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ON THE HOMOGENIZED ENVELOPING ALGEBRA OF THE LIE ALGEBRA Sℓ(2,$\mathbb{C}$) II

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Abstract. In a previous paper, we studied the homogenized enveloping algebra of the Lie algebra $s\ell(2, \mathbb{C})$ and the homogenized Verma modules. The aim of this paper is to study the homogenization $\mathcal{O}_B$ of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of $s\ell(2, \mathbb{C})$, and to apply the ideas developed jointly with J. Mondragón in our work on Groebner basis algebras, to give the relations between the categories $\mathcal{O}_B$ and $\mathcal{O}$ as well as, between the derived categories $\mathcal{D}^b(\mathcal{O}_B)$ and $\mathcal{D}^b(\mathcal{O})$.

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1. Introduction. In a series of three papers, the author studied jointly with J. Mondragón homogeneous Groebner basis algebras, or homogeneous G-algebras, B_n and its deshomogenized G-algebra $A_n = B_n/(z-1)B_n$. We proved B_n has a Poincare–Birkoff–Witt basis it is Koszul noetherian Artin–Schelter regular of global dimension $n+1$, in particular its Yoneda algebra $B_n^!$ is finite dimensional selfinjective. We described the structure of the algebras B_n and $B_n^!$ and analyzed the relations among the algebras B_n, $B_n^!$ and A_n. The first two are related by Koszul duality and B_n and A_n are related by a homogenization–deshomogenization process. We studied these connections, both at the level of module categories, and of derived categories.

An application of these ideas to the enveloping algebra U of the Lie algebra $s\ell(2, \mathbb{C})$ was given in a previous paper [**18**]. We studied homogenized Verma modules $V(\lambda)$ over the homogenization B of the algebra U, we proved they are Koszul of projective dimension two. We then describe the structure of the Koszul $B^!$-modules $W(\lambda)$ corresponding to $V(\lambda)$ under Koszul duality. It is well known that the graded Auslander–Reiten components of selfinjective Koszul algebra are of type $\mathbb{Z}A_\infty$, we proved that each $W(\lambda)$ is in a different component, and that it lies at the mouth. In this way, we obtain a family of Auslander–Reiten components of a wild algebra parametrized by $\mathbb{C}$.

The aim of this paper is to study the homogenization $\mathcal{O}_B$ of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of $s\ell(2, \mathbb{C})$ and to apply the ideas developed jointly with J. Mondragón in the study of Groebner basis algebras, to give the relations between the categories $\mathcal{O}_B$ and $\mathcal{O}$ as well as, between the derived categories $\mathcal{D}^b(\mathcal{O}_B)$ and $\mathcal{D}^b(\mathcal{O})$.

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Groebner basis algebras or G-algebras were considered by Levandosky [**16**] generalizing results from Apel [**1**], Berger [**2**] and [**3**]. They include important classes of algebras like the Weyl algebras, [**5, 26**] the enveloping algebras of the finite dimensional Lie algebras [**14**] and the quantum polynomial ring. We considered in [**20**] and [**21**] homogeneous versions of these algebras, as well as the homogenization deshomogenization process. We proved they are Koszul, Artin–Schelter regular and noetherian, and gave the structure of both, the homogeneous G-algebra B_n and its Yoneda algebra $B_n^!$.

For the convenience of the reader, we recall the definition and some basic properties of G-algebras.

Let $\Bbbk$ be a field and $T = \Bbbk\langle X_1, X_2, \ldots X_n\rangle$ the free algebra with n generators and suppose there is a set $F = \{f_{ji} \mid 1 \leq i < j \leq n\}$, where for all $j > i f_{ji} = X_jX_i - c_{ij}X_iX_j - d_{ij}$, $d_{ij} = \sum_{k=1}^{n} b_{ij}^k X_k + a_{ij}$, with $b_{ij}^k, a_{ij} \in \Bbbk$, $c_{ij} \in \Bbbk - \{0\}$, we denote by A_n the quadratic algebra $T/\langle F\rangle$, with $I = \langle F\rangle$ the two sided ideal generated by F and let B_n be the homogenization of the quadratic algebra A_n defined by generators and relations as follows: $B_n = \Bbbk\langle X_1, X_2, \ldots X_{\mathrm{n}}, Z\rangle/I_h$, where I_h is the ideal generated by the homogenized relations of I:

$f_{ji}^h = X_jX_i - c_{ij}X_iX_j - \sum_{k=1}^{n} b_{ij}^k ZX_k - a_{ij}Z^2$, and the commutators $X_iZ - ZX_i$.

Conversely, given an homogeneous quadratic algebra $B_n = \Bbbk\langle X_1, X_2, \ldots X_{\mathrm{n}}, Z\rangle/I_h$ where I_h is the ideal generated by the homogenized relations of I:

$f_{ji}^h = X_jX_i - c_{ij}X_iX_j - \sum_{k=1}^{n} b_{ij}^k ZX_k - a_{ij}Z^2$, and the commutators $X_iZ - ZX_i$.

