BULL. AUSTRAL. MATH. SOC. 26A45
VOL. 18 (1978), 407-420.

Further results on an integral representation of functions of generalised variation

A.M. Russell

In this paper we present further properties of the kth variation of a function, and obtain an integral representation for a function having bounded kth variation and an absolutely continuous $(k-1)$th derivative. The absolute continuity requirement replaces a previous stronger condition that required the kth derivative of a function to be continuous except on a set of Lebesgue measure zero.

1. Introduction

It is a well known result that if f is an absolutely continuous function on $[a, b]$, then f is of bounded variation, and its variation is given by

$$V_1(f; a, b) = \int_a^b |f'(t)|dt .$$

In [3] the author extended this result to functions which have bounded kth variation and which have the additional restriction that the kth derivative is continuous except on a set of Lebesgue measure zero. In this paper we weaken the additional restriction by showing that the kth total variation of a function f can be written in the form

Received 6 March 1978. The author thanks Dr C.J.F. Upton for stimulating discussions relating to the contents of this paper.

$$(k-1)!V_k(f;\ a,\ b) = \int_a^b |f^{(k)}(t)|dt\ ,$$

when $f^{(k-1)}$ is absolutely continuous on $[a,\ b]$.

In order to arrive at the more general result just outlined it was found expedient to work with two definitions of bounded kth variation, one defined with quite arbitrary subdivisions $a = x_0,\ x_1,\ \ldots,\ x_n = b$ of $[a,\ b]$, and the other using subdivisions in which all subintervals $[x_{i-1},\ x_i]$ are of equal length. We show first that provided continuous functions are used, we obtain the same class of functions irrespective of which subdivisions are used.

2. Notation and preliminaries

DEFINITION 1. We shall say that a set of points $x_0,\ x_1,\ \ldots,\ x_n$ is a π-subdivision of $[a,\ b]$ when $a \le x_0 < x_1 < \ldots < x_n = b$.

Before introducing the two definitions of bounded kth variation, we need the definition and some properties of kth divided differences, and for this purpose we refer the reader to [2].

DEFINITION 2. The total kth variation of a function f on $[a,\ b]$ is defined by

$$V_k(f;\ a,\ b) = \sup_{\pi} \sum_{i=0}^{n-k} \left(x_{i+k}-x_i\right) \left|Q_k\left(f;\ x_i,\ \ldots,\ x_{i+k}\right)\right|\ .$$

If $V_k(f;\ a,\ b) < \infty$, we say that f is of bounded kth variation on $[a,\ b]$, and write $f \in BV_k[a,\ b]$. The summations over which the supremum is taken are called approximating sums for $V_k(f;\ a,\ b)$.

We now concern ourselves with subdivisions of $[a,\ b]$ in which all sub-intervals are of equal length. More formally, if $h > 0$, then we will denote by π_h a subdivision $x_0,\ x_1,\ \ldots,\ x_n$ of $[a,\ b]$ such that $a = x_0 < x_1 < \ldots < x_n \le b$, where $x_i - x_{i-1} = h$, $i = 1,\ 2,\ \ldots,\ n$, and $0 \le b-x_n \le h$. In order to introduce the second definition of bounded kth

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

variation we make use of the difference operator Δ_h^k defined by

$$\Delta_h^1 f(x) = f(x+h) - f(x) ,$$

and

$$\Delta_h^k f(x) = \Delta_h^1 \left(\Delta_h^{k-1} f(x) \right) .$$

DEFINITION 3. If f is continuous on $[a, b]$, then we define total kth variation of f on $[a, b]$ (restricted form) by

$$\overline{V}_k(f; a, b) = \sup_{\pi_h} \sum_{i=0}^{n-k} \left| \frac{\Delta_h^k f(x_i)}{h^{k-1}} \right| .$$

If $\overline{V}_k(f; a, b) < \infty$ we say that f is of restricted bounded kth variation on $[a, b]$, and write $f \in \overline{BV}_k[a, b]$.

