H}(b)$$

is a unitary map. It could be useful to compute its adjoint. We have the following lemma.

Lemma 23.19 *Let b be a nonextreme point of the closed unit ball of H^∞. For any $h \in \mathcal{H}(b)$, we have*

$$Q_b^* h = h \oplus h^+,$$

where we recall that h^+ is the unique function in H^2 such that $T_{\bar{b}} h = T_{\bar{a}} h^+$.

Proof Let $\varphi \oplus \psi \in \mathbb{K}_b'$ and let $h \in \mathcal{H}(b)$. According to Lemma 23.18, $\varphi, \psi \in H^2$ and

$$\langle \varphi, bf \rangle_2 = \langle \psi, af \rangle_2 \qquad (f \in H^2). \tag{23.21}$$

Using Theorem 23.8, we have

$$\begin{aligned}\langle \varphi \oplus \psi,\ Q_b^* h \rangle_{\mathbb{K}_b'} &= \langle Q_b(\varphi \oplus \psi),\ h \rangle_b \\ &= \langle \varphi, h \rangle_b = \langle \varphi, h \rangle_2 + \langle \varphi^+, h^+ \rangle_2.\end{aligned}$$

Let us check that $\varphi^+ = \psi$. Using (23.21), for any $f \in H^2$, we have

$$\langle \bar{b}\varphi, f \rangle_2 = \langle \bar{a}\psi, f \rangle_2,$$

which means that $\bar{b}\varphi - \bar{a}\psi \perp H^2$. In other words, $P_+(\bar{b}\varphi) = P_+(\bar{a}\psi)$. By the uniqueness of φ^+, we get that $\varphi^+ = \psi$. Thus,

$$\langle \varphi \oplus \psi,\ Q_b^* h \rangle_{\mathbb{K}_b'} = \langle \varphi, h \rangle_2 + \langle \psi, h^+ \rangle_2 = \langle \varphi \oplus \psi,\ h \oplus h^+ \rangle_{\mathbb{H}_b}.$$

It remains to note that $h \oplus h^+ \in \mathbb{K}_b'$. We have $h \oplus h^+ \in H^2 \oplus H^2$. Moreover, for any $f \in H^2$, we have

$$\begin{aligned}\langle h \oplus h^+,\ bf \oplus (-af) \rangle_{\mathbb{H}_b} &= \langle h, bf \rangle_2 - \langle h^+, af \rangle_2 \\ &= \langle P_+(\bar{b}h), f \rangle_2 - \langle P_+(\bar{a}h^+), f \rangle_2,\end{aligned}$$

and since $P_+(\bar{b}h) = P_+(\bar{a}h^+)$, we get that $h \oplus h^+ \perp bf \oplus (-af)$ for any $f \in H^2$. According to Lemma 23.18, we can conclude that $h \oplus h^+ \in \mathbb{K}_b'$ and $Q_b^* h = h \oplus h^+$. □

Let

$$\begin{aligned} W:\quad H^2 \oplus H^2 &\longmapsto H^2 \oplus H^2 \\ f \oplus g &\longmapsto zf \oplus zg. \end{aligned}$$

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Then W defines a bounded and linear operator on $H^2 \oplus H^2$ and it is clear that W leaves the (closed) subspace $\{bf \oplus (-af) : f \in H^2\}$ invariant. Hence, W^* leaves $\mathbb{K}_b'$ invariant. Furthermore, it is easy to check that

$$\begin{array}{rccc} W^* : & H^2 \oplus H^2 & \longmapsto & H^2 \oplus H^2 \\ & f \oplus g & \longmapsto & P_+(\bar{z}f) \oplus P_+(\bar{z}g). \end{array}$$

In other words, $W^* = S^* \oplus S^*$.

Theorem 23.20 *Let b be a nonextreme point of the closed unit ball of H^∞. Then the following diagram is commutative.*

$$\begin{array}{ccc} \mathbb{K}_b' & \xrightarrow{\;Q_b\;} & \mathcal{H}(b) \\ {\scriptstyle S^*\oplus S^*}\big\downarrow & & \big\downarrow{\scriptstyle X_b} \\ \mathbb{K}_b' & \xrightarrow[\;Q_b\;]{} & \mathcal{H}(b) \end{array} \qquad (23.22)$$

In particular, X_b is unitarily equivalent to $(S^* \oplus S^*)_{|\mathbb{K}_b'}$.

Proof Let $f \oplus g \in \mathbb{K}_b'$. Then

$$\begin{aligned} Q_b W^*(f \oplus g) &= Q_b(S^*f \oplus S^*g) \\ &= S^*f \\ &= X_b f \\ &= X_b Q_b(f \oplus g). \end{aligned}$$

This completes the proof. □

In Theorem 19.11, we have given a different representation of $\mathcal{H}(b)$ and a different model for X_b. It is interesting to explore the link between these two representations. This will be done in Exercise 23.6.2.

Exercises

Exercise 23.6.1 Let b be a nonextreme point of the closed unit ball of H^∞ and define

$$\begin{array}{rccc} T_B : & H^2 & \longrightarrow & H^2 \oplus H^2 \\ & f & \longmapsto & bf \oplus (-af). \end{array}$$

Show that T_B is an isometry and check that $\mathcal{H}(T_B) = \mathbb{K}_b'$.

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Exercise 23.6.2 Let b be a nonextreme point of the closed unit ball of H^∞, let $\Delta = (1 - |b|^2)^{1/2}$ on $\mathbb{T}$, let $\mathbb{K}_b'$ be defined as in Lemma 23.18, and let

$$\mathcal{K}_b' := H^2 \oplus \mathrm{Clos}(\Delta H^2) \ominus \{bf \oplus \Delta f : f \in H^2\}.$$

For $f, g \in H^2$, define

$$\Omega(f \oplus (-ag)) = f \oplus \Delta g.$$

(i) Show that Ω can be extended into a unitary operator from $H^2 \oplus H^2$ onto $H^2 \oplus \mathrm{Clos}(\Delta H^2)$.
(ii) Show that $\Omega \mathbb{K}_b' = \mathcal{K}_b'$.
(iii) Show that $(S^* \oplus S^*)_{|\mathbb{K}_b'}$ and $(S^* \oplus V_\Delta^*)_{\mathcal{K}_b'}$ are unitarily equivalent and the unitary equivalence is given by Ω.

This result explains the link between the models of X_b given by Theorem 19.11 and Theorem 23.20.

23.7 A characterization of $\mathcal{H}(b)$

In this section, we treat an analog of Theorem 17.24 that characterizes $\mathcal{H}(b)$ spaces when b is a nonextreme point of the closed unit ball of H^∞. To give the motivation, we gather some properties of S^* on $\mathcal{H}(b)$.

Lemma 23.21 *Let b be a nonextreme point of the closed unit ball of H^∞, and $b \not\equiv 0$. Then the following assertions hold.*

(i) *$\mathcal{H}(b)$ is S^*-invariant (we recall that the restriction of S^* to $\mathcal{H}(b)$ was denoted by X_b).*
(ii) *$I - X_b X_b^*$ and $I - X_b^* X_b$, respectively, are operators of rank one and rank two.*
(iii) *For every $f \in \mathcal{H}(b)$,*

$$\|X_b f\|_b^2 \le \|f\|_b^2 - |f(0)|^2.$$

(iv) *There is an element $f \in \mathcal{H}(b)$, with $f(0) \neq 0$, such that*

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2.$$

Proof (i) This was established in Theorem 18.13.

(ii) This follows from Corollaries 18.23 and 23.15.

(iii) According to Theorem 23.14, for every function $f \in \mathcal{H}(b)$, we have

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2 - |a(0)|^2 |\langle f, b\rangle_b|^2. \tag{23.23}$$

This gives the required inequality.

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(iv) Define

$$f = \|b\|_b^2 k_0^b - \overline{b(0)} b.$$

By Corollary 23.9, this function belongs to $\mathcal{H}(b)$. Moreover, we have

$$\langle b, f \rangle_b = \|b\|_b^2 \langle b, k_0^b \rangle_b - b(0)\langle b, b \rangle_b = \|b\|_b^2 b(0) - b(0)\|b\|_b^2 = 0,$$

and thus, by (23.23),

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2.$$

It remains to check that $f(0) \neq 0$. Remembering that $\|b\|_b^2 = |a(0)|^{-2} - 1$ (Corollary 23.9), an easy computation shows that

$$f(0) = \frac{1 - |a(0)|^2 - |b(0)|^2}{|a(0)|^2},$$

and thus $f(0) \neq 0$, because $|a(0)|^2 + |b(0)|^2 < 1$. In fact,

$$a(0) = \int_{\mathbb{T}} a(\zeta)\, dm(\zeta) \quad \text{and} \quad b(0) = \int_{\mathbb{T}} b(\zeta)\, dm(\zeta),$$

and thus, using the Cauchy–Schwarz inequality, we get

$$|a(0)|^2 + |b(0)|^2 \le \int_{\mathbb{T}} (|a(\zeta)|^2 + |b(\zeta)|^2)\, dm(\zeta) = 1.$$

Hence, we have $|a(0)|^2 + |b(0)|^2 = 1$ if and only if

$$\left| \int_{\mathbb{T}} a(\zeta)\, dm(\zeta) \right|^2 = \int_{\mathbb{T}} |a(\zeta)|^2\, dm(\zeta)$$

and

$$\left| \int_{\mathbb{T}} b(\zeta)\, dm(\zeta) \right|^2 = \int_{\mathbb{T}} |b(\zeta)|^2\, dm(\zeta).$$

The last two identities hold provided that b is a constant function, which is absurd. □

Lemma 23.21 provides the motivation for the following characterization of $\mathcal{H}(b)$ spaces.