For any element $a \in \Bbbk$, there is a deshomogenized algebra $A_{n,a}$ defined as $A_{n,a} = \Bbbk\langle X_1, X_2, \ldots X_{\mathrm{n}}\rangle/I_a$, with I_a the ideal generated by the deshomogenized relations $f_{ji}^a = X_jX_i - c_{ij}X_iX_j - a\sum_{k=1}^{n} b_{ij}^k X_k - a_{ij}a^2$. When $a = 1$ we write A_n instead of $A_{n,1}$, and for $a = 0$, $A_{n,0}$ is just the quantum polynomial ring $\Bbbk_q[X_1, X_2, \ldots X_{\mathrm{n}}] = \Bbbk\langle X_1, X_2, \ldots X_{\mathrm{n}}\rangle/\langle X_jX_i - c_{ij}X_iX_j \mid \mathrm{j} > i, c_{ij} \in \Bbbk - \{0\}\rangle$.

In the following proposition, we establish the relations between $A_{n,a}$ and B_n.

Proposition 1. *Given an homogeneous quadratic algebra* $\Bbbk\langle X_1, X_2, \ldots X_n, Z\rangle/I_h = B_n$ *with* I_h *is the ideal generated by the homogenized relations:* $f_{ji}^h = X_jX_i - c_{ij}X_iX_j - \sum_{k=1}^{n} b_{ij}^k ZX_k - \mathrm{a}_{ij}Z^2$, *and the commutators* $X_iZ - ZX_i$, *and for* $a \in \Bbbk$ *its deshomogenization* $A_{n,a} = \Bbbk\langle X_1, X_2, \ldots X_n\rangle/I_a$, *there is an isomorphism of* $\Bbbk$*-algebras:* $B_n/(Z-a)B_n \cong A_{n,a}$.

Corollary 1. *For an homogeneous quadratic algebra* $B_n = \Bbbk\langle X_1, X_2, \ldots X_n, Z\rangle/I_h$, *there is an isomorphism of (graded)* $\Bbbk$*-algebras* $B_n/ZB_n \cong \Bbbk_q[X_1, X_2, \ldots X_n]$.

Definition 1. Let $T = \Bbbk\langle X_1, X_2, \ldots X_{\mathrm{n}}\rangle$ be the free algebra with n generators and $A = \Bbbk\langle X_1, X_2, \ldots X_{\mathrm{n}}\rangle/I$ the quotient by a two sided ideal. We say that $A = T/I$ has a Poincare–Birkof–Witt basis if every non-zero element of A can be written in a unique way as a polynomial $\sum c_\alpha X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n}$, where the sum is finite and $c_\alpha \in \Bbbk - \{0\}$.

In the next proposition [**20**], we proved that the existence of a Poincare–Birkoff–Witt Basis (PBW for short) is preserved under the homogenization-deshomogenization process.

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PROPOSITION 2. *Let $B_n = \Bbbk\langle X_1, X_2, \dots X_n, Z\rangle/I_h$ be a quadratic homogeneous algebra, $a \in \Bbbk$-$\{0\}$ and $A_{n,a} = \Bbbk\langle X_1, X_2, \dots X_n\rangle/I_a$ its deshomogenization. Then, $A_{n,a}$ has a PBW basis if and only if B_n has a PBW basis.*

For a quadratic algebra, to have a PBW basis is equivalent to have a finite Groebner basis. Since we do not want to get involved in this paper with the theory of noncommutative Groebner basis [**9–11**] we will use the following equivalent definition, and refer to [**9**], or to [**20**] for the proof of the equivalence with the standard definition, as well as for the main properties of G-algebras.

DEFINITION 2. A quadratic algebra of the form A_n (B_n) with a Poincare–Birkoff–Witt Basis will be called a Groebner basis algebra or G-algebra (homogeneous G-algebra).

We recall now the main results in our previous paper [**18**].

Through the paper $\mathbb{C}$ will denote the complex numbers, with $\mathbb{N}$ we denote the positive integers and with $\mathbb{N}_0$ the non-negative integers. Our main concern is sℓ(2,$\mathbb{C}$), the $\mathbb{C}$-vector subspace of the space of two by two matrices $M_{2\times 2}(\mathbb{C})$ consisting of the matrices with zero trace. sℓ(2,$\mathbb{C}$) is a Lie algebra with bracket product $[X, Y] = XY - YX$. The enveloping algebra U of sℓ(2,$\mathbb{C}$) is given by generators and relations by $U = \mathbb{C}\langle e, f, h\rangle/L$, where $\mathbb{C}\langle e, f, h\rangle$ is the free algebra with three generators: e,f,h and L is the ideal generated by the relations: $[e, f] - h$, $[h, e] - 2e$, $[h, f] + 2f$. It is well known that U has a Poincare–Birkoff basis [**7, 25**], this means that every element u$\in U$ can be written in a unique way as a combination $u = \sum_\ell \sum_{i+j+k=\ell} c_{i,j,k} e^i f^j h^k$ and $c_{i,j,k} \in \mathbb{C}$.