If we denote, for brevity, $C[a, b]$ by C, $BV_k[a, b]$ by BV_k, and $\overline{BV}_k[a, b]$ by $\overline{BV}_k$, then we show subsequently that

$$C \cap BV_k = \overline{BV}_k .$$

We point out at this stage that the restriction to continuous functions is not nearly as severe as it first may appear, because functions belonging to $BV_k[a, b]$ when $k \geq 2$ are automatically continuous. (See Theorem 4 of [2].)

Our final definition deals with synchronized sets of points.

DEFINITION 4. Let $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ be two sets of points belonging to $[a, b]$ such that $x_0 < x_1 < \ldots < x_n$ and $y_0 < y_1 < \ldots < y_n$. If

$$y_i = x_{i+1} , \quad i = 0, 1, \ldots, n-1 ,$$

or

$$x_i = y_{i+1} , \quad i = 0, 1, \ldots, n-1 ,$$

we say that the two sets of points are synchronized; otherwise, we say

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

that the two sets of points are not synchronized.

The following theorem will be a useful result. Since it is well known, and appears in the literature, for example, in §18 of [1], a proof will not be given.

THEOREM 1. *Let* F *be absolutely continuous on* $[a, b]$ *, written in the form* $F(x) = \int_a^x f(t)dt$ *,* $a \le x \le b$ *. Then* F *is of bounded variation on* $[a, b]$ *, and*

$$V_1(F; a, b) = \int_a^b |f(t)|dt .$$

We now direct our attention to establishing the result

$$C \cap BV_k = \overline{BV}_k , \quad k \ge 1 .$$

LEMMA 1. *Let* $I_1, I_2, \ldots, I_n$ *be a set of* n *adjoining closed intervals on the real line having lengths* $p_1/q_1, p_2/q_2, \ldots, p_n/q_n$ *respectively, where* $p_1, p_2, \ldots, p_n$ *,* $q_1, q_2, \ldots, q_n$ *are positive integers. Then it is possible to subdivide the intervals* $I_1, I_2, \ldots, I_n$ *into sub-intervals of equal length.*

The proof is easy and will be omitted.

LEMMA 2. *If* $k \ge 1$ *, then* $C \cap BV_k \subset \overline{BV}_k$ *, using abbreviated notation.*

Proof. This is easy and will not be included.

LEMMA 3. *If* $k \ge 1$ *, then* $C \cap BV_k \supset \overline{BV}_k$.

Proof. Let us suppose that f is continuous, belongs to $\overline{BV}_k[a, b]$, but $f \notin BV_k[a, b]$. Then for an arbitrarily large number K , and an arbitrarily small positive number ε , there exists a subdivision $\pi_1(x_0, x_1, \ldots, x_n)$ of $[a, b]$ such that

$$S_{\pi_1} \equiv \sum_{i=0}^{n-k} (x_{i+k}-x_i) |Q_k(f; x_i, \ldots, x_{i+k})| > K + \varepsilon .$$

If not all the lengths $(x_{i+1}-x_i)$, $i = 0, 1, \ldots, n-1$ are rational, then because f is continuous we can obtain a subdivision $\pi_2(y_0, y_1, \ldots, y_n)$ of $[a, b]$ in which all the lengths $(y_{i+1}-y_i)$, $i = 0, 1, \ldots, n-1$ are rational, and such that $|S_{\pi_1}-S_{\pi_2}| < \varepsilon$, S_{π_2} being the approximating sum of $V_k(f; a, b)$ corresponding to the π_2 subdivision. Consequently,

$$\begin{aligned} S_{\pi_2} &\geq S_{\pi_1} - |S_{\pi_1}-S_{\pi_2}| \\ &> K . \end{aligned}$$