Theorem 23.22 *Let $\mathcal{H}$ be a Hilbert space contained in H^2. Assume that the following hold.*

(i) *$\mathcal{H}$ is S^*-invariant (and denote the restriction of S^* to $\mathcal{H}$ by T).*
(ii) *The operators $I - TT^*$ and $I - T^*T$, respectively, are of rank one and rank two.*
(iii) *For each $f \in \mathcal{H}$,*

$$\|Tf\|_{\mathcal{H}}^2 \le \|f\|_{\mathcal{H}}^2 - |f(0)|^2. \tag{23.24}$$

(iv) *There is an element* $f \in \mathcal{H}$, *with* $f(0) \neq 0$, *such that*

$$\|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2.$$

Then there is a nonextreme point b *in the closed unit ball of* H^∞, *unique up to a unimodular constant, such that* $\mathcal{H} \mathrel{\overline{\underline{=}}} \mathcal{H}(b)$.

Proof According to Theorem 16.29, we know that $\mathcal{H}$ is contained contractively in H^2 and, if $\mathcal{M}$ denotes its complementary space, then S acts as a contraction on $\mathcal{M}$ (note that the notation is different in this theorem, and in fact the roles of $\mathcal{M}$ and $\mathcal{H}$ are exchanged). Our strategy is quite simple. We show that S acts as an isometry on $\mathcal{M}$. Then we apply Theorem 17.24 to deduce that there exists a function b in the closed unit ball of H^∞ such that $\mathcal{M} \mathrel{\overline{\underline{=}}} \mathcal{M}(b)$, and then Corollary 16.27 enables us to conclude that $\mathcal{H} \mathrel{\overline{\underline{=}}} \mathcal{H}(b)$. However, the proof is very long. To show that S acts as an isometry, we decompose the proof into several steps, 14 in all.

Step 1: T *is onto.*

This is equivalent to saying that $\ker T^* = \{0\}$ and T has a closed range. Assume that $\ker T^* \neq \{0\}$. Since $\ker T^* \subset \mathcal{R}(I - TT^*)$, by an argument of dimension, we get $\ker T^* = \mathcal{R}(I - TT^*)$. It follows from Theorem 7.22 that T^* is a partial isometry and $\ker T = \mathcal{R}(I - T^*T)$. Hence, by hypothesis, $\dim \ker T = 2$. But, this is impossible because $\ker T \subset \ker S^* = \mathbb{C}$. Thus, $\ker T^* = \{0\}$.

Now, we show that T^*T has a closed range. Indeed, according to the decomposition $\mathcal{H} = \ker(I - T^*T) \oplus \mathcal{R}(I - T^*T)$, the operator T^*T admits the matrix representation

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & T^*T \end{pmatrix},$$

where T^*T is restricted to $\mathcal{R}(I - T^*T)$. But, since $\mathcal{R}(I - T^*T)$ is of finite dimension, the operator $T^*T_{|\mathcal{R}(I - T^*T)}$ has a closed range and then, by Lemma 1.38, the operator T^*T also has a closed range. Then Corollary 1.35 ensures that T is onto.

Step 2: $1 \in \mathcal{H}$ *and* $f \in \mathcal{H} \implies Sf \in \mathcal{H}$. *In particular, all analytic polynomials belong to* $\mathcal{H}$.

Argue by absurdity and assume that $1 \notin \mathcal{H}$. Then we would have

$$\ker T = \ker S^* \cap \mathcal{H} = \mathbb{C} \cap \mathcal{H} = \{0\},$$

i.e. T is a bijection. But, since $T(I - T^*T) = (I - TT^*)T$, we would obtain $\dim \mathcal{R}(I - T^*T) = \dim \mathcal{R}(I - TT^*)$, which is a contradiction. Therefore, $1 \in \mathcal{H}$. Furthermore, if $f \in \mathcal{H}$, then $S^*Sf = f - f(0) \in \mathcal{H}$. Since T is

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onto, there exists $h \in \mathcal{H}$ such that $S^*Sf = Th = S^*h$. This is equivalent to $Sf - h \in \ker S^* = \mathbb{C}$. Thus, $Sf = h - h(0)$, which implies that $Sf \in \mathcal{H}$.

Step 3: The set

$$\mathcal{D} = \{f \in \mathcal{H} : \|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2\}$$

is a closed subspace of $\mathcal{H}$. *Moreover,* $\ker(I - T^*T) \subset \{f \in \mathcal{D} : f(0) = 0\}$.

It is clear that, if $f \in \mathcal{D}$ and $\lambda \in \mathbb{C}$, then $\lambda f \in \mathcal{D}$. Now, let $f, g \in \mathcal{D}$. We use the parallelogram law twice below. First,

$$\|Tf + Tg\|_{\mathcal{H}}^2 + \|T(f-g)\|_{\mathcal{H}}^2 = 2\|Tf\|_{\mathcal{H}}^2 + 2\|Tg\|_{\mathcal{H}}^2.$$

Second, by the definition of $\mathcal{D}$,

$$\begin{aligned} &2\|Tf\|_{\mathcal{H}}^2 + 2\|Tg\|_{\mathcal{H}}^2 \\ &\quad = 2\|f\|_{\mathcal{H}}^2 - 2|f(0)|^2 + 2\|g\|_{\mathcal{H}}^2 - 2|g(0)|^2 \\ &\quad = \|f+g\|_{\mathcal{H}}^2 + \|f-g\|_{\mathcal{H}}^2 - |(f+g)(0)|^2 - |(f-g)(0)|^2. \end{aligned}$$

Thus,

$$\begin{aligned} &\|Tf + Tg\|_{\mathcal{H}}^2 - \|f+g\|_{\mathcal{H}}^2 + |(f+g)(0)|^2 \\ &\quad = \|f-g\|_{\mathcal{H}}^2 - |(f-g)(0)|^2 - \|T(f-g)\|_{\mathcal{H}}^2. \end{aligned}$$

According to (23.24), on the one hand, we have

$$\|f-g\|_{\mathcal{H}}^2 - |(f-g)(0)|^2 - \|T(f-g)\|_{\mathcal{H}}^2 \geq 0$$

and, on the other,

$$\|Tf + Tg\|_{\mathcal{H}}^2 = \|T(f+g)\|_{\mathcal{H}}^2 \leq \|f+g\|_{\mathcal{H}}^2 - |(f+g)(0)|^2,$$

which is equivalent to

$$\|Tf + Tg\|_{\mathcal{H}}^2 - \|f+g\|_{\mathcal{H}}^2 + |(f+g)(0)|^2 \leq 0.$$

Hence, we get

$$\|T(f+g)\|_{\mathcal{H}}^2 = \|f+g\|_{\mathcal{H}}^2 - |(f+g)(0)|^2,$$

which means that $f + g \in \mathcal{D}$. Therefore, $\mathcal{D}$ is a vector subspace of $\mathcal{H}$.

We proceed to prove that $\mathcal{D}$ is closed. Let $f \in \overline{\mathcal{D}}$. Then there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{D}$ that converges to f in $\mathcal{H}$. Since T is continuous (in fact, according to (23.24), it is a contraction), the sequence $(Tf_n)_{n\geq 1}$ converges to Tf in $\mathcal{H}$ and, since $\mathcal{H}$ is contractively contained in H^2, the sequence $(f_n)_{n\geq 1}$ is also convergent to f in H^2. In particular, since evaluations at points of $\mathbb{D}$ are continuous on $\mathbb{D}$, the scalar sequence $(f_n(0))_{n\geq 1}$ converges to $f(0)$. Since $f_n \in \mathcal{D}$, we have

$$\|Tf_n\|_{\mathcal{H}}^2 = \|f_n\|_{\mathcal{H}}^2 - |f_n(0)|^2.$$

Letting n tend to ∞, we thus get

$$\|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2,$$

which means that $f \in \mathcal{D}$. Therefore, $\mathcal{D}$ is a closed subspace of $\mathcal{H}$.