We denote by B the homogenized enveloping algebra of sℓ(2,$\mathbb{C}$) [**15**] defined by generators and relations as: $B = \mathbb{C}\langle e, f, h, z\rangle/I$, where $\mathbb{C}\langle e, f, h, z\rangle$ is the free algebra in four generators and I is the ideal generated by the relations: $[e, f] - hz$, $[h, e] - 2ez$, $[h, f] + 2fz$, $[e, z]$, $[f, z]$, $[h, z]$. By the above proposition, the algebra B has a Poincare–Birkof–Witt basis. Using the ideas and results form [**20**] and [**21**] it was proved in [**18**] it is Koszul noetherian Artin–Schelter regular of global dimension four. Its Yoneda algebra is $B^! = \mathbb{C}\langle e, f, h, z\rangle/I^\perp$, where $\mathbb{C}\langle e, f, h, z\rangle$ is the free algebra in four generators and $I^{\perp}$ is the ideal generated by the relations $e^2, f^2, h^2, z^2, (e, f), (e, h), (f, h), (h, z) + ef, (e, z) - 2eh, (f, z) - 2hf$, where $(u, v) = uv + vu$ is the anti commutator.

We recover U by deshomogenization, this is; $U \cong B/(z - 1)B$ and the polynomial algebra $C = \mathbb{C}[e, f, h]$ is obtained as $B/zB \cong \mathbb{C}[e, f, h]$. If we denote by $C^!$ the exterior algebra $C^! = \mathbb{C}\langle e, f, h, \rangle/\langle e^2, f^2, h^2, (e, f), (e, h), (f, h)\rangle$, then $C^!$ is a subalgebra of $B^!$ and there is a decomposition $B^! = C^! \oplus C^! z (B^! = C^! \oplus zC^!)$ as left (right) $C^!$-modules.

By construction B is a $\mathbb{C}[z]$-algebra, consider the multiplicative set $S = \{1, z, z^2, \dots z^k, \dots\}$ and denote by $\mathbb{C}[z]_z$ the graded localization $\mathbb{C}[z]_S$. There is an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[z]_z \cong \mathbb{C}[z, z^{-1}]$, where $\mathbb{C}[z, z^{-1}]$ is the ring of Laurent polynomials. The algebra $B_z = B \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}]$ is a $\mathbb{Z}$-graded algebra with homogeneous elements b/z^k with b an homogenous element of B and degree $(b/z^k) =$ degree$(b) - k$.

There is a commutative diagram:

$$\begin{array}{ccccc} B & \overset{\varphi}{\to} & B_z & \overset{\pi}{\to} & B_z/(z-1)B_z \\ & \searrow & & \nearrow & \\ q & & B/(z-1)B & & \psi \end{array},$$

with φ, π, q the natural maps and ψ an isomorphism of $\mathbb{C}$-algebras.

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Therefore, $U \cong B/(z-1)B \cong B_z/(z-1)B_z$.

There is an isomorphism $\theta : (B_z)_0 \to B_z/(z-1)B_z$, and isomorphisms of graded algebras:

$U[z, z^{-1}] = U \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \cong B \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}]$.

The ring $U[z, z^{-1}]$ is a strongly $\mathbb{Z}$-graded. The ring homomorphism $U \to U[z, z^{-1}]$ induces functors:

$U[z, z^{-1}] \otimes_U - : Mod_U \to Gr_{U[z,z^{-1}]}$ and $res_U : Gr_{U[z,z^{-1}]} \to Mod_U$.

By Dade's theorem [**6**] we have:

THEOREM 1. *The functors $U[z, z^{-1}] \otimes_U -$ and res_U are inverse exact equivalences, which induce by restriction equivalences between the corresponding categories of finitely generated modules* mod_U *and* $gr_{U[z,z^{-1}]}$.

COROLLARY 2. *The equivalences $U[z, z^{-1}] \otimes_U -$ and res_U preserve projective and irreducible modules and send left ideals to left ideals giving an order preserving bijection.*

In view of the previous statements, the study of the U-modules reduces to the study of the graded B_z-modules. We consider next the relation between the graded B-modules and the graded modules over the localization B_z.

We will make use of the following:

DEFINITION 3. Given a B-module M we define the z-torsion of M as $t_z(M) = \{m \in M \mid$ there exists $n \geq 0$, with $z^n m = 0\}$. The module M is of z-torsion if $t_z(M) = M$ and z-torsion free if $t_z(M)$=0.

The module $t_z(M)$ is a submodule of M and a map φ:$M \to N$ restricts to a map $\varphi_{|t_z(M)} : t_z(M) \to t_z(N)$ in this way $t_z(-)$ is a subfunctor of the identity with $t_z(t_z(M)) = t_z(M)$,

For any B-module M there is an exact sequence:

$$0 \to \quad t_z(M) \quad \to \quad M \quad \to \quad M/t_z(M) \quad \to 0 \ ,$$

with $t_z(M)$ of z-torsion and $M/t_z(M)$ z-torsion free.

The kernel of the natural morphism $M \to M_z = B_z \otimes_B M$ is $t_z(M)$.

In the next proposition, we describe as a particular case known facts concerning any localization [**28**].

