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APPROXIMATION BY PARTIAL ISOMETRIES

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1. Introduction

Let $B(H)$ be the algebra of bounded linear operators on a complex separable Hilbert space H. The problem of operator approximation is to determine how closely each operator $T\in B(H)$ can be approximated in the norm by operators in a subset L of $B(H)$. This problem is initiated by P. R. Halmos [**3**] when he considered approximating operators by the positive ones. Since then, this problem has been attacked with various classes L: the class of normal operators whose spectrum is included in a fixed nonempty closed subset of the complex plane [**4**], the classes of unitary operators [**6**] and invertible operators [**1**]. The purpose of this paper is to study the approximation by partial isometries.

In Section 2 below, after some preliminary preparations we first determine the distance from an arbitrary operator T to the class of isometries in terms of some operator parameters of T. This is based on, and closely related to, the work of D. D. Rogers [**6**] on unitary approximations. We also settle the related approximation problem for the larger class consisting of isometries and coisometries.

Section 3 contains our main result on approximation by partial isometries. We determine the distance from T to such operators in terms of the "singular values" of T. The proof is inspired by the work of Halmos on the normal spectral approximation [**4**]. In this case, the distance is always attained by some partial isometry.

2. Isometries

We start by considering polar decompositions of an operator. Recall that for any operator T, $\operatorname{ind} T=\dim\ker T-\dim\ker T^*$ if at least one of these numbers is finite and $\operatorname{ind} T=0$ otherwise. The proof of the next lemma is contained in [**5**, Solution 135].

Lemma 2.1. *Let T be an operator on H. Then $T=VP$, where V is a nonunitary isometry, a nonunitary coisometry or a unitary operator according as whether* $\operatorname{ind} T<0$, >0 *or* $=0$, *and* $P=(T^*T)^{1/2}$ *is a positive operator.*

For an operator T, let $\sigma(T)$ (resp. $\sigma_e(T)$) denote its spectrum (resp. essential spectrum); let $m(T)=\inf\{\lambda:\lambda\in\sigma((T^*T)^{1/2})\}$ (resp. $m_e(T)=\inf\{\lambda:\lambda\in\sigma_e((T^*T)^{1/2})\}$) be its *minimum*

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modulus (resp. *essential minimum modulus*). It is known that (1) $m(T)=\inf\{\|Tx\|:\|x\|=1\}$, (2) $m(T)>0$ (resp. $m_e(T)>0$) if and only if T is left invertible (resp. left Fredholm) and (3) if L is a left inverse of T (resp. L is such that $LT-1$ is compact), then

$$m(T)=\frac{1}{\|L\|}\left(\text{resp. } m_e(T)=\frac{1}{\|L\|_e}\right).$$

Analogous results hold when T is replaced by T^* and "left" by "right". (Readers are referred to [**1**, Theorems 1 and 2] for these and other properties.) The next lemma relates the distance between two operators to their minimum moduli.

Lemma 2.2. *Let T and S be operators on H. Then*

(1) $\|T-S\|\geqq|m(T)-m(S)|$;

(2) $\|T-S\|\geqq|m(T)-m(S^*)|$ *if T is left invertible and S is right invertible.*

Proof. (1) Let $\{x_n\}$ be a sequence of unit vectors in H such that $\lim\|Sx_n\|=m(S)$. Then $\|T-S\|\geqq\|(T-S)x_n\|\geqq\|Tx_n\|-\|Sx_n\|\geqq m(T)-\|Sx_n\|$. Letting $n\to\infty$, we obtain $\|T-S\|\geqq m(T)-m(S)$. Since $\|T-S\|=\|S-T\|\geqq m(S)-m(T)$ from above, we have $\|T-S\|\geqq|m(T)-m(S)|$.

(2) Let L be a left inverse of T and R be a right inverse of S. Since

$$\|L\|\cdot\|T-S\|\cdot\|R\|\geqq\|R-L\|\geqq|\,\|R\|-\|L\|\,|,$$

we have

$$\|T-S\|\geqq\left|\frac{1}{\|L\|}-\frac{1}{\|R\|}\right|=|m(T)-m(S^*)| \quad \text{as asserted.}$$

Now we are ready for the approximation by isometries.

Theorem 2.3. *For any operator T,*

$$\inf\{\|T-V\|:V\text{ isometry}\}=\begin{cases}\max\{\|T\|-1,1-m(T)\} & \text{if ind } T\leqq 0\\ \max\{\|T\|-1,1+m_e(T^*)\} & \text{otherwise.}\end{cases}$$

The infimum is attained if ind $T\leqq0$.