It remains to check that $\ker(I - T^*T) \subset \{f \in \mathcal{D} : f(0) = 0\}$. Fix an element $f \in \ker(I - T^*T)$. Then we have $f = T^*Tf$, which implies that

$$\|f\|_{\mathcal{H}}^2 = \langle f, T^*Tf\rangle_{\mathcal{H}} = \|Tf\|_{\mathcal{H}}^2 \le \|f\|_{\mathcal{H}}^2 - |f(0)|^2 \le \|f\|_{\mathcal{H}}^2.$$

Thus, $\|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2$ and $f(0) = 0$. In particular, $f \in \mathcal{D}$.

Step 4: There exists $f_0 \in \mathcal{D}$ with $f_0(0) \neq 0$ and $f_0 \perp \ker(I - T^*T)$.

By hypothesis, we know that there is a function $f \in \mathcal{D}$ such that $f(0) \neq 0$. Decompose $f = f_0 + f_1$ such that $f_0 \perp \ker(I - T^*T)$ and $f_1 \in \ker(I - T^*T)$. Using Step 3, we know that $f_1 \in \mathcal{D}$ and $f_1(0) = 0$. Thus, $f_0 \in \mathcal{D}$ and $f_0(0) = f(0) \neq 0$. The function f_0 satisfies the required conditions.

To prove that S acts as an isometry on $\mathcal{M}$, we now consider two situations: $1 \notin \mathcal{D}$ and $1 \in \mathcal{D}$. The verification of the latter is longer (Steps 6–13).

Step 5: S acts as an isometry on $\mathcal{M}$ (case $1 \notin \mathcal{D}$).

Denote by $\mathsf{V}(1, f_0)$ the vector space generated by 1 and f_0. This vector space is of dimension 2 because 1 and f_0 are linearly independent ($1 \notin \mathcal{D}$ and $f_0 \in \mathcal{D}$). Moreover, since $1 = (I - T^*T)1$, the inclusion $\mathsf{V}(1, f_0) \subset \mathcal{R}(I - T^*T)$ holds. Then, with an argument on dimension, we get

$$\mathsf{V}(1, f_0) = \mathcal{R}(I - T^*T),$$

and this implies that

$$\mathcal{H} = \ker(I - T^*T) \oplus \mathsf{V}(1, f_0). \tag{23.25}$$

Using Steps 3 and 4, we have

$$\ker(I - T^*T) \oplus \mathbb{C}f_0 \subset \mathcal{D}.$$

Thus, appealing to Step 1 and (23.25)), we deduce that

$$\mathcal{H} = T\mathcal{H} = T(\ker(I - T^*T) \oplus \mathbb{C}f_2) = T\mathcal{D}.$$

Now, for each $g \in \mathcal{M}$, we have

$$\begin{aligned}
\|g\|_{\mathcal{M}}^2 &= \sup_{f \in \mathcal{H}} (\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2) \\
&= \sup_{f \in \mathcal{D}} (\|g + Tf\|_2^2 - \|Tf\|_{\mathcal{H}}^2) \\
&= \sup_{f \in \mathcal{D}} (\|Sg + STf\|_2^2 - \|Tf\|_{\mathcal{H}}^2) \\
&= \sup_{f \in \mathcal{D}} (\|Sg + f - f(0)\|_2^2 - \|Tf\|_{\mathcal{H}}^2).
\end{aligned}$$

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But, for each $f \in \mathcal{D}$,

$$\begin{aligned}\|Sg + f - f(0)\|_2^2 &= \|Sg + f\|_2^2 + |f(0)|^2 - 2\,\Re\langle Sg + f,\ f(0)\rangle\\ &= \|Sg + f\|_2^2 - |f(0)|^2\\ &= \|Sg + f\|_2^2 + \|Tf\|_{\mathcal{H}}^2 - \|f\|_{\mathcal{H}}^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|g\|_{\mathcal{M}}^2 &= \sup_{f\in\mathcal{D}}(\|Sg + f\|_2^2 - \|f\|_{\mathcal{H}}^2)\\ &\le \sup_{f\in\mathcal{H}}(\|Sg + f\|_2^2 - \|f\|_{\mathcal{H}}^2) = \|Sg\|_{\mathcal{M}}^2.\end{aligned}$$

But, from Theorem 16.29, we already know that S acts as a contraction on $\mathcal{M}$ and hence we conclude that S acts as an isometry on $\mathcal{M}$.

For the rest of proof, we assume that $1 \in \mathcal{D}$ and our goal is to show that S still acts as an isometry on $\mathcal{M}$.

Step 6: Suppose that there exists an integer $n \ge 1$ *such that* $z^m \in \mathcal{D}$, *with* $0 \le m \le n-1$. *Then*

$$\|z^m\|_{\mathcal{H}} = 1 \qquad (0 \le m \le n-1).$$

In particular, $i_{\mathcal{H}}^*(z^m) = z^m$, *for all* $0 \le m \le n-1$, *where* $i_{\mathcal{H}}$ *is the canonical injection from* $\mathcal{H}$ *into* H^2.

We argue by induction. For $m = 0$, since $1 \in \mathcal{D}$, we have

$$\|T1\|_{\mathcal{H}}^2 = \|1\|_{\mathcal{H}}^2 - 1.$$

But, $T1 = S^*1 = 0$, which gives $\|1\|_{\mathcal{H}} = 1$. Assume that, for some m_0 with $0 \le m_0 < n-1$, the identity $\|z^m\|_{\mathcal{H}} = 1$ holds for all $0 \le m \le m_0$. Then, using the fact that $z^{m_0+1} \in \mathcal{D}$, we get

$$\|Tz^{m_0+1}\|_{\mathcal{H}} = \|z^{m_0+1}\|_{\mathcal{H}}.$$

However, $Tz^{m_0+1} = z^{m_0}$, and we deduce that $\|z^{m_0+1}\|_{\mathcal{H}} = \|z^{m_0}\|_{\mathcal{H}} = 1$. Hence, the identity $\|z^m\|_{\mathcal{H}} = 1$ holds for all $0 \le m \le m_0 + 1$. Therefore, by induction, it holds for all $0 \le m \le n-1$.

In the trivial decomposition $z^m = z^m + 0$, we have $z^m \in \mathcal{H}$, $0 \in \mathcal{M}$ and $\|z^m\|_2^2 = \|z^m\|_{\mathcal{H}}^2 + \|0\|_{\mathcal{M}}^2$. Thus, by Corollary 16.28, we have $i_{\mathcal{H}}^* z_m = z_m$ for all $0 \le m \le n-1$.

Step 7: There exists an integer $n \ge 1$ *such that* $z^m \in \mathcal{D}$, *for all* $0 \le m \le n-1$, *but* $z^n \notin \mathcal{D}$.

Assume on the contrary that, for all $k \ge 0$, $z^k \in \mathcal{D}$. Then, according to Step 6, we get $i_{\mathcal{H}}^* z_k = z_k$, for all $k \ge 0$. Therefore, $i_{\mathcal{H}} i_{\mathcal{H}}^* z^k = z^k$, for all $k \ge 0$.

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But, z^k is an orthonormal basis of H^2 and thus $i_{\mathcal{H}} i_{\mathcal{H}}^* = I_{H^2}$. In particular, using Corollary 16.8, we get

$$\mathcal{H} = \mathcal{M}(i_{\mathcal{H}}) = \mathcal{M}((i_{\mathcal{H}} i_{\mathcal{H}}^*)^{1/2}) = \mathcal{M}(I_{H^2}) = H^2.$$

Thus, we have $T = S^*$, or equivalently $T^* = S$, which gives $I - TT^* = 0$. This is absurd.

Step 8: Let n be as in Step 7. Then $(I - TT^*)z^{n-1} \neq 0$ and $T^{*n}1 \neq z^n$. Moreover, if $n > 1$, we also have

$$\begin{aligned} T^* z^{m-1} &= z^m, \\ (I - TT^*) z^{m-1} &= 0, \\ T^{*k} z^{m-k} &= z^m, \end{aligned}$$

for all $1 \leq m \leq n-1$ and $0 \leq k \leq m$.

To prove the first relation, we again argue by absurdity. Assume that $(I - TT^*)z^{n-1} = 0$. Since

$$(I - TT^*)z^{n-1} = (I - TT^*)Tz^n = T((I - T^*T)z^n),$$

it would imply that $(I - T^*T)z^n \in \ker T$. But the function $(I - T^*T)z^n$ is also orthogonal to the kernel of T. Indeed, we have $\ker T = \ker S^* \cap \mathcal{H} = \mathbb{C}1$ and, since $n \geq 1$,

$$\begin{aligned} \langle (I - T^*T)z^n,\ 1\rangle_{\mathcal{H}} &= \langle z^n,\ (I - T^*T)1\rangle_{\mathcal{H}} \\ &= \langle z^n, 1\rangle_{\mathcal{H}} \\ &= \langle z^n, i_{\mathcal{H}}^* 1\rangle_{\mathcal{H}} \\ &= \langle i_{\mathcal{H}}(z^n), 1\rangle_2 \\ &= \langle z^n, 1\rangle_2 \\ &= 0. \end{aligned}$$

Thus, $(I - T^*T)z^n \perp \ker T$, which is equivalent to $(I - T^*T)z^n = 0$. This means that $z^n \in \ker(I - T^*T)$. But, by Step 3, we conclude that $z^n \in \mathcal{D}$, a contradiction with the definition of n. Therefore, $(I - TT^*)z^{n-1} \neq 0$.