In the π_2 subdivision, all sub-intervals have rational length, so we can apply Lemma 1 to obtain a π_h subdivision of $[a, b]$ in which each sub-interval has length h . If S_{π_h} is the corresponding approximating sum for $\overline{V}_k(f; a, b)$, then it follows from Theorem 3 of [2] that

$$\frac{1}{(k-1)!} S_{\pi_h} \geq S_{\pi_2} > K ,$$

since for any π_h subdivision, and each $i = 0, 1, \ldots, n-k$,

$$\frac{\Delta_h^k f(x_i)}{h^{k-1}} = (k-1)!(x_{i+k}-x_i)Q_k(f; x_i, \ldots, x_{i+k}) .$$

Thus $S_{\pi_h} > (k-1)!K$, and this is a contradiction to the assumption that $f \in \overline{BV}_k[a, b]$. Hence $f \in \overline{BV}_k[a, b]$, and so $\overline{BV}_k \subset C \cap BV_k$.

THEOREM 2. *If* $k \geq 1$ *, then* $C \cap BV_k = \overline{BV}_k$ *; and if* f *is a continuous function on* $[a, b]$ *, then*

(1) $$\overline{V}_k(f; a, b) = (k-1)!V_k(f; a, b) , \quad k \geq 1 .$$

Proof. The first part follows from Lemmas 2 and 3. For the second part we first observe that

(2) $$\overline{V}_k(f; a, b) \leq (k-1)!V_k(f; a, b) .$$

Let $\varepsilon > 0$ be arbitrary. Then there exists a π_1 subdivision of $[a, b]$

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

and the corresponding approximating sum S_{π_1} to $V_k(f;\, a,\, b)$ such that

$$S_{\pi_1} > V_k(f;\, a,\, b) - \frac{\varepsilon}{2(k-1)!} .$$

If not all the sub-intervals of π_1 have rational lengths, then we can proceed as in Lemma 3 to obtain a π_h subdivision of $[a,\, b]$ in which all sub-intervals are of equal length h. Then, if S_{π_h} is the corresponding approximating sum to $\overline{V}_k(f;\, a,\, b)$, we can show that

$$\begin{aligned} \frac{1}{(k-1)!} S_{\pi_h} &\geq S_{\pi_1} - \frac{\varepsilon}{2(k-1)!} \\ &> V_k(f;\, a,\, b) - \frac{\varepsilon}{(k-1)!} . \end{aligned}$$

Consequently,

$$\begin{aligned} \overline{V}_k(f;\, a,\, b) &> S_{\pi_h} \\ &> (k-1)!V_k(f;\, a,\, b) - \varepsilon , \end{aligned}$$

from which it follows that $\overline{V}_k(f;\, a,\, b) \geq (k-1)!V_k(f;\, a,\, b)$. This inequality together with (2) gives (1).

We now proceed towards an application of the result, $C \cap BV_k = \overline{BV}_k$.

3. Main results

Let the set of points $a = x_0,\, x_1,\, \ldots,\, x_{n-1}$, $x_n = b$ be a π subdivision of $[a,\, b]$, and let t be a real number such that $0 \leq t \leq 1$. We shall have need to consider the two related sets of points

$$(3) \qquad \begin{cases} x_{i+1} + t\left(x_s - x_{i+1}\right) , & \text{where } s = i+2,\, \ldots,\, i+k , \\ \text{and} & \\ x_i + t\left(x_s - x_i\right) , & \text{where } s = i+1,\, \ldots,\, i+k-1 . \end{cases}$$

In relation to the sets of points (3) we shall consider the sum

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

$$(4)\quad \sum_{i=0}^{n-k} |Q_{k-2}(f;\, x_{i+1}+t(x-x_{i+1});\, x_{i+2},\, \ldots,\, x_{i+k}) - - Q_{k-2}(f;\, x_i+t(x-x_i);\, x_{i+1},\, \ldots,\, x_{i+k-1})| \;.$$