PROPOSITION 3. *The following statements hold:*

(*i*) *Given a graded map of B-modules $\varphi : M \to N$ the map induced in the localization: $\varphi_z : M_z \to N_z$ is zero if and only if φ factors through a z-torsion module.*

(*ii*) *Assume the localized modules M_z is finitely generated and let $\phi : M_z \to N_z$ be a morphism of graded B_z-modules. Then, there exists an integer $k \geq 0$ and a morphism of B-modules $\varphi : z^k M \to N$ such that the composition $M_z \xrightarrow{\sigma} (z^k M)_z \xrightarrow{\varphi_z} N_z$, $\varphi_z \sigma = \phi$ and σ is an isomorphism of graded B_z-modules.*

(*iii*) *Let M be a finitely generated B_z-module. The module M is by restriction a B-module and there exists a finitely generated graded B-submodule $\overline{M}$ of M such that $\overline{M}_z \cong M$.*

(*iv*) *Let M, N be finitely generated B-modules a map $\phi : M_z \to N_z$ is an isomorphism if and only if its lifting φ: $z^k M \to N$, $\varphi_z = \phi$ has kernel and cokernel of z-torsion.*

1.1. Verma modules. In the representation theory of semisimple Lie algebras Verma modules have a special role, in this subsection we recall the definition and some of the main properties of Verma modules over the enveloping algebra U of sℓ(2,$\mathbb{C}$), as well as the properties of the homogenized version of the Verma B-modules, and refer to [**18, 25**] for the proofs.

For any $\lambda \in \mathbb{C}$ the Verma module $M(\lambda)$ is $M(\lambda)= U/(Ue + U(h-\lambda))$. It is well known which Verma U-modules are irreducible.

PROPOSITION 4. *A Verma U-module M is irreducible if and only if $\lambda \notin \mathbb{N}_0$. If $n \in \mathbb{N}_0$, then the Verma U-module $M(n)$ is indecomposable. Furthermore, the module $M(-n-2)$ is the unique simple submodule of $M(n)$ and $M(n)/M(-n-2) = V^{(n+1)}$ is the unique finite dimensional simple of dimension $n+1$.*

For each $\lambda \in \mathbb{C}$ we define the homogenized left ideal I_λ of B by $I_\lambda = Be + B(h-\lambda z)$ and the homogenized Verma module $V(\lambda) = B/I_\lambda$.

We proved in [**18**] the following:

PROPOSITION 5. *For each $\lambda \in \mathbb{C}$ the monomials $\{f^i z^m\}$ form a $\mathbb{C}$- basis of the homogenized Verma module $V(\lambda)= B/I_\lambda$, where $I_\lambda = Be + B(h-\lambda z)$.*

COROLLARY 3. *The homogenized Verma module $V(\lambda) = B/I_\lambda$ is z-torsion free.*

We call the module $(V(\lambda)_z)_0$ the deshomogenized Verma module, it is isomorphic to the usual Verma U-module, which we denoted by $M(\lambda)$.

The next proposition follows by Proposition 4 and by Corollary 2.

PROPOSITION 6. *The localization of the homogenized Verma B-module $V(\lambda)_z$ is irreducible if and only if $\lambda \notin \mathbb{N}_0$.*

As a consequence of this proposition we obtain the following properties of the homogenized Verma modules.

PROPOSITION 7.

(*i*) *Given a non-zero submodule X of the homogenized Verma B-module $V(\lambda)$ with $\lambda \notin \mathbb{N}_0$ the module $V(\lambda)/X$ is z-torsion.*

(*ii*) *The module $V(\lambda)$ is indecomposable for any $\lambda \in \mathbb{C}$.*

Consider the case $V(n)$ with $n \in \mathbb{N}_0$ and a map $\varphi : B \to V(n)$ with $Be + B(h - \lambda z) \subset \ker\varphi$. An non-zero element v of V(n) is of the form $v = \sum c_i(z) f^i$ with $c_i(z) \in \mathbb{C}[z]$ and for some i, $c_i(z) \neq 0$.

$(h-\lambda z)v = \sum c_i(z)(h-\lambda z)f^i = \sum c_i(z) f^i (h-nz) + \sum c_i(z)(n-(\lambda+2i)) f^i z = 0$ if and only if for all i with $c_i(z) \neq 0$, $n = \lambda + 2i$. This means, there is a unique i with $c_i(z) \neq 0$, $\lambda = n + 2i$ and $v = c_i(z) f^i$.

Now, $ev = c_i(z) ef^i = c_i(z) f^i e + izc_i(z) f^{i-1}(h-(i-1)z) = c_i(z) f^i e + ic_i(z) f^{i-1}(h-nz) + ic_i(z) f^{i-1} z^2 (n-(i-1))$.

Hence; $ev = 0$ if and only if $n = i-1$ or $i = 0$. in the first case $\lambda = -n-2$, $\varphi(1) = c(z) f^{n+1}$, in the second case $\lambda = n$, $\varphi(1) = c(z)1$.

We have proved:

LEMMA 1. *Let $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, then*

$$Hom_B(V(\lambda), V(n)) = \begin{cases} 0 & \text{if} \quad \lambda \neq n \text{ and } \lambda \neq \text{-n-2} \\ \mathbb{C}[z] & \qquad \text{otherwise} \end{cases}.$$

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LEMMA 2. *Given the exact sequence:* $0 \to V(-n-2) \xrightarrow{\varphi} V(n) \to V(n)/V(-n-2) \to 0$, *with* $\varphi(1) = f^{n+1}$, $V(n)/V(-n-2)$ *is z-torsion free.*

Proof. Let $\sum c_i(z) f^i + V(-n-2)$ be an element of $V(n)/V(-n-2)$ and assume for some $k \geq 0$, $z^k \sum c_i(z) f^i \in V(-n-2)$. Therefore: $\sum z^k c_i(z) f^i = \sum b_j(z) f^{j+n+1}$ and $c_i(z) = 0$ for $0 \leq i \leq n$.