If $n = 1$, then $(I - TT^*)1 \neq 0$, that is $1 \neq TT^*1$. Hence, $z \neq T^*1$. Now, assume that $n > 1$. We first prove that

$$T^* z^{m-1} = z^m, \qquad \text{for every } 1 \leq m \leq n-1. \tag{23.26}$$

We have

$$\|T^* z^{m-1} - z^m\|_{\mathcal{H}}^2 = \|T^* z^{m-1}\|_{\mathcal{H}}^2 + \|z^m\|_{\mathcal{H}}^2 - 2\,\Re\langle T^* z^{m-1}, z^m\rangle_{\mathcal{H}}$$

and

$$\langle T^* z^{m-1}, z^m\rangle_{\mathcal{H}} = \langle z^{m-1}, Tz^m\rangle_{\mathcal{H}} = \|z^{m-1}\|_{\mathcal{H}}^2.$$

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Hence, using Step 6, we get

$$\|T^*z^{m-1} - z^m\|_{\mathcal{H}}^2 = \|T^*z^{m-1}\|_{\mathcal{H}}^2 + 1 - 2 = \|T^*z^{m-1}\|_{\mathcal{H}}^2 - 1. \quad (23.27)$$

But, since T is a contraction on $\mathcal{H}$, we have

$$\|T^*z^{m-1}\|_{\mathcal{H}} \leq \|T^*\| \, \|z^{m-1}\| \leq 1.$$

Thus (23.27) implies that $\|T^*z^{m-1} - z^m\|_{\mathcal{H}} \leq 0$, which gives (23.26).

Since $T^*z^{m-1} = z^m$, we have $TT^*z^{m-1} = z^{m-1}$, and thus

$$(I - TT^*)z^{m-1} = 0 \qquad (1 \leq m \leq n-1).$$

To prove that $T^{*n}1 \neq z^n$, we argue by absurdity. Assume that $T^{*n}1 = z^n$. Then

$$\|z^n\|_{\mathcal{H}}^2 = \langle z^n, z^n\rangle_{\mathcal{H}} = \langle z^n, T^{*n}1\rangle_{\mathcal{H}} = \langle T^n z^n, 1\rangle_{\mathcal{H}}.$$

But, $T^n z^n = 1$, whence

$$\|z^n\|_{\mathcal{H}}^2 = \|1\|_{\mathcal{H}}^2 = 1.$$

In particular, we deduce that

$$\|z^n\|_{\mathcal{H}} = \|z^{n-1}\|_{\mathcal{H}} = \|Tz^n\|_{\mathcal{H}}.$$

This means that $z^n \in \mathcal{D}$, which is a contradiction. Thus, we have $T^{*n}1 \neq z^n$.

Finally, it remains to prove that

$$T^{*k}z^{m-k} = z^m \qquad (0 \leq k \leq m). \quad (23.28)$$

We argue by induction. For $k = 0$, it is obvious. Now, assume that, for some $0 \leq k < m$, we have $T^{*k}z^{m-k} = z^m$. Then using (23.26), we have

$$T^{*(k+1)}z^{m-(k+1)} = T^{*k}(T^*z^{m-k-1}) = T^{*k}z^{m-k} = z^m,$$

which proves (23.28).

Step 9: Let $f \in \mathcal{H}$ and write

$$f(z) = \sum_{m=0}^{n-1} a_m z^m + z^m T^m f(z) \qquad (z \in \mathbb{D}).$$

Then

$$\|f\|_{\mathcal{H}}^2 = \sum_{m=0}^{n-1} |a_m|^2 + \|z^n T^n f\|_{\mathcal{H}}^2.$$

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We have

$$\|f\|_{\mathcal{H}}^2 = \left\| \sum_{m=0}^{n-1} a_m z^m \right\|_{\mathcal{H}}^2 + \|z^n T^n f\|_{\mathcal{H}}^2 + 2 \sum_{m=0}^{n-1} \Re(a_m \langle z^m, z^n T^n f \rangle_{\mathcal{H}}).$$

But, using Step 6,

$$\begin{aligned} \langle z^k, z^\ell \rangle_{\mathcal{H}} &= \langle i_{\mathcal{H}}^*(z^k), z^\ell \rangle_{\mathcal{H}} \\ &= \langle z^k, i_{\mathcal{H}}(z^\ell) \rangle_2 \\ &= \langle z^k, z^\ell \rangle_2 \\ &= \delta_{k,\ell} \qquad (0 \le k, \ell \le n-1). \end{aligned}$$

Hence,

$$\left\| \sum_{m=0}^{n-1} a_m z^m \right\|_{\mathcal{H}}^2 = \sum_{m=0}^{n-1} |a_m|^2.$$

Moreover,

$$\begin{aligned} \langle z^m, z^n T^n f \rangle_{\mathcal{H}} &= \langle i_{\mathcal{H}}^*(z^m), z^n T^n f \rangle_{\mathcal{H}} \\ &= \langle z^m, i_{\mathcal{H}}(z^n T^n f) \rangle_2 \\ &= \langle z^m, z^n T^n f \rangle_2 = 0 \qquad (0 \le m \le n-1). \end{aligned}$$

This proves Step 9.

Step 10: For every $f \in \mathcal{H}$ and $g \in \mathcal{M}$, we have

$$\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 = \|g + z^n T^n f\|_2^2 - \|z^n T^n f\|_{\mathcal{H}}^2.$$

Write

$$f = \sum_{m=0}^{n-1} a_m z^m + z^n T^n f.$$

Then

$$\begin{aligned} \|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 &= \left\| g + z^n T^n f + \sum_{m=0}^{n-1} a_m z^m \right\|_2^2 - \|f\|_{\mathcal{H}}^2 \\ &= \|g + z^n T^n f\|_2^2 + \sum_{m=0}^{n-1} |a_m|^2 - \|f\|_{\mathcal{H}}^2 \\ &\quad + 2 \sum_{m=0}^{n-1} \Re(a_m \langle z^m,\ g + z^n T^n f \rangle_2). \end{aligned}$$

Using Step 9, we get

$$\begin{aligned}
&\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 \\
&\quad = \|g + z^n T^n f\|_2^2 - \|z^n T^n f\|_{\mathcal{H}}^2 + 2 \sum_{m=0}^{n-1} \Re(a_m \langle z^m,\ g + z^n T^n f\rangle_2).
\end{aligned}$$

But, for every $0 \le m \le n-1$, we have

$$\begin{aligned}
\langle z^m,\ g + z^n T^n f\rangle_2 &= \langle z^m, g\rangle_2 \\
&= \langle z^m, i_{\mathcal{M}}(g)\rangle_2 \\
&= \langle i_{\mathcal{M}}^*(z^m), g\rangle_{\mathcal{M}} = 0,
\end{aligned}$$

because $i_{\mathcal{M}}^*(z^m) = z^m - i_{\mathcal{H}}^*(z^m) = z^m - z^m = 0$. This proves Step 10.

Step 11. For every $f \in \mathcal{H}$, there exists $\hat{f} \in \ker(I - T^n T^{*n})$ such that

$$\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 = \|g + \hat{f}\|_2^2 - \|\hat{f}\|_{\mathcal{H}}^2 \qquad (g \in \mathcal{M}).$$

Let $f \in \mathcal{H}$, and define the constants $c_0, c_1, \dots, c_{n-1}$ recursively by the formulas

$$\alpha_n = \langle z^{n-1},\ (I - TT^*) z^{n-1}\rangle_{\mathcal{H}},$$

$$c_{n-1} = -\langle f,\ (I - TT^*) z^{n-1}\rangle_{\mathcal{H}} / \alpha_n$$

and, if $n > 1$,

$$c_{n-k} = -\left\langle f + \sum_{m=n-k+1}^{n-1} c_m z^m,\ T^{k-1}(I - TT^*) T^{*k-1} z^{n-k} \right\rangle_{\mathcal{H}} \Big/ \alpha_n,$$

for $2 \le k \le n$. Note that $\alpha_n \ne 0$ and thus the sequence $c_0, c_1, \dots, c_{n-1}$ is well defined. Indeed, since $I - TT^*$ is a self-adjoint operator of rank one, there exists an element $g \in \mathcal{H}$ such that $I - TT^* = g \otimes g$, and thus $\alpha_n = |\langle z^{n-1}, g\rangle_{\mathcal{H}}|^2$. If $\alpha_n = 0$, then it would imply that $\langle z^{n-1}, g\rangle_{\mathcal{H}} = 0$ and that $(I - TT^*) z^{n-1} = 0$, a contradiction with Step 8.