Proof. Let $\alpha=\inf\{\|T-V\|:V\text{ isometry}\}$. If ind $T\leqq0$, then $T=VP$, where V is an isometry and $P=(T^*T)^{1/2}\geqq0$ by Lemma 2.1. For this V we have

$$\|T-V\|=\|VP-V\|=\|P-1\|=\max\{\|P\|-1,1-m(P)\}=\max\{\|T\|-1,1-m(T)\}.$$

This shows that $\alpha\leqq\max\{\|T\|-1,1-m(T)\}$. For the other direction, since any isometry

V satisfies $\|V\| = m(V) = 1$, we have

$$\|T - V\| \geqq \|T\| - \|V\| = \|T\| - 1$$

and

$$\|T - V\| \geqq m(V) - m(T) = 1 - m(T)$$

by Lemma 2.2(1). It follows that $\alpha \geqq \max\{\|T\| - 1, 1 - m(T)\}$.

Next assume that $\operatorname{ind} T > 0$. We have $\alpha \leqq \inf\{\|T - U\| : U \text{ unitary}\} = \max\{\|T\| - 1, 1 + m_e(T^*)\}$ by [**6**, Theorem 1.3]. Since $\operatorname{ind} T > 0 \geqq \operatorname{ind} V$ for any isometry V, [**6**, Theorem 2.1] is applicable showing that $\|T - V\| \geqq 1 + m_e(T^*)$. Thus $\alpha \geqq \max\{\|T\| - 1,\ 1 + m_e(T^*)\}$ completing the proof.

In the following, we show that when $\operatorname{ind} T > 0$ the distance from T to the class of isometries may not be attained. The proof is modified from the one for [**6**, Theorem 1.4 (ii)]. But first we need some lemmas.

Lemma 2.4. *For any contraction* T *(i.e.,* $\|T\| \leqq 1$*)*, $\operatorname{ind}(T - 1) = 0$.

Proof. This follows from the fact that $\ker(T - 1) = \ker(T^* - 1)$ (cf. [**7**, Proposition I.3.1]).

Lemma 2.5. *Let T be an operator and V be an isometry on H. Then* $\operatorname{ind} T \leqq \operatorname{ind} V^*T$.

Proof. Since $\ker T \subseteq \ker V^*T$, we have $\dim\ker T \leqq \dim\ker V^*T$. On the other hand, $V(\ker T^*V) \subseteq \ker T^*$ implies that, since V is an isometry, $\dim\ker T^*V \leqq \dim\ker T^*$. Thus $\operatorname{ind} T = \dim\ker T - \dim\ker T^* \leqq \dim\ker V^*T - \dim\ker T^*V = \operatorname{ind} V^*T$ as asserted.

Theorem 2.6. *If T is an operator with* $\operatorname{ind} T > 0$ *and* $\inf\{\|T - V\| : V \text{ isometry}\} = 1$, *then this infimum is not attained for any isometry.*

Proof. Assume that W is an isometry such that $\|T - W\| = \inf\{\|T - V\| : V \text{ isometry}\} = 1$. Then $\|W^*T - 1\| \leqq 1$ and it follows from Lemma 2.4 that $\operatorname{ind} W^*T = 0$. By Lemma 2.5, this leads to $\operatorname{ind} T \leqq 0$, a contradiction.

If T is the adjoint of a unilateral weighted shift with weights α_n, $0 < \alpha_n \leqq 2$, such that $\lim \alpha_n = 0$, then T is not one-to-one while T^* is and $\operatorname{ran} T^*$ is not closed (cf. [**5**, Solution 96]). Hence $\operatorname{ind} T = \dim\ker T > 0$ and $m_e(T^*) = 0$ (cf. [**1**, Theorem 2(vi)]). Therefore

$$\inf\{\|T - V\| : V \text{ isometry}\} = \max\{\|T\| - 1, 1 + m_e(T^*)\} = 1.$$

This shows that the assumption on T in the preceding theorem is not vacuously satisfied.

To obtain the distance from an arbitrary operator to the class consisting of isometries and coisometries, we need the following lemma.

Lemma 2.7. *Let T be an operator on H.*

(1) *If* ind $T=0$, *then* $m(T)=m(T^*)$ *whence T is one-sided invertible if and only if T is invertible.*

(2) *If* ind $T\leqq 0$, *then* $m(T)\geqq m(T^*)$ whence *T is right invertible if and only if T is invertible.*

Proof. We only prove (1). If T is left invertible, then $\ker T=\{0\}$ and ran T is closed. Thus $\operatorname{ind} T=0$ implies that $\ker T^*=\{0\}$ whence $\overline{\operatorname{ran} T}=H$. It follows that T is invertible. If T is right invertible, then T^* is left invertible and $\operatorname{ind} T^*=0$. By the arguments above, T^* is invertible and so is T.

As for the minimum moduli of T and T^*, either both are 0 or at least one is positive. In the latter case, T is one-sided invertible whence invertible and we have $m(T)=m(T^*)$ (cf. [**1**, Theorem 1]).