It follows $\sum c_i(z) f^i = \sum c_{j+n+1}(z) f^{j+n+1} \in V(-n-2)$. □

LEMMA 3. *Let* $\mu \in \mathbb{C}$ *and* $\lambda \in \mathbb{C}\text{-}\mathbb{N}_0$. *Then,*

$$Hom_B(V(\mu), V(\lambda)) = \begin{cases} 0 & \text{if} \quad \lambda \neq \mu \\ \mathbb{C}[z] & \text{if} \quad \lambda = \mu. \end{cases}$$

Proof. Let $\varphi \colon B \to V(\lambda)$ be the map $\varphi(1) = v = \sum c_i(z) f^i$, then $hv = \sum c_i(z) f^i (h - 2iz)$, then $0 = (h - \mu z)v = \sum c_i(z) f^i (h - \lambda z) + \sum c_i(z)(\lambda - \mu - 2i) f^i z$ and $\sum c_i(z)(\lambda - \mu - 2i) f^i z = 0$ in $V(\lambda)$.

Therefore: $c_i(z) \neq 0$ implies $\lambda = \mu + 2i$, $v = c(z) f^i$.

Assume $i \neq 0$.

$ev = c(z)(f^i e + izf^{i-1}(h - \lambda z) + i(\lambda - (i-1)) f^{i-1} z^2) = 0$.

It follows $\lambda = i - 1$, a contradiction.

Therefore, $i = 0$ and $\lambda = \mu$, $v = c(z)$. □

We will assume the reader is familiar with basic results on Koszul algebras as developed in [**12, 13**].

The homogenized Verma modules are Koszul [**18**].

THEOREM 2. *Let* $V(\lambda)$ *be a homogenized Verma B-module. Then* $V(\lambda)$ *has a minimal projective resolution:*

$$0 \to B[-2] \xrightarrow{d_2} B \oplus B[-1] \xrightarrow{d_1} B \to V(\lambda) \to 0,$$

with $d_1(a, b) = ae + b(h - \lambda z)$ *and* $d_2(b) = (b(\lambda + 2)z - h, e)$. *In particular,* $V(\lambda)$ *is a Koszul module.*

In [**20, 21**] we gave the structure of both, the homogeneous G-algebras B_n, and their Koszul duals $B_n^!$. The algebras $B_n^!$ have a structure similar to the exterior algebra, in fact the exterior algebra $C_n^!$ is a sublagebra of $B_n^!$, and $B_n^!$ decomposes as left (right) $C_n^!$ module: $B_n^! = C_n^! \oplus C_n^! z$ ($B_n^! = C_n^! \oplus zC_n^!$).

Given a selfinjective algebra Λ, there is an automorphism $\sigma : \Lambda \to \Lambda$, such that it induces an isomorphism of Λ-Λ bimodules $\Lambda \cong D(\Lambda)\sigma$, where $D(\Lambda) = \mathrm{Hom}_{\mathbb{k}}(\Lambda, \mathbb{k})$ has the usual left Λ-module structure, but on the right is the multiplication given by twisting with the automorphism σ. The algebra is symmetric, if and only if $\sigma = 1$. We refer to Yamagata´s notes [**29**] for the details.

In the Koszul case, any automorphism $\sigma \colon \Lambda \to \Lambda$ induces a graded automorphism of the Yoneda algebra $\tau \colon \Gamma \to \Gamma$ as follows:

$\Gamma = \bigoplus_{k \geq 0} \mathrm{Ext}^k_\Lambda(\Lambda_0, \Lambda_0)$, let $\gamma \in \mathrm{Ext}^k_\Lambda(\Lambda_0, \Lambda_0)$ be the extension:

$$\gamma \colon 0 \to \Lambda_0 \to \mathrm{E}_1 \to \mathrm{E}_2 \to \ldots \to \mathrm{E}_k \to \Lambda_0 \to 0.$$

Define $\tau(\gamma)$ as the extension:

$$\tau(\gamma)\colon 0 \to \sigma\Lambda_0 \to \sigma \mathrm{E}_1 \to \sigma \mathrm{E}_2 \to \dots \to \sigma \mathrm{E}_k \to \sigma\Lambda_0 \to 0,$$

where given a Λ-module M the module σM is equal to M as $\Bbbk$-vector space and for $\lambda \in \Lambda$, m$\in \sigma M$, the product is defined by $\lambda *$m$=\sigma(\lambda)$m.

Since the Λ-modules Λ_0 and $\sigma\Lambda_0$ are isomorphic, $\tau(\gamma)\in \mathrm{Ext}^k_\Lambda(\Lambda_0, \Lambda_0)$.

In the case, Λ is selfinjective and $\sigma{:}\Lambda \to \Lambda$ is the Nakayama automorphism, we denote the induced automorphism also by σ, and call it the Nakayama automorphism of Γ.

It was proved in [**21**] that for a homogeneous G-algebra B_n and its Yoneda algebra $B_n^!$ the Nakayama automorphism has a simple form. It is defined as $\sigma(X_i) = u_iX_i$ and $\sigma(Z) = u_0Z$, with $u_i \in \Bbbk - \{0\}$.