Then we define

$$\hat{f} = f + \sum_{m=0}^{n-1} c_m z^m,$$

and we show that $\hat{f}$ satisfies the required properties. We obviously have $T^n \hat{f} = T^n f$, whence, according to Step 10, we have

$$\begin{aligned}
\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 &= \|g + z^n T^n f\|_2^2 - \|z^n T^n f\|_{\mathcal{H}}^2 \\
&= \|g + z^n T^n \hat{f}\|_2^2 - \|z^n T^n \hat{f}\|_{\mathcal{H}}^2 \\
&= \|g + \hat{f}\|_2^2 - \|\hat{f}\|_{\mathcal{H}}^2 \qquad (g \in \mathcal{M}).
\end{aligned}$$

Thus, it remains to check that $\hat{f} \in \ker(I - T^n T^{*n})$, which is equivalent to $\hat{f} \perp \mathcal{R}(I - T^n T^{*n})$. But

$$I - T^n T^{*n} = \sum_{k=1}^{n} T^{k-1}(I - TT^*)T^{*k-1},$$

whence it is sufficient to prove that $\hat{f} \perp \mathcal{R}(T^{k-1}(I - TT^*)T^{*k-1})$. Define $u_k = T^{k-1}(I - TT^*)T^{*k-1} z^{n-k}$ and note that $u_k \neq 0$. In fact, according to Step 8, we have

$$\begin{aligned}\langle z^{n-k}, u_k\rangle_{\mathcal{H}} &= \langle T^{*k-1} z^{n-k},\ (I - TT^*)T^{*k-1} z^{n-k}\rangle_{\mathcal{H}}\\ &= \langle z^{n-1},\ (I - TT^*) z^{n-1}\rangle_{\mathcal{H}}\\ &= \alpha_n \neq 0.\end{aligned}$$

Hence, $T^{k-1}(I - TT^*)T^{*k-1}$ is an operator of rank one and its range is generated by u_k. Therefore, $\hat{f} \perp \mathcal{R}(T^{k-1}(I - TT^*)T^{*k-1})$ is equivalent to $\hat{f} \perp u_k$, $1 \le k \le n$. Now, note that

$$\langle \hat{f}, u_k\rangle_{\mathcal{H}} = \langle f, u_k\rangle_{\mathcal{H}} + \sum_{m=0}^{n-1} c_m \langle z^m, u_k\rangle_{\mathcal{H}}.$$

But, according to the definitions of c_m, we have

$$c_{n-k}\alpha_n = -\left\langle f + \sum_{m=n-k+1}^{n-1} c_m z^m,\ u_k \right\rangle_{\mathcal{H}},$$

whence

$$\langle f, u_k\rangle_{\mathcal{H}} = -c_{n-k}\alpha_n - \sum_{m=n-k+1}^{n-1} c_m \langle z^m, u_k\rangle_{\mathcal{H}} = -\sum_{m=n-k}^{n-1} c_m \langle z^m, u_k\rangle_{\mathcal{H}}.$$

Thus, we get

$$\langle \hat{f}, u_k\rangle_{\mathcal{H}} = \sum_{m=0}^{n-k-1} c_m \langle z^m, u_k\rangle_{\mathcal{H}}.$$

For every $0 \le m \le n - k - 1$, we have

$$\begin{aligned}\langle z^m, u_k\rangle_{\mathcal{H}} &= \langle z^m,\ T^{k-1}(I - TT^*)T^{*(k-1)} z^{n-k}\rangle_{\mathcal{H}}\\ &= \langle z^{m+k-1},\ (I - TT^*) z^{n-1}\rangle_{\mathcal{H}}\\ &= \langle (I - TT^*) z^{m+k-1},\ z^{n-1}\rangle_{\mathcal{H}},\end{aligned}$$

and, according to Step 8, we have $(I - TT^*)z^{m+k-1} = 0$ (and note that $m + k - 1 \le n - 2$). Thus, $\langle z^m, u_k\rangle_{\mathcal{H}} = 0$ and $\langle \hat{f}, u_k\rangle_{\mathcal{H}} = 0$, for every $1 \le k \le n$. This proves Step 11.

Step 12: If $h \in \ker(I - T^n T^{*n})$*, then*

$$\|h\|_{\mathcal{H}} = \|z^n h\|_{\mathcal{H}}. \tag{23.29}$$

Moreover, for every $g \in \mathcal{M}$*, we have*

$$\|g\|_{\mathcal{M}}^2 = \sup\{\|g+f\|_2^2 - \|z^n f\|_{\mathcal{H}}^2 : f \in \mathcal{H} \text{ and } (I - T^n T^{*n})f = 0\}. \tag{23.30}$$

Take any $h \in \ker(I - T^n T^{*n})$. Then, for every $0 \le m \le n-1$, we have

$$\begin{aligned}\langle (I - T^{*n}T^n)(z^n h),\ z^m\rangle_{\mathcal{H}} &= \langle z^n h,\ (I - T^{*n}T^n)(z^m)\rangle_{\mathcal{H}} \\ &= \langle z^n h, z^m\rangle_{\mathcal{H}} \\ &= \langle z^n h, i_{\mathcal{H}}^*(z^m)\rangle_{\mathcal{H}} \\ &= \langle z^n h, z^m\rangle_2 = 0.\end{aligned}$$

This proves that $(I - T^{*n}T^n)(z^n h) \perp \ker T^n$. Moreover,

$$T^n((I - T^{*n}T^n)(z^n h)) = (I - T^n T^{*n})(T^n z^n h) = (I - T^n T^{*n})h = 0.$$

Therefore, $(I - T^{*n}T^n)(z^n h) = 0$, that is $z^n h = T^{*n}T^n(z^n h)$. Thus,

$$\begin{aligned}\|z^n h\|_{\mathcal{H}}^2 &= \langle z^n h, T^{*n}T^n(z^n f)\rangle_{\mathcal{H}} \\ &= \|T^n(z^n h)\|_{\mathcal{H}}^2 = \|h\|_{\mathcal{H}}^2.\end{aligned}$$

Now, using Step 11 and (23.29), we get

$$\begin{aligned}\|g\|_{\mathcal{M}}^2 &= \sup\{\|g+f\|_2^2 - \|f\|_{\mathcal{H}}^2 : f \in \mathcal{H}\} \\ &= \sup\{\|g+f\|_2^2 - \|f\|_{\mathcal{H}}^2 : f \in \mathcal{H} \text{ and } f \in \ker(I - T^n T^{*n})\} \\ &= \sup\{\|g+f\|_2^2 - \|z^n f\|_{\mathcal{H}}^2 : f \in \mathcal{H} \text{ and } f \in \ker(I - T^n T^{*n})\},\end{aligned}$$

which proves (23.30).

Step 13: S acts as an isometry on $\mathcal{M}$ (case $1 \in \mathcal{D}$).

Since $\|zg\|_{\mathcal{M}} \le \|g\|_{\mathcal{M}}$, for every $g \in \mathcal{M}$ and $\mathcal{H} = T^n\mathcal{H}$, using Step 12, we have

$$\|z^n g\|_{\mathcal{M}}^2 \le \|zg\|_{\mathcal{M}}^2 \le \|g\|_{\mathcal{M}}^2.$$

But

$$\begin{aligned}\|g\|_{\mathcal{M}}^2 &= \sup_{\substack{f \in \mathcal{H}, \\ (I - T^n T^{*n})(T^n f) = 0}} \|g + T^n f\|_2^2 - \|z^n T^n f\|_{\mathcal{H}}^2 \\ &= \sup_{\substack{f \in \mathcal{H}, \\ (I - T^n T^{*n})(T^n f) = 0}} \|z^n g + z^n T^n f\|_2^2 - \|z^n T^n f\|_{\mathcal{H}}^2 \\ &\le \|z^n g\|_{\mathcal{M}}^2.\end{aligned}$$

Hence, $\|zg\|_{\mathcal{M}} = \|g\|_{\mathcal{M}}$, which proves Step 13.

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Step 14: There is a nonextreme point b in the closed unit ball of H^∞, unique up to a unimodular constant, such that $\mathcal{H} \doteq \mathcal{H}(b)$.

According to Steps 5 and 13, S acts as an isometry on $\mathcal{M}$. Therefore, Theorem 17.24 implies that there exists a function b in the closed unit ball of H^∞ such that $\mathcal{M} \doteq \mathcal{M}(b)$. Now Corollary 16.27 implies that $\mathcal{H} \doteq \mathcal{H}(b)$. Finally, b cannot be an extreme point of the closed unit ball of H^∞, since for instance the analytic polynomials belongs to $\mathcal{H}(b)$ (see Exercise 18.9.4).