Normally, the sum (4) would be an approximating sum for $V_{k-1}(f;\, a,\, b)$, but since the two sets of points (3) are not synchronized subdivisions, further investigation is required to determine the relationship between (4) and $V_{k-1}(f;\, a,\, b)$. In view of Theorem 2, we simplify our procedure by considering π_h subdivisions in which each sub-interval $[x_{i-1},\, x_i]$ is of length h . When $k \geq 2$ and $f \in BV_k[a,\, b]$, f is continuous, and so by Theorem 2, there is no loss of generality in considering π_h subdivisions. Thus we can write (3) in the more convenient form

$$x_{i+1}+th,\, x_{i+1}+2th,\, \ldots,\, x_{i+1}+(k-1)th \;,$$

and

$$x_i+th,\, x_i+2th,\, \ldots,\, x_i+(k-1)th \;.$$

The relative distribution of these two sets of points depends upon the value of t , so we now discuss various cases, starting with the simplest.

The case $t = 0$. This is trivial as each divided difference in (4) is zero when $t = 0$.

The case $0 < t \leq \frac{1}{k-2}$. This gives rise to the distribution

$$x_i+th < x_i+2th < \ldots < x_i+(k-1)th \leq x_{i+1}+th < x_{i+1}+2th < \ldots < x_{i+1}+(k-1)th \;.$$

That (4) is again dominated by $V_{k-1}(f;\, a,\, b)$ follows readily. The cases $\frac{1}{p} < t \leq \frac{1}{p-1}$, $p = k-3,\, \ldots,\, 2$ are similar in character, with the "overlap" of the two sets "increasing" as p decreases. We discuss in some detail the situation when $p = 2$.

The case $\frac{1}{2} < t \leq 1$. First of all if $t = 1$, (4) is clearly dominated by $V_{k-1}(f;\, a,\, b)$. Hence we suppose that $\frac{1}{2} < t < 1$, and present the following:

THEOREM 3. *If* $\frac{1}{2} < t < 1$, *then*

(5) $$\sum_{i=0}^{n-k} |Q_{k-2}(f;\ x_{i+1}+t(x-x_{i+1});\ x_{i+2},\ \ldots,\ x_{i+k}) - Q_{k-2}(f;\ x_i+t(x-x_i);\ x_{i+1},\ \ldots,\ x_{i+k-1})| \le V_{k-1}(f;\ a,\ b)\ .$$

Proof. Suppose that t is irrational, so that points of different sub-divisions do not coincide.

Let $n - k = 1$, so that we consider the three sets of points

$$x_i+th,\ x_i+2th,\ \ldots,\ x_i+(k-1)th\ ,\quad i = 0,\ 1,\ 2\ .$$

The sets of points corresponding to $i = 0$ and $i = 1$ are distributed relative to one another as follows:

$$x_0+th < x_0+2th < x_1+th < x_0+3th < x_1+2th < \ldots$$
$$\ldots < x_1+(k-3)th < x_0+(k-1)th < x_1+(k-2)th < x_1 + (k-1)th.$$

In other words, after the first two points $x_0 + th$ and $x_0 + 2th$, the points alternate until $x_0 + (k-1)th$, and this is finally followed by $x_1 + (k-2)th$ and $x_1 + (k-1)th$. However, when the third set of points is added some ambiguity occurs because $x_2 + th$, definitely greater than $x_1 + 2th$, may be either greater than or less than $x_0 + 4th$, depending upon the value of t in $(\frac{1}{2},\ 1)$. To be definite, let us assume that $x_1+2th < x_2+th < x_0+4th$, and proceed. An analysis similar to the following will apply if we assume $x_2+th > x_0+4th$. Accordingly, relabel the set of $(3k-3)$ points $y_1,\ y_2,\ \ldots,\ y_{3k-3}$, where

$$y_1 = x_0+th,\ y_2 = x_0+2th,\ y_3 = x_1+th,\ y_4 = x_0+3th,$$
$$y_5 = x_1+2th,\ y_6 = x_2+th,\ y_7 = x_0+4th,\ \ldots,\ y_{3k-3} = x_2+(k-1)th\ .$$