Since the homogenized enveloping algebra of a finite dimensional Lie algebra is a G-algebra, for the particular case of sℓ(2,$\mathbb{C}$) the Nakayama automorphism is defined in B as $\sigma(z) = u_0z$, $\sigma(e) = u_1e$, $\sigma(f) = u_2f$, $\sigma(h) = u_3h$, with $u_i \in \mathbb{C} - \{0\}$.

We remarked above, [**24**] that graded Auslander–Reiten components of selfinjective Koszul algebras are of type $\mathbb{Z}A_\infty$. In [**18**], we proved the following:

Theorem 3.
(i) *For any complex $\lambda \in \mathbb{C}$ the $B^!$-module $W(\lambda) = B^!/(B^!f + B^!(z + \lambda h))$ is the module corresponding to $V(\lambda)$ under Koszul duality.*
(ii) *For each $\lambda \in \mathbb{C}$ the $B^!$-module $W(\lambda)$ is at the mouth of the component.*
(iii) *For $\lambda,\mu \in \mathbb{C}$, $\lambda \neq \mu$ the modules $W(\lambda)$ and $W(\mu)$ are in different components.*

2. Weight modules over the enveloping algebra U of sℓ(2,$\mathbb{C}$) and BGG category $\mathcal{O}$. In this section, we recall for the enveloping algebra U of sℓ(2,$\mathbb{C}$), some basic results on Bernstein–Gelfand–Gelfand category $\mathcal{O}$. We refer to Mazorchuk [**25**] for the proofs and more results on category $\mathcal{O}$.

Let M be a module over the enveloping algebra U of sℓ(2,$\mathbb{C}$). For $\lambda \in \mathbb{C}$ the weight space is $M_\lambda = \{m \in M \mid hm = \lambda m\}$ and λ is the weight. M is a weight module if $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, in particular the Verma modules are weight modules.

Proposition 8.
(i) *Every submodule of a weight module is a weight module.*
(ii) *Every quotient of a weight module is a weight module.*
(iii) *A direct sum of weight modules is a weight module.*
(iv) *The tensor product of two weight modules is a weight module.*

Denote by $\mathfrak{M}$ the category of weight modules, by the previous proposition it is abelian. Denote by $\overline{\mathfrak{M}}$ the subcategory of all weight modules M with $\dim_{\mathbb{C}} M_\lambda < \infty$. The Verma modules belong to $\overline{\mathfrak{M}}$.

Given a weight module M, define for any $\xi \in \mathbb{C}/2\mathbb{Z}$ the submodule $M^\xi = \bigoplus_{\lambda\in\xi} M_\lambda$. If we denote by $\mathfrak{M}^\xi$ the subcategory of weight modules of the form M^ξ, then $\mathfrak{M} = \bigoplus_{\xi\in\mathbb{C}/2\mathbb{Z}} \mathfrak{M}^\xi$. We denote by $\overline{\mathfrak{M}}^\xi$ the category $\mathfrak{M}^\xi \cap \overline{\mathfrak{M}}$.

Definition 4. The category $\mathcal{O}$ consists of all U-modules M satisfying the following conditions:
(i) M is finitely generated.

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(ii) M is a weight module.
(iii) For all v$\in M$ dim$\mathbb{C}[e]$v$< \infty$.

PROPOSITION 9. *The category $\mathcal{O}$ is closed with respect to taking submodules, quotients and finite direct sums. In particular, the category $\mathcal{O}$ is an abelian Krull Schmidt category with the usual kernels and cokernels, and any simple in $\mathcal{O}$ is a simple U-module.*

We have next the following:

PROPOSITION 10.
(i) *For any $\lambda \in \mathbb{C}$ the Verma module $M(\lambda)$ is in $\mathcal{O}$.*
(ii) *For any $M \in \mathcal{O}$ $dim_{\mathbb{C}} M_\lambda < \infty$, this is: $\mathcal{O} \subset \overline{\mathfrak{M}}$.*

DEFINITION 5. The Casimir element of U is $c = (h+1)^2 + 4fe$.

LEMMA 4. *The Casimir element is in the centre of U.*

As a consequence of the lemma, given a U-module M multiplication by c is a homomorphism $c : M \to M$.

Let τ be an element of $\mathbb{C}$. Then for any given U-module M we define $M(c,\tau) = \{m \in M \mid \text{there is } k \geq 0 \text{ with } (c-\tau)^k m = 0\}$. Then $M(c,\tau)$ is a U-submodule of M.

For any $\lambda \in \mathbb{C}$ and $M \in \overline{\mathfrak{M}}$, multiplication by c is a endomorphism of the finite dimensional $\mathbb{C}$-vector space M_λ and there is a Jordan decomposition $M_\lambda = \bigoplus_{\tau\in\mathbb{C}} M_\lambda(\tau)$, where $M_\lambda(\tau) = \{m \in M_\lambda \mid (c-\tau)^k m = 0 \text{ for } k \geq 0\}$.

The module $M(\tau) = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda(\tau)$ is a submodule of $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$.

Denote by $\overline{\mathfrak{M}}^{\,\xi,\tau}$ the full subcategory of $\overline{\mathfrak{M}}^{\xi}$ consisting of all M such that $M = M(\tau)$. By definition, $\overline{\mathfrak{M}}^{\xi} = \bigoplus_{\tau\in\mathbb{C}} \overline{\mathfrak{M}}^{\,\xi,\tau}$.