This completes the proof of Theorem 23.22. □

23.8 More inhabitants of $\mathcal{H}(b)$

In Section 18.6, we showed that

$$Q_w b \in \mathcal{H}(b) \qquad (w \in \mathbb{D}).$$

It is trivial that the reproducing kernel k_w^b is also in $\mathcal{H}(b)$. In Section 23.4, we saw that the analytic polynomials form a dense manifold in $\mathcal{H}(b)$. Now, we use this information to find more objects in $\mathcal{H}(b)$. Moreover, we also discuss some properties on the newly found elements.

Theorem 23.23 *Let b be a nonextreme point of the closed unit ball of H^∞, and let $w \in \mathbb{D}$. Then*

$$k_w \in \mathcal{H}(b) \quad \text{and} \quad bk_w \in \mathcal{H}(b).$$

Moreover, for every $f \in \mathcal{H}(b)$, we have

$$\langle f, k_w \rangle_b = f(w) + \frac{b(w)}{a(w)}\, f^+(w) \tag{23.31}$$

and

$$\langle f, bk_w \rangle_b = \frac{f^+(w)}{a(w)}. \tag{23.32}$$

Proof According to Theorems 17.8 and 23.2, the Cauchy kernel k_w belongs to $\mathcal{H}(b)$ if and only if $T_{\bar{b}} k_w$ belongs to $\mathcal{M}(\bar{a})$. But, by (12.7), we have

$$T_{\bar{b}} k_w = \overline{b(w)} k_w \quad \text{and} \quad T_{\bar{a}} k_w = \overline{a(w)} k_w,$$

which implies that

$$T_{\bar{b}} k_w = T_{\bar{a}} \left(\frac{\overline{b(w)}}{\overline{a(w)}}\, k_w \right).$$

This identity shows that $k_w \in \mathcal{H}(b)$ and, moreover, that

$$k_w^+ = \frac{\overline{b(w)}}{\overline{a(w)}}\, k_w. \tag{23.33}$$

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Thus, by Theorem 23.8, for every $f \in \mathcal{H}(b)$, we have

$$\begin{aligned}\langle f, k_w\rangle_b &= \langle f, k_w\rangle_2 + \langle f^+, k_w^+\rangle_2 \\ &= \langle f, k_w\rangle_2 + \frac{b(w)}{a(w)}\,\langle f^+, k_w\rangle_2 \\ &= f(w) + \frac{b(w)}{a(w)}\,f^+(w).\end{aligned}$$

Remember that k_w is the reproducing kernel of H^2.

Similarly, the function bk_w belongs to $\mathcal{H}(b)$ if and only if the function $T_{\bar{b}}(bk_w)$ belongs to $\mathcal{M}(\bar{a})$. But, once more using $T_{\bar{a}}k_w = \overline{a(w)}k_w$, we obtain

$$\begin{aligned}T_{\bar{b}}(bk_w) &= P_+(|b|^2k_w) \\ &= P_+((1-|a|^2)k_w) \\ &= k_w - T_{\bar{a}}(ak_w) \\ &= T_{\bar{a}}\left(\frac{k_w}{\overline{a(w)}} - ak_w\right),\end{aligned}$$

which shows that $bk_w \in \mathcal{H}(b)$ and, moreover, that

$$(bk_w)^+ = \left(\frac{1}{\overline{a(w)}} - a\right)k_w. \tag{23.34}$$

Thus, by Theorem 23.8, for every $f \in \mathcal{H}(b)$, we have

$$\begin{aligned}\langle f, bk_w\rangle_b &= \langle f, bk_w\rangle_2 + \langle f^+, (bk_w)^+\rangle_2 \\ &= \langle f, bk_w\rangle_2 + \frac{1}{a(w)}\,\langle f^+, k_w\rangle_2 - \langle f^+, ak_w\rangle_2 \\ &= \langle f, bk_w\rangle_2 - \langle f^+, ak_w\rangle_2 + \frac{f^+(w)}{a(w)}.\end{aligned}$$

To finish the proof and get the equality (23.32), it remains to notice that, by Lemma 4.8,

$$\begin{aligned}\langle f, bk_w\rangle_2 &= \langle \bar{b}f, k_w\rangle_2 \\ &= \langle T_{\bar{b}}f, k_w\rangle_2 \\ &= \langle T_{\bar{a}}f^+, k_w\rangle_2 \\ &= \langle f^+, ak_w\rangle_2.\end{aligned}$$

This completes the proof. □

If we take $w = 0$ in Theorem 23.23, we obtain the following special case. However, note that the first conclusion was already obtained in Corollary 23.9.

Corollary 23.24 *Let b be a nonextreme point of the closed unit ball of H^∞. Then*

$$b \in \mathcal{H}(b).$$

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Moreover, for every $f \in \mathcal{H}(b)$*, we have*

$$\langle f, 1\rangle_b = f(0) + \frac{b(0)}{a(0)}\, f^+(0)$$

and

$$\langle f, b\rangle_b = \frac{f^+(0)}{a(0)}.$$

Corollary 23.25 *Let* $z, w \in \mathbb{D}$*. Then we have*

$$\langle k_z, k_w\rangle_b = \left(1 + \frac{\overline{b(z)}b(w)}{\overline{a(z)}a(w)}\right)k_z(w), \tag{23.35}$$

$$\langle k_z, bk_w\rangle_b = \frac{\overline{b(z)}}{\overline{a(z)}a(w)}k_z(w), \tag{23.36}$$

$$\langle bk_z, bk_w\rangle_b = \left(\frac{1}{\overline{a(z)}a(w)} - 1\right)k_z(w). \tag{23.37}$$

Proof Using (23.31) with $f = k_z$, we get

$$\langle k_z, k_w\rangle_b = k_z(w) + \frac{b(w)}{a(w)}\, k_z^+(w).$$

Now, apply (23.33) to obtain (23.35).

If we put $f = k_z$ in (23.32), we obtain

$$\langle k_z, bk_w\rangle_b = \frac{k_z^+(w)}{a(w)} = \frac{\overline{b(z)}}{\overline{a(z)}a(w)}\, k_z(w).$$

Finally, to prove (23.37), we apply (23.32) with $f = bk_z$ and use (23.34). Hence, we have

$$\langle bk_z, bk_w\rangle_b = \frac{(bk_z)^+(w)}{a(w)} = \frac{1}{a(w)}\left(\frac{1}{\overline{a(z)}} - a(w)\right)k_z(w). \qquad \square$$

Note that if we take $z = w$ in (23.35), then we get

$$\|k_w\|_b^2 = \frac{1}{1-|w|^2}\left(1 + \frac{|b(w)|^2}{|a(w)|^2}\right). \tag{23.38}$$

In Theorem 23.13, we showed that analytic polynomials form a dense manifold in $\mathcal{H}(b)$. Knowing that Cauchy kernels are also in $\mathcal{H}(b)$ (Theorem 23.23), we expect to have a similar result for the manifold they create. The following result provides an affirmative answer.

Corollary 23.26 *Let* b *be a nonextreme point of the closed unit ball of* H^∞*. Then*

$$\text{Span}(k_w : w \in \mathbb{D}) = \mathcal{H}(b).$$

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Proof Let $f \in \mathcal{H}(b)$ be such that $f \perp \text{Span}(k_w : w \in \mathbb{D})$. Then, according to Theorem 23.23, we have

$$f(w) + \frac{b(w)}{a(w)} f^+(w) = 0 \qquad (w \in \mathbb{D}).$$

This is equivalent to $fa = -bf^+$ on $\mathbb{T}$. Multiplying this equality by $\bar{b}$ and using the identity $|a|^2 + |b|^2 = 1$, we obtain

$$a(\bar{b}f - \bar{a}f^+) = -f^+. \tag{23.39}$$

The relation $T_{\bar{b}}f = T_{\bar{a}}f^+$ can be rewritten as $P_+(\bar{b}f - \bar{a}f^+) = 0$, which means that the function $\bar{b}f - \bar{a}f^+$ belongs to $\overline{H_0^2}$. In particular, by (23.39), we deduce that f^+/a belongs to L^2. Now, on the one hand, it follows from Corollary 4.28 that f^+/a belongs to H^2, because a is outer. On the other hand, (23.39) also implies that f^+/a belongs to $\overline{H_0^2}$, whence $f^+/a = 0$. That is, $f^+ = 0$ and then $f = 0$, which proves that the linear span of Cauchy kernels k_w, $w \in \mathbb{D}$, is dense in $\mathcal{H}(b)$. □

Exercise

Exercise 23.8.1 Let (a, b) be a pair. Show that

$$(k_w^b)^+ = \overline{b(w)} a k_w \qquad (w \in \mathbb{D}).$$

Hint: Note that $k_w^b = k_w - \overline{b(w)} b k_w$. Then use (23.33) and (23.34).