Consequently, using Theorem 1, Corollary of [2], and writing $Q_{k-2}(y_i,\ \ldots,\ y_{i+k-2})$ instead of $Q_{k-2}(f;\ y_i,\ \ldots,\ y_{i+k-2})$, we obtain

$$\sum_{i=0}^{n-k} |Q_{k-2}(f;\ x_i+t(x-x_i);\ x_{i+1},\ \ldots,\ x_{i+k-1}) -$$

$$-Q_{k-2}(f;\ x_{i+1}+t(x-x_{i+1});\ x_{i+2},\ \ldots,\ x_{i+k})|$$

$$= |\beta_1 Q_{k-2}(y_1,\ \ldots,\ y_{k-1}) + \beta_2 Q_{k-2}(y_2,\ \ldots,\ y_k) + \ldots$$

$$\ldots + \beta_{2k-5}Q_{k-2}(y_{2k-5},\ \ldots,\ y_{3k-7}) - \alpha_3 Q_{k-2}(y_3,\ \ldots,\ y_{k+1}) -$$

$$- \alpha_4 Q_{k-2}(y_4,\ \ldots,\ y_{k+2}) - \ldots - \alpha_{2k-3}Q_{k-2}(y_{2k-3},\ \ldots,\ y_{3k-5})| +$$

$$+ |\alpha_3 Q_{k-2}(y_3,\ \ldots,\ y_{k+1}) + \alpha_4 Q_{k-2}(y_4,\ \ldots,\ y_{k+2}) +$$

$$\ldots + \alpha_{2k-3}Q_{k-2}(y_{2k-3},\ \ldots,\ y_{3k-5}) - \gamma_5 Q_{k-2}(y_5,\ \ldots,\ y_{k+3}) -$$

$$-\gamma_6 Q_{k-2}(y_6,\ \ldots,\ y_{k+4}) - \ldots - \gamma_{2k-1}Q_{k-2}(y_{2k-1},\ \ldots,\ y_{3k-3})|\ ,$$

where the α's, β's , and γ's are all non-negative, and

$$\beta_1 + \beta_2 + \ldots + \beta_{2k-5} = \alpha_3 + \alpha_4 + \ldots + \alpha_{2k-3} = \gamma_5 + \gamma_6 + \ldots + \gamma_{2k-1} = 1\ .$$

After some re-arrangement, the summation can be shown to equal

$$|\beta_1\{Q(y_1,\ \ldots,\ y_{k-1}) - Q(y_2,\ \ldots,\ y_k)\} +$$

$$+ (\beta_1+\beta_2)\{Q(y_2,\ \ldots,\ y_k) - Q(y_3,\ \ldots,\ y_{k+1})\}$$

$$+ (\beta_1+\beta_2+\beta_3-\alpha_3)\{Q(y_3,\ \ldots,\ y_{k+1}) - Q(y_4,\ \ldots,\ y_{k+2})\} +$$

$$\ldots + (\beta_1 + \beta_2 + \beta_3 + \ldots + \beta_{2k-5} - \alpha_3 - \alpha_4 - \ldots - \alpha_{2k-5}) \times$$

$$\{Q(y_{2k-5},\ \ldots,\ y_{3k-7}) - Q(y_{2k-4},\ \ldots,\ y_{3k-6})\} +$$

$$+ (\beta_1 + \ldots + \beta_{2k-5} - \alpha_3 - \alpha_4 - \ldots - \alpha_{2k-4}) \times$$

$$\{Q(y_{2k-4},\ \ldots,\ y_{3k-6}) - Q(y_{2k-3},\ \ldots,\ y_{3k-5})\}|$$

$$+ |\alpha_3\{Q(y_3,\ \ldots,\ y_{k+1}) - Q(y_4,\ \ldots,\ y_{k+2})\} +$$

$$+ (\alpha_3+\alpha_4)\{Q(y_4,\ \ldots,\ y_{k+2}) - Q(y_5,\ \ldots,\ y_{k+3})\} +$$