Theorem 2.8. *For any operator T,*

$$\inf\{\|T-V\|: V \text{ isometry } \textit{or} \text{ coisometry}\}=\max\{\|T\|-1, \min\{1-m(T), 1-m(T^*)\}\}.$$

Moreover, the infimum is always attained.

Proof. Let α and β denote the quantities on the left and right sides of the above equality, respectively. First assume that ind $T\leqq 0$. By Lemma 2.7(2), we have $m(T^*)\leqq m(T)$. Hence $\alpha\leqq\max\{\|T\|-1, 1-m(T)\}=\beta$ by Theorem 2.3. For the reverse inequality, we have, by Theorem 2.3, $\|T-V\|\geqq\max\{\|T\|-1, 1-m(T)\}$ or $\max\{\|T\|-1, 1-m(T^*)\}$ depending on whether V is an isometry or a coisometry. It is easily seen that either quantity is not smaller than β. Hence $\|T-V\|\geqq\beta$ and it follows that $\alpha=\beta$. If ind $T>0$, the conclusion follows by applying the above arguments to T^*.

3. Partial isometries

In this section we prove our main result on the approximation by partial isometries. We start with the following lemma. Recall that for any operator T, $\Pi(T)$ denotes its approximate point spectrum.

Lemma 3.1. *If T is an arbitrary operator on H, then*

$$\|T-S\|\geqq \sup_{z\in\Pi(T)}\inf_{w\in\sigma(S)}|z-w|.$$

Proof. For any $z\in\Pi(T)$, there exists a sequence $\{x_n\}$ of unit vectors in H such that $\lim\|(T-z)x_n\|=0$. Hence

$$\|T-S\|\geqq\|(T-S)x_n\|\geqq\|(S-z)x_n\|-\|(T-z)x_n\|\geqq\inf_{w\in\sigma(S)}|z-w|-\|(T-z)x_n\|,$$

where the last inequality follows from the spectral theorem for normal operators (cf. [**4**, p. 56]). Letting $n\to\infty$, we obtain the required inequality.

Corollary 3.2. *If T and S are normal operators, then*

$$\|T-S\| \geqq \sup_{z\in\sigma(T)} \inf_{w\in\sigma(S)} |z-w|.$$

Proof. This follows from the preceding lemma and the fact that $\sigma(T)=\Pi(T)$ for any normal T.

The preceding results have been obtained before by Rogers [**8**, Theorem 1.6] and Halmos [**4**, p. 57], respectively. When T and S are arbitrary operators, we need to consider their "singular values" instead of spectra. This is achieved through the following lemma which was noted by H. Wielandt in the case of finite-dimensional spaces (cf. [**2**, p. 113]).

Lemma 3.3. *For any operator T, let*

$$P_T=(T^*T)^{1/2} \quad \textit{and} \quad \tilde{T}=\begin{bmatrix} 0 & T \\ T^* & 0 \end{bmatrix}.$$

Then $\sigma(\tilde{T})\supseteq\{\pm\lambda:\lambda\in\sigma(P_T)\}$. *These two sets differ at most by the number* 0 *and they are equal if* $\operatorname{ind} T\geqq 0$.

Proof. Let α be a nonzero real number. We prove that $\alpha\notin\sigma(\tilde{T})$ if and only if $\alpha\neq\pm\lambda$ for any $\lambda\in\sigma(P_T)$. If $\alpha\notin\sigma(\tilde{T})$, then $\tilde{T}-\alpha$ is invertible with inverse, say,

$$S=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

Carrying out the matrix multiplication of $(\tilde{T}-\alpha)S=1$, we obtain, for the (1,2)- and (2,2)-entries, $-\alpha S_{12}+TS_{22}=0$ and $T^*S_{12}-\alpha S_{22}=1$. From the first equation, we have $S_{12}=(TS_{22})/\alpha$, which, when substituted into the second one, gives $(T^*T-\alpha^2)(S_{22}/\alpha)=1$. Since $T^*T-\alpha^2$ is Hermitian, this shows that $T^*T-\alpha^2$ is invertible. Hence $\alpha^2\notin\sigma(T^*T)$ or, equivalently, $\pm\alpha\notin\sigma(P_T)$.

Conversely, if $\alpha\neq\pm\lambda$ for any $\lambda\in\sigma(P_T)$, then $\alpha^2\notin\sigma(T^*T)$ whence $T^*T-\alpha^2$ is invertible. Let R be its inverse. A little computation shows that

$$\begin{bmatrix} (TRT^*-1)/\alpha & TR \\ RT^* & \alpha R \end{bmatrix}$$

is the inverse of $\tilde{T}-\alpha$. Hence $\alpha\notin\sigma(\tilde{T})$.