23.9 Unbounded Toeplitz operators and $\mathcal{H}(b)$ spaces

In this section, we explain the close relation between $\mathcal{H}(b)$ spaces and unbounded Toeplitz operators with symbols in the Smirnov class. We first recall that the Nevanlinna class $\mathcal{N}$ consists of holomorphic functions in $\mathbb{D}$ that are quotients of functions in H^∞, and the Smirnov class $\mathcal{N}^+$ consists of such quotients in which the denominators are outer functions; see Section 5.1. The representation of such functions as the quotient of two H^∞ functions, even if we assume the denominator is outer, is not unique. However, if we impose some extra conditions, then the representation becomes unique.

Lemma 23.27 *Let φ be a nonzero function in the Smirnov class $\mathcal{N}^+$. Then there exists a unique pair (a, b) such that $\varphi = b/a$.*

Proof By definition, we can write φ as $\varphi = \psi_1/\psi_2$, where $\psi_1, \psi_2 \in H^\infty$, $\psi_1 \neq 0$ and ψ_2 is outer. If the required pair (a, b) exists then, because $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, the function a must satisfy the identity

$$\frac{1 - |a|^2}{|a|^2} = \frac{|\psi_1|^2}{|\psi_2|^2} \qquad (\text{a.e. on } \mathbb{T}),$$

that is,

$$|a|^2 = \frac{|\psi_2|^2}{|\psi_1|^2 + |\psi_2|^2} \qquad \text{(a.e. on } \mathbb{T}). \tag{23.40}$$

Since $\psi_2 \in H^\infty$, the function $|\psi_2|^2$ is log-integrable on $\mathbb{T}$ and hence $|\psi_1|^2 + |\psi_2|^2$ is also log-integrable on $\mathbb{T}$. Thus there is a unique function $a \in H^\infty$ that satisfies (23.40) and is positive at the origin. For the function $b = a\varphi$, then we have

$$|a|^2 + |b|^2 = \frac{|\psi_2|^2}{|\psi_1|^2 + |\psi_2|^2} + \frac{|\psi_2|^2}{|\psi_1|^2 + |\psi_2|^2}\,\frac{|\psi_1|^2}{|\psi_2|^2} = 1 \qquad \text{(a.e. on } \mathbb{T}).$$

Hence (a, b) is a pair and the existence of the desired representation of φ is established. The uniqueness holds because the outer function a is uniquely determined by (23.40) and $a(0) > 0$. □

The representation of $\varphi \in \mathcal{N}^+$ given by Lemma 23.27 is called the *canonical representation* of φ.

We start now with a function φ that is holomorphic in $\mathbb{D}$ and define T_φ to be the operator of multiplication by φ on the domain

$$\mathcal{D}(T_\varphi) = \{f \in H^2 : \varphi f \in H^2\}.$$

It is easily seen that T_φ is a closed operator; see Section 7.7. Indeed, let $f_n \in \mathcal{D}(T_\varphi)$ such that $f_n \longrightarrow f$ in H^2 and $\varphi f_n \longrightarrow g$ in H^2. In particular, for each $z \in \mathbb{D}$, we have $f_n(z) \longrightarrow f(z)$ and $(\varphi f_n)(z) \longrightarrow g(z)$. Since $(\varphi f_n)(z)$ also tends to $\varphi(z)f(z)$, we deduce that $\varphi f = g$. In other words, $f \in \mathcal{D}(T_\varphi)$ and $T_\varphi f = g$. Hence, the graph of T_φ, $\mathcal{G}(T_\varphi) = \{f \oplus \varphi f : f \in H^2, \varphi f \in H^2\}$, is closed in $H^2 \oplus H^2$, which means that T_φ is a closed operator.

Lemma 23.28 *Let φ be a function holomorphic on $\mathbb{D}$. Then the following are equivalent:*

(i) $\mathcal{D}(T_\varphi) \neq \{0\}$*;*
(ii) *φ is in the Nevanlinna class $\mathcal{N}$.*

Proof Suppose that there exists a function $f \neq 0$ that belongs to $\mathcal{D}(T_\varphi)$. Thus $\varphi = \varphi f/f$ is the quotient of two H^2 functions, hence the quotient of two functions in $\mathcal{N}$. Thus, $\varphi \in \mathcal{N}$. Conversely, if φ is in the Nevanlinna class, then we can write $\varphi = \psi_1/\psi_2$, where ψ_1 and ψ_2 are in H^∞. Then $\mathcal{D}(T_\varphi)$ contains the set $\psi_2 H^2$. □

Lemma 23.29 *Let φ be a function holomorphic on $\mathbb{D}$. Then the following are equivalent:*

(i) *$\mathcal{D}(T_\varphi)$ is dense in H^2;*
(ii) *φ is in the Smirnov class $\mathcal{N}^+$.*

Proof (i) $\Longrightarrow$ (ii) Since $\mathcal{D}(T_\varphi)$ is dense, it is in particular not reduced to $\{0\}$. Hence, according to Lemma 23.28, φ is in the Nevanlinna class. Write $\varphi = \psi/\chi$, where ψ and χ are functions in H^∞, whose inner factors are relatively prime. Assume that f is in $\mathcal{D}(T_\varphi)$ and let $g = \varphi f$. Then $\psi f = \chi g$. Write $\psi = \psi_i\psi_o$, $f = f_i f_o$, $\chi = \chi_i\chi_o$ and $g = g_i g_o$, where $\psi_i,\ f_i,\ \chi_i,\ g_i$ are inner and $\psi_o,\ f_o,\ \chi_o,\ g_o$ are outer. By the uniqueness of the canonical factorization for the inner and outer parts, we have $\psi_i f_i = \chi_i g_i$. Since $GCD(\psi_i, \chi_i) = 1$, then χ_i divides f_i, which means that there is an inner function θ_i such that $f_i = \theta_i\chi_i$. Hence, $\psi_o f = \psi_o f_i f_o = \chi_i\theta_i\psi_o f_o$. We get from this relation that $\psi_o f \in \chi H^2$. Using once more the uniqueness of the canonical factorization, we deduce that $f \in \chi_i H^2$. Thus $\mathcal{D}(T_\varphi) \subset \chi_i H^2$. Now, since $\mathcal{D}(T_\varphi)$ is dense in H^2, we conclude by Theorem 8.16 that χ_i must be a constant. In other words, χ must be outer and then $\varphi \in \mathcal{N}^+$.

(ii) $\Longrightarrow$ (i) If $\varphi = \psi/\chi$, where ψ and χ are in H^∞ and χ is outer, then, as noted above, $\mathcal{D}(T_\varphi)$ contains χH^2, which is dense in H^2 by Theorem 8.16. Hence $\mathcal{D}(T_\varphi)$ is also dense in H^2. □

We just have seen that, when $\varphi \in \mathcal{N}^+$, then the domain of T_φ is dense in H^2. Using the canonical representation of φ, we can precisely identify $\mathcal{D}(T_\varphi)$.

Lemma 23.30 *Let φ be a nonzero function in $\mathcal{N}^+$ with canonical representation $\varphi = b/a$. Then*

$$\mathcal{D}(T_\varphi) = aH^2.$$

Proof The inclusion $aH^2 \subset \mathcal{D}(T_\varphi)$ is clear (as noted above). Suppose now that $f \in \mathcal{D}(T_\varphi)$. Then we have

$$|\varphi f|^2 = \frac{|b|^2|f^2|}{|a|^2} = \left|\frac{f}{a}\right|^2 - |f|^2 \qquad \text{(a.e. on } \mathbb{T}),$$

which implies that f/a is in $L^2(\mathbb{T})$. Since a is outer, Corollary 4.28 implies that f/a is in H^2, giving the inclusion $\mathcal{D}(T_\varphi) \subset aH^2$. □

Since, whenever $\varphi \in \mathcal{N}^+$, the operator T_φ is densely defined and closed, its adjoint T_φ^* is also densely defined and closed. The next result shows that de Branges–Rovnyak spaces naturally occur as the domain of the adjoint of Toeplitz operators with symbols in $\mathcal{N}^+$.

Theorem 23.31 *Let φ be a nonzero function in $\mathcal{N}^+$ with canonical representation $\varphi = b/a$. Then the following assertions hold.*

(i) $\mathcal{D}(T_\varphi^*) = \mathcal{H}(b)$.