$$+ (\alpha_3+\alpha_4+\alpha_5-\gamma_5)\{Q(y_5,\ \ldots,\ y_{k+3}) - Q(y_6,\ \ldots,\ y_{k+4})\} +$$

$$\ldots + (\alpha_3 + \alpha_4 + \ldots + \alpha_{2k-3} - \gamma_5 - \gamma_6 - \ldots - \gamma_{2k-3}) \times$$

$$\{Q(y_{2k-3},\ \ldots,\ y_{3k-5}) - Q(y_{2k-2},\ \ldots,\ y_{3k-4})\} +$$

$$+ (\alpha_3 + \alpha_4 + \ldots + \alpha_{2k-3} - \gamma_5 - \gamma_6 - \ldots - \gamma_{2k-2}) \times$$

$$\{Q(y_{2k-2},\ \ldots,\ y_{3k-4}) - Q(y_{2k-1},\ \ldots,\ y_{3k-3})\}|$$

$$\le \beta_1|Q(y_1,\ \ldots,\ y_{k-1}) - Q(y_2,\ \ldots,\ y_k)| + (\beta_1+\beta_2) \times$$

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

$$
\begin{aligned}
&|Q(y_2, \ldots, y_k)-Q(y_3, \ldots, y_{k+1})| + (\beta_1+\beta_2+\beta_3-\alpha_3+\alpha_3) \times \\
&|Q(y_3, \ldots, y_{k+1})-Q(y_4, \ldots, y_{k+2})| + (\beta_1+\beta_2+\beta_3+\beta_4-\alpha_3-\alpha_4+\alpha_3+\alpha_4) \times \\
&|Q(y_4, \ldots, y_{k+2})-Q(y_5, \ldots, y_{k+3})| + \ldots + \\
&+ (1 - \alpha_3 - \alpha_4 - \ldots - \alpha_{2k-4} + \alpha_3 + \alpha_4 + \ldots + \alpha_{2k-4} - \gamma_5 - \gamma_6 - \gamma_{2k-4}) \times \\
&|Q(y_{2k-4}, \ldots, y_{3k-6})-Q(y_{2k-3}, \ldots, y_{3k-5})| + \\
&+ (1 - \gamma_5 - \gamma_6 - \ldots - \gamma_{2k-3})|Q(y_{2k-3}, \ldots, y_{3k-5})-Q(y_{2k-2}, \ldots, y_{3k-4})| \\
&+ (1 - \gamma_5 - \gamma_6 - \ldots - \gamma_{2k-2})|Q(y_{2k-2}, \ldots, y_{3k-4})-Q(y_{2k-1}, \ldots, y_{3k-3})| \\
\le &\sum_{i=0}^{2k-2} |Q(y_i, \ldots, y_{i+k-2})-Q(y_{i+1}, \ldots, y_{i+k-1})| \le V_{k-1}(f; a, b) .
\end{aligned}
$$

A similar, but longer, analysis applies for higher values of $n - k$.

Finally, let t be a rational number. Then, since f is continuous, sets of points $x_i + st'h$ and $x_{i+1} + st'h$, $s = 1, 2, \ldots, k-1$, where t' is irrational, exist such that the sums (4) corresponding to t and t' differ by an arbitrarily small specified ε. Thus (5) is still valid, and we conclude the proof.

THEOREM 4. *If* $k \ge 3$, *and* $f \in BV_k[a, b]$, *then* $f' \in BV_{k-1}[a, b]$ *and*

$$V_{k-1}(f'; a, b) \le (k-1)V_k(f; a, b) . \tag{6}$$

Proof. That $f' \in BV_{k-1}[a, b]$ follows from Theorem 12 of [2]. Now see Theorem 9 of [2], but observe that the "k^2" in the second last line of the proof of that theorem can be replaced by "k".