As for the number 0, note that T is invertible if and only if $\tilde{T}$ is. Hence $0\in\sigma(T)$ if and only if $0\in\sigma(\tilde{T})$. On the other hand, $0\in\sigma(P_T)$ if and only if $m(T)=0$ which is, in term, equivalent to the fact that T is not left invertible. Thus $0\in\sigma(P_T)$ implies that $0\in\sigma(\tilde{T})$. If $\operatorname{ind} T\geqq 0$, then, from Lemma 2.7(2), T is left invertible if and only if T is invertible. Thus $0\in\sigma(P_T)$ if and only if $0\in\sigma(\tilde{T})$, completing the proof.

In case $\operatorname{ind} T<0$, these two sets may differ. Indeed, if T is a unilateral shift, then $\sigma(\tilde{T})=\{0, \pm 1\}$ and $\sigma(P_T)=\{1\}$.

Theorem 3.4. *Let T and S be operators on H and $P=(T^*T)^{1/2}$ and $Q=(S^*S)^{1/2}$. Then*

$$\|T-S\| \geqq \sup_{\lambda \in \sigma(P)} \inf_{\eta \in \sigma(Q)} \{\lambda, |\lambda-\eta|\}.$$

Moreover, if $\operatorname{ind} S \geqq 0$, *then*

$$\|T-S\| \geqq \sup_{\lambda \in \sigma(P)} \inf_{\eta \in \sigma(Q)} |\lambda-\eta|.$$

Proof. Let

$$\tilde{T}=\begin{bmatrix} 0 & T \\ T^* & 0 \end{bmatrix} \quad \text{and} \quad \tilde{S}=\begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}.$$

Since both are Hermitian, we have

$$\|\tilde{T}-\tilde{S}\| \geqq \sup_{z \in \sigma(\tilde{T})} \inf_{w \in \sigma(\tilde{S})} |z-w|$$

by Corollary 3.2. However, it is easily seen that $\|\tilde{T}-\tilde{S}\|=\|T-S\|$. Thus we may conclude from Lemma 3.3 that

$$\|T-S\| \geqq \sup_{\lambda \in \sigma(P)} \inf_{\eta \in \sigma(Q)} \{\lambda, |\lambda-\eta|\} \quad \text{if} \quad \sigma(\tilde{S}) \neq \{\pm\eta : \eta \in \sigma(Q)\}$$

and

$$\|T-S\| \geqq \sup_{\lambda \in \sigma(P)} \inf_{\eta \in \sigma(Q)} |\lambda-\eta| \quad \text{otherwise.}$$

Lemma 3.5. *Let T be an arbitrary operator and S be a partial isometry. Then* $\|T-S\| \geqq \min\{\lambda, |\lambda-1|\}$ *for any* $\lambda \in \sigma((T^*T)^{1/2})$.

Proof. S is a partial isometry implies that $(S^*S)^{1/2}$ is an orthogonal projection. Thus $\sigma((S^*S)^{1/2}) \subseteq \{0, 1\}$. Our assertion follows from Theorem 3.4.

Now we are ready for our main result.

Theorem 3.6. *For any operator T,*

$$\inf\{\|T-S\| : S \text{ partial isometry}\} = \sup_{\lambda \in \sigma((T^*T)^{1/2})} \min\{\lambda, |\lambda-1|\}.$$

Moreover, this infimum is always attained.

Proof. First assume that $\operatorname{ind} T \leqq 0$. By Lemma 2.1, we have $T=VP$, where V is an isometry and $P=(T^*T)^{1/2}$. Let ϕ be the function from $\mathbb{C}$ to $\{0,1\}$ defined by

$$\phi(z)=\begin{cases}0 & \text{if } |z|\leqq|z-1| \\ 1 & \text{otherwise}\end{cases},$$

and let $S=V\phi(P)$. Then S is a partial isometry and

$$\|T-S\|=\|V(P-\phi(P))\|=\|P-\phi(P)\|=\sup_{\lambda\in\sigma(P)}\min\{\lambda,|\lambda-1|\}$$

by the spectral theorem. If $\operatorname{ind} T>0$, then $\operatorname{ind} T^*<0$. The arguments above show that there exists a partial isometry R such that

$$\|T^*-R\|=\sup_{\eta\in\sigma((TT^*)^{1/2})}\min\{\eta,|\eta-1|\}.$$

Note that R^* is also a partial isometry and the nonzero elements of $\sigma(P)$ and $\sigma((TT^*)^{1/2})$ are the same (cf. [5, Problem 76]). We conclude that

$$\|T-R^*\|=\|T^*-R\|=\sup_{\lambda\in\sigma(P)}\min\{\lambda,|\lambda-1|\}.$$

In both cases, we have

$$\inf\{\|T-S\|:S\text{ partial isometry}\}\leqq\sup_{\lambda\in\sigma(P)}\min\{\lambda,|\lambda-1|\}.$$

The reverse inequality follows from Lemma 3.5. This completes the proof.

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