Since $\mathcal{O}$ is a subcategory of $\overline{\mathfrak{M}}$ we can define $\mathcal{O}^{\xi,\tau}$ as $\mathcal{O}^{\xi,\tau} = \mathcal{O} \cap \overline{\mathfrak{M}}^{\,\xi,\tau}$. and the above decomposition induces a decomposition $\mathcal{O} = \bigoplus_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} \mathcal{O}^{\xi,\tau}$.

PROPOSITION 11.
(i) *Every object in $\mathcal{O}$ has finite length.*
(ii) *The category $\mathcal{O}$ has enough projective objects and the Verma modules $M(\lambda)$ are projective.*

We call the categories $\mathcal{O}^{\xi,\tau}$ the blocks of $\mathcal{O}$.
There is the following description of the blocks:

THEOREM 4. *Let ξ be an element of $\mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$.*
(i) *If $(\lambda+1)^2 \neq \tau$, for all $\lambda \in \mathbb{C}$, then $\mathcal{O}^{\xi,\tau} = 0$.*
(ii) *If $(\lambda+1)^2 = \tau$, for a unique $\lambda \in \mathbb{C}$, then $\mathcal{O}^{\xi,\tau}$ is semisimple, this is: the category $\mathcal{O}^{\xi,\tau}$ is equivalent to the category of complex vector spaces.*
(iii) *If $(\lambda_1+1)^2 = (\lambda_2+1)^2 = \tau$, for $\lambda_1,\lambda_2 \in \mathbb{C}$, with $\lambda_1 \neq \lambda_2$, then $\tau = n^2$, for some $n \in \mathbb{N}_0$ and the block $\mathcal{O}^{\xi,\tau}$ is equivalent to the category of finitely generated modules over the algebra $\mathfrak{D}$ with quiver $\underset{\beta}{\overset{\alpha}{\leftrightarrows}}$ and relations $\alpha\beta = 0$.*

Observe that in case (iii) we have an algebra with a "node" in the sense of [**22**] such that removing the node we obtain an algebra with quiver $\to\to$ and no relations, but

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this is isomorphic to the algebra of triangular matrices of size 3×3, which is hereditary, hence, Koszul. It follows from **[22]** that the algebra $\mathfrak{D}$ is Koszul.

The algebra $\mathfrak{D}$ is of finite representation type and using the functors given in **[20]** we can describe completely its module structure. In fact we have up to isomorphism only five indecomposable modules.

The blocks of type (iii) are called regular blocks, and since they are all equivalent, it is enough to consider the principal block $\mathcal{O}^{2\mathbb{Z},1}$,which is denoted by $\mathcal{O}_0$.

The module structure of the block both as graded and non-graded object is fully understood and we can find the details in Mazorchuk book **[25]**.

We want to describe next the homogenized version $\mathcal{O}_B$ of the category $\mathcal{O}$.

3. The homogenized category $\mathcal{O}_B$.

Before describing the homogenized category $\mathcal{O}$ we will give some results concerning the graded localization $M_z = B_z \otimes_B M$ of a graded left B-module M and of the deshomogenization process $M/(z-1)M$. We give in this section an explicit description of the graded left B-modules such that $M/(z-1)M$ is in $\mathcal{O}$.

We remarked in Section 1, **[18, 20]** that B_z is a $\mathbb{Z}$-graded algebra which has in degree zero $U \cong (B_z)_0 \cong B/(z-1)B \cong B_z/(z-1)B_z$. This result extends to the finitely generated B-modules.

Proposition 12. *Let M be a finitely generated graded B-module and $M_z = B_z \otimes_B M$ be its graded localization. For any integer k, the degree k part of M_z, given by $(M_z)_k = \{m/z^\ell \,|deg(m)\text{-}\ell = k\}$, is isomorphic to $M/(z-1)M$ as $(B_z)_0$-modules.*

Proof. Let's assume first M is of z-torsion. Then, there is a positive integer ℓ such that $z^\ell M = 0$. Given $m \in M$, $z^\ell m = 0 = m + (z-1)h(z)m$, with $h(z) \in \mathbb{C}[z]$, hence $m + (z-1)M = 0 + (z-1)M$ and $M/(z-1)M = 0$.

In the general case, applying the tensor product functor $B/(z-1)B \otimes_B -$ to the exact sequence:

$$*)\ 0 \to \mathrm{t}_z(M) \to M \to M/\mathrm{t}_z(M) \to 0,$$

we obtain an exact sequence:

$$t_z(M)/(z-1)t_z(M) \to M/(z-1)M \to (M/\mathrm{t}_z(M))/(z-1)(M/t_z(M)) \to 0,$$

and $t_z(M)/(z-1)t_z(M) = 0$ implies $M/(z-1)M \cong (M/t_z(M))/(z-1)(M/t_z(M))$.

If we apply now the tensor functor $B\,_z\otimes_B$- to the exact sequence *) we obtain the exact sequence:

$$0 \to t_z(M)_z \to M_z \to (M/t_z(M))_z \to 0,$$

and $t_z(M)_z = 0$ implies $M_z \cong (M/t_z(M))_z$.

In view of these remarks, we may assume M is z-torsion free.