THEOREM 5. *If* $k \ge 3$, *and* $f \in BV_k[a, b]$, *then* $f' \in BV_{k-1}[a, b]$ *and*

$$V_{k-1}(f'; a, b) \ge (k-1)V_k(f; a, b) .$$

Proof of inequality. It follows from Theorem 11 of [2] that f' is continuous in $[a, b]$, so we can write

$$f(x) = f(a) + \int_a^x f'(t)dt .$$

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

Hence, using a property of kth divided differences,

(7)

$$|Q_{k-1}(f;\ x_{i+1},\ \ldots,\ x_{i+k})-Q_{k-1}(f;\ x_i,\ \ldots,\ x_{i+k-1})|$$

$$= \left|Q_{k-2}\left\{\frac{f(x)-f(x_{i+1})}{x-x_{i+1}};\ x_{i+2},\ \ldots,\ x_{i+k}\right\}-Q_{k-2}\left\{\frac{f(x)-f(x_i)}{x-x_i};\ x_{i+1},\ \ldots,\ x_{i+k-1}\right\}\right|$$

$$= \left|Q_{k-2}\left\{\int_0^1 f'\left(x_{i+1}+t\left(x-x_{i+1}\right)\right)dt;\ x_{i+2},\ \ldots,\ x_{i+k}\right\}\right.$$

$$\left.-Q_{k-2}\left\{\int_0^1 f'\left(x_i+t\left(x-x_i\right)\right)dt;\ x_{i+1},\ \ldots,\ x_{i+k-1}\right\}dt\right|$$

$$= \left|\int_0^1 \{Q_{k-2}(f'\left(x_{i+1}+t\left(x-x_{i+1}\right)\right);\ x_{i+2},\ \ldots,\ x_{i+k})\right.$$

$$\left.-Q_{k-2}(f'\left(x_i+t\left(x-x_i\right)\right);\ x_{i+1},\ \ldots,\ x_{i+k-1})\}dt\right|$$

$$= \left|\int_0^1 \{Q_{k-2}(f'(x);\ x_{i+1}+t\left(x_{i+2}-x_{i+1}\right),\ \ldots,\ x_{i+1}+t\left(x_{i+k}-x_{i+1}\right))\right.$$

$$\left.-Q_{k-2}(f'(x);\ x_i+t\left(x_{i+1}-x_i\right),\ \ldots,\ x_i+t\left(x_{i+k-1}-x_i\right))\}\right|t^{k-2}dt\ .$$

Therefore, using Theorem 3, we obtain

$$\sum_{i=0}^{n-k} |Q_{k-1}(f;\ x_{i+1},\ \ldots,\ x_{i+k})-Q_{k-1}(f;\ x_i,\ \ldots,\ x_{i+k-1})|$$

$$\le V_{k-1}(f';\ a,\ b)\int_0^1 t^{k-2}dt = \frac{1}{k-1}V_{k-1}(f';\ a,\ b)\ .$$

We can now conclude that

(8) $$(k-1)V_k(f;\ a,\ b) \le V_{k-1}(f';\ a,\ b)\ ,$$

as required.

Combining (6) and (8) gives us

THEOREM 6. *If* $k \ge 3$, *and* $f \in BV_k[a,\ b]$, *then* $f' \in BV_{k-1}[a,\ b]$, *and*

(9) $$V_{k-1}(f';\ a,\ b) = (k-1)V_k(f;\ a,\ b)\ .$$

We now treat the case $k = 2$ separately, this case requiring the

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

extra hypothesis that f' exists throughout $[a, b]$.