(ii) *For each $f \in \mathcal{H}(b)$, we have $T_\varphi^* f = f^+$ and*

$$\|f\|_b^2 = \|f\|_2^2 + \|T_\varphi^* f\|_2^2. \tag{23.41}$$

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Proof (i) By definition, a function $f \in H^2$ belongs to $\mathcal{D}(T_\varphi^*)$ if and only if there is a function $g \in H^2$ such that

$$\langle T_\varphi h, f \rangle_2 = \langle h, g \rangle_2 \tag{23.42}$$

for all $h \in \mathcal{D}(T_\varphi)$. By Lemma 23.30, $\mathcal{D}(T_\varphi) = aH^2$, which means that $f \in \mathcal{D}(T_\varphi^*)$ if and only if there is $g \in H^2$ such that

$$\langle T_\varphi(a\psi), f \rangle_2 = \langle a\psi, g \rangle_2 \tag{23.43}$$

for all $\psi \in H^2$. But

$$\langle T_\varphi(a\psi), f \rangle_2 = \langle b\psi, f \rangle_2.$$

Hence, (23.43) is equivalent to

$$\langle b\psi, f \rangle_2 = \langle a\psi, g \rangle_2 \qquad (\psi \in H^2),$$

which can be written as

$$\langle \psi,\, \bar{b}f - \bar{a}g \rangle_2 = 0 \qquad (\psi \in H^2).$$

In other words, $f \in \mathcal{D}(T_\varphi^*)$ if and only if there exists a function $g \in H^2$ such that

$$T_{\bar{b}} f = T_{\bar{a}} g. \tag{23.44}$$

It follows from Theorems 17.8 and 23.2 that this is equivalent to saying that $f \in \mathcal{H}(b)$.

(ii) If we compare (23.44) and (23.42), we have

$$f^+ = g = T_\varphi^* f.$$

Then, (23.41) follows from Theorem 23.8. □

Exercises

Exercise 23.9.1 Let φ be a rational function in the Smirnov class. Show that the functions a and b in the canonical representation of φ are rational functions. Hint: Assume that $\varphi = p/q$, where p and q are polynomials with GCD $(p, q) = 1$, q has no roots in $\mathbb{D}$ and $q(0) > 0$. Note that the function $|p|^2 + |q|^2$ is a nonnegative trigonometric polynomial. Apply the Fejér–Riesz theorem to get a polynomial r without roots in $\mathbb{D}$, with $r(0) > 0$ and such that $|r|^2 = |p|^2 + |q|^2$; see Theorem 27.19. Note now that $a = q/r$ is a rational function and $b = a\varphi = p/r$ is also a rational function. Verify that (a, b) is a pair and $\varphi = b/a$.

Exercise 23.9.2 Let $\varphi \in \mathcal{N}^+$ and $\psi \in H^\infty$. We denote $T_{\bar{\varphi}} = T_\varphi^*$.

(i) Show that $\mathcal{D}(T_\varphi) \subset \mathcal{D}(T_{\bar{\varphi}})$.
Hint: Use Theorem 23.31 and Lemma 23.30.

(ii) Show that, for any $g \in \mathcal{D}(T_\varphi)$, we have

$$T_{\bar{\varphi}} g = P_+(\bar{\varphi} g).$$

Hint: Note that, for any $f \in \mathcal{D}(T_\varphi)$,

$$\langle T_{\bar{\varphi}} g, f\rangle_2 = \langle g, \varphi f\rangle_2 = \langle \bar{\varphi} g, f\rangle_2 = \langle P_+(\bar{\varphi} g), f\rangle_2.$$

Exercise 23.9.3 Let $\varphi \in \mathcal{N}^+$ and $\psi \in H^\infty$. Show that, for any $f \in \mathcal{D}(T_{\bar{\varphi}})$, we have

$$T_{\bar{\varphi}} T_{\bar{\psi}} f = T_{\bar{\varphi}\bar{\psi}} f = T_{\bar{\psi}} T_{\bar{\varphi}} f.$$

Hint: Note that, if $\varphi = a/b$ is the canonical representation of φ, then $\mathcal{D}(T_{\bar{\varphi}}) = \mathcal{H}(b)$ is invariant under $T_{\bar{\psi}}$. Hence $T_{\bar{\psi}} f \in \mathcal{D}(T_{\bar{\varphi}})$. For $g \in \mathcal{D}(T_\varphi)$, we have

$$\begin{aligned}\langle T_{\bar{\varphi}} T_{\bar{\psi}} f, g\rangle_2 &= \langle T_{\bar{\psi}} f, \varphi g\rangle_2 \\ &= \langle f, \psi\varphi g\rangle_2 \\ &= \langle T_{\bar{\psi}\bar{\varphi}} f, g\rangle_2,\end{aligned}$$

which shows that $T_{\bar{\varphi}} T_{\bar{\psi}} f = T_{\bar{\varphi}\bar{\psi}} f$. Argue similarly to prove that $T_{\bar{\psi}} T_{\bar{\varphi}} f = T_{\bar{\varphi}\bar{\psi}} f$.

Notes on Chapter 23

Section 23.1

Theorems 23.2 and 23.3 are due to Sarason [159, lemmas 3, 4 and 5].

Section 23.3

Theorem 23.8 is due to Sarason [159, lemma 2]. The idea of using the element f^+ to compute the norm is very useful and has also been introduced by Sarason in [159]. The power of the method is illustrated by Corollary 23.9. It illustrates very well that the computation of the norm of an element $f \in \mathcal{H}(b)$ is transformed into the resolution of a system $T_{\bar{b}} f = T_{\bar{a}} g$, where we are looking for a solution $g \in H^2$. For instance, the norm of $S^* b$ has been computed by Sarason in [160] using another more difficult method; see Exercise 18.9.5. The computation presented here and based on f^+ is from Sarason's book [166].

In [159], Sarason proved the density of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$, when b is nonextreme; see Corollary 23.10.

The formula of Theorem 23.11 to find the element f^+ by a limiting process is due to Sarason [159].

Exercises 23.3.1, 23.3.2 and 23.3.3 come also from [159].

Section 23.4

The density of polynomials in $\mathcal{M}(\bar{a})$ and $\mathcal{H}(b)$ (in the nonextreme case) proved in Theorem 23.13 is due to Sarason [159, corollary 1].

Section 23.5

Theorem 23.14 and Corollary 23.15 are due to Sarason [160]. In that paper, he is motivated by relating de Branges and Rovnyak's model theory with that of Sz.-Nagy and Foiaş. Thus, he constructs the Sz.-Nagy–Foiaş model of X_b and, for that, he needs to determine the defect operators of the contraction X_b.

Section 23.6

Lemma 23.19 is from [166]. Theorem 23.20 is also due to Sarason [160] and can be rephrased in the context of Sz.-Nagy–Foiaş theory. Indeed, in the case when b is nonextreme, then $\dim \mathcal{D}_{X_b} = 2$ and $\dim \mathcal{D}_{X_b^*} = 1$. Let u_1 and u_2 be a pair of orthogonal unit vectors in $\mathcal{D}_{X_b}$ and let $v = \|S^*b\|_b^{-1} S^*b$ be the unit vector spanning $\mathcal{D}_{X_b^*}$. Then, the operator function Θ_{X_b} is determined by the 1×2 matrix function (θ_1, θ_2), where θ_j is defined by

$$\Theta_{X_b}(\lambda)u_j = \theta_j(\lambda)v \qquad (j = 1, 2).$$

If we replace u_1, u_2 by another orthonormal basis for $\mathcal{D}_{X_b}$, then it will multiply the matrix function (θ_1, θ_2) from the right by a constant 2×2 unit matrix. In [160], Sarason shows that there is a choice of basis (u_1, u_2) such that $\theta_1(\lambda) = \overline{b(\bar{\lambda})}$ and $\theta_2(\lambda) = \overline{a(\bar{\lambda})}$. In this context, Theorem 23.20 says exactly that $S^* \oplus S^*_{|\mathbb{K}'_b}$ is the Sz.-Nagy–Foiaş model of X_b and the projection Q_b implements the unitary equivalence between the operator X_b and its Sz.-Nagy–Foiaş model.

Section 23.7

Theorem 23.22 is due to Guyker [96]. It answers a question raised by de Branges and Rovnyak [65, p. 39]. See also the paper of Leech [116], who obtained other equivalent conditions for a Hilbert space $\mathcal{H}$ to coincide with a de Branges–Rovnyak space $\mathcal{H}(b)$ for some nonextreme function b.

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Section 23.8

The fact that the Cauchy kernel k_w belongs to $\mathcal{H}(b)$ when b is nonextreme, as well as the computation of the norm of k_w, are due to Sarason [160, proposition 1]. The two formulas (23.31) and (23.32) that appear in Theorem 23.23 are also due to Sarason [164, proposition].

Corollary 23.26 is from [159], but we have given a different proof.

Section 23.9

Unbounded Toeplitz operators on the Hardy space H^2 arise often with symbols belonging to $L^2(\mathbb{T})$. However, there are natural questions that lead to Toeplitz operators having more restrictive symbols, in particular with symbols in the Smirnov class. We mention interesting works of Helson [101], Suárez [182] and Seubert [174]. The links between $\mathcal{H}(b)$ spaces and the domain of the adjoint of Toeplitz operators with symbols in the Smirnov class are due to Sarason [170].