THEOREM 7. *If* $f \in BV_2[a, b]$ *and* f' *exists in* $[a, b]$ *, then* $f' \in BV[a, b]$ *and*

$$V_2(f; a, b) = V_1(f'; a, b) .$$

Proof. It follows from Theorem 9 of [2] that

$$V_1(f'; a, b) \le V_2(f; a, b) . \tag{10}$$

To establish the reverse inequality, let $a = x_0, x_1, \ldots, x_n = b$ be any subdivision of $[a, b]$. Then

$$\sum_{i=0}^{n-2} |Q_1(f; x_{i+1}, x_{i+2}) - Q_1(f; x_i, x_{i+1})|$$

$$= \sum_{i=0}^{n-2} |f'(\eta_{i+1}) - f'(\eta_i)| , \text{ where } x_i < \eta_i < x_{i+1} , \quad i = 0, 1, \ldots, n-2 ,$$

$$\le V_1(f'; a, b) .$$

Therefore,

$$V_2(f; a, b) \le V_1(f'; a, b) . \tag{11}$$

From (10) and (11) it is now clear that

$$V_2(f; a, b) = V_1(f'; a, b) . \tag{12}$$

We are now in a position to offer more general versions of Theorems 3 and 4 of [3].

THEOREM 8. *If* $f \in BV_k[a, b]$, $k \ge 3$ *, then*

$$(k-1)!V_k(f; a, b) = V_2(f^{(k-2)}; a, b) .$$

Furthermore, if $k \ge 2$ *, and* $f^{(k-1)} \in BV_1[a, b]$ *, then*

$$(k-1)!V_k(f; a, b) = V_1(f^{(k-1)}; a, b) .$$

Proof. It follows from Theorem 12 of [2] that $f^{(r)} \in BV_{k-r}[a, b]$, $r = 1, 2, \ldots, k-2$. Successive applications of (9), and a final

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

application of Theorem 7, give the required results.

THEOREM 9. *Let* f *be a function such that* $f^{(k-1)}$ *is absolutely continuous on* $[a, b]$. *Then* $f^{(k-s)} \in BV_s[a, b]$, $s = 1, 2, \ldots, k$, *and, in particular,*

$$(13)\qquad (k-1)!V_k(f; a, x) = \int_a^x |f^{(k)}(t)|dt , \quad a \le x \le b .$$

Proof. Since $f^{(k-1)}$ is absolutely continuous on $[a, b]$, it is also of bounded (first) variation on that interval. It follows from repeated applications of Lemma 3 of [4] that

$$f^{(k-s)} \in BV_s[a, b] , \quad s = 1, 2, \ldots, k .$$

Consequently, from the second part of the previous theorem, we conclude that

$$(k-1)!V_k(f; a, b) = V_1\left(f^{(k-1)}; a, b\right) ,$$

and

$$V_1\left(f^{(k-1)}; a, b\right) = \int_a^b |f^{(k)}(t)|dt ,$$

using Theorem 1.

REMARK. In view of (1), (13) can be written in the more elegant form

$$\overline{V}_k(f; a, x) = \int_a^x |f^{(k)}(t)|dt , \quad a \le x \le b .$$

References

[1] Edwin Hewitt, Karl Stromberg, *Real and abstract analysis. A modern treatment of the theory of functions of a real variable* (Springer-Verlag, Berlin, Heidelberg, New York, 1965. Second printing corrected: Springer-Verlag, Berlin, Heidelberg, New York, 1969. Third printing: Graduate Texts in Mathematics, 25. Springer-Verlag, New York, Heidelberg, Berlin, 1975).

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press

[2] A.M. Russell, "Functions of bounded kth variation", *Proc. London Math. Soc.* (3) **26** (1973), 547-563.

[3] A.M. Russell, "An integral representation for a generalised variation of a function", *Bull. Austral. Math. Soc.* **11** (1974), 225-229.

[4] A.M. Russell, "Stieltjes-type integrals", *J. Austral. Math. Soc. Ser. A* **20** (1975), 431-448.

Department of Mathematics,
University of Melbourne,
Parkville,
Victoria.

https://doi.org/10.1017/S0004972700008273 Published online by Cambridge University Press