Consider the composition $\varphi = \pi j$ of the canonical maps: $(M_z)_k \xrightarrow{j} M_z \xrightarrow{\pi} M_z/(z-1)M_z$. The map φ is defined as $\varphi(m/z^\ell) = m/z^\ell + (z-1)M_z$, where deg(m)-$\ell = k$.

We claim φ is an isomorphism of $(B_z)_0$-modules.

It is clear that φ is a morphism of $(B_z)_0$-modules and $\varphi(m/z^\ell) = 0$ if and only if $m/z^\ell \in (z-1)M_z$.

Assume $m/z^{\ell} = (z-1)n/z^{t}$ with $m \neq 0$. Since we are assuming M is z-torsion free, $z^{t}m = (z-1)z^{\ell}n$, where $n = n_1 + n_2 + \cdots n_s$ and $\deg(n_i) > \deg(n_{i+1})$. It follows $z^{t}m = zn_1 + zn_2 + \cdots zn_s - (n_1 + n_2 + \cdots n_s)$, in the right zn_1 is of maximal degree and n_s of minimal degree, using the fact that M is z-torsion free we get a contradiction.

To prove that φ is surjective, consider an element $n/z^{t} + (z-1)M_z$, where $n = n_1 + n_2 + \cdots n_s$ and $\deg(n_i) > \deg(n_{i+1})$ and consider the homogenization m of n given by $m = n = n_1 + z^{t_2}n_2 + \cdots z^{t_s}n_s$, where $t_i = deg(n_1)$-$\deg(n_i)$. It follows as above that $m + (z-1)M = n + (z-1)M$ and $\varphi(m/z^{t}) = n/z^{t} + (z-1)M_z$. □

Corollary 4. *A finitely generated graded B-module M is of z-torsion if and only if M/(z-1) M* $= 0$.

Proof. In the proof of the proposition we proved that if M is of z-torsion, then $M/(z\text{-}1)\ M = 0$. As a consequence $M/(z-1)M \cong (M/t_z(M))/(z-1)(M/t_z(M))$ and $M/(z-1)M = 0$ implies $(M/t_z(M))/(z-1)(M/t_z(M)) = 0$.

Assume N is a finitely generated graded z-torsion free B-module with $N/(z-1)N = 0$.

We saw in the proof of the proposition that if n is an homogeneous element of N, then $n \in (z-1)N$ implies $n = 0$.

Therefore, $(M/t_z(M))/(z-1)(M/t_z(M)) = M/(z-1)M = 0$ implies $M/t_z(M) = 0$, and M is of z-torsion. □

As a consequence of the proposition we have $(B_z)_k \cong B_z/(z-1)B_z$ as $(B_z)_0$-modules.

We also have the following:

Proposition 13. *Let M be a finitely generated graded B-module, there is an isomorphism of $(B_z)_0$-modules $M/(z-1)M \cong M_z/(z-1)M_z$.*

Proof. We may assume M is z-torsion free.

We have a commutative diagram:

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr, "q"'] & M_z \arrow[r, "\pi"] & M_z/(z-1)M_z \\ & M/(z-1)M \arrow[ur, "\psi"'] & \end{tikzcd}$$

It is clear that $\ker\pi j \supset (z-1)M$. If $m \in ker\pi j$, then $m/1 = (z-1)w/z^{\ell}$ and $z^{\ell}m = (z-1)w$, and $z^{\ell}m = m + (z-1)h(z)m$. Therefore, $m \in (z-1)M$ and ψ is injective.

Let $w/z^{\ell} + (z-1)M_z$ be an element of $M_z/(z-1)M_z$. As above, $w/1 = z^{\ell}(w/z^{\ell}) = w/z^{\ell} + (z-1)v/z^{\ell}$. Therefore: $\psi(w + (z-1)M) = w/z^{\ell} + (z-1)M_z$. □

Corollary 5. *Let $0 \to L \to N \to M \to 0$ be an exact sequence of finitely generated graded B-modules. Then applying the functor $B/(z-1)B \otimes_B$- to the sequence we obtain an exact sequence:*

$$0 \to L/(z-1)L \to N/(z-1)N \to M/(z-1)M \to 0.$$

Proof. The sequence $0 \to L_z \to N_z \to M_z \to 0$ is exact, hence in degree zero we have an exact sequence: $0 \to (L_z)_0 \to (N_z)_0 \to (M_z)_0 \to 0$ which is isomorphic to the exact sequence: $0 \to L/(z-1)L \to N/(z-1)N \to M/(z-1)M \to 0$. □

We are interested in the subcategory of gr_B consisting of all finitely generated graded left B-modules M such that its deshomogenization $M/(z-1)M$ is isomorphic to a U-module in the category $\mathcal{O}$. If we have an exact sequence of finitely generated graded B-modules $0 \to K \to L \to M \to N \to 0$, then the sequence $0 \to K/(z-1)K \to L/(z-1)L \to M/(z-1)M \to N/(z-1)N \to 0$ is exact. From this, it follows that the category is abelian and has the same kernels and cokernels as in gr_B.

In the last part of the section, we characterize this category as the homogenized version $\mathcal{O}_B$ of the category $\mathcal{O}$.

3.1. Homogenized weight modules. This subsection will be dedicated to the study of the homogenized weight modules over the homogenized enveloping algebra B of sℓ(2,$\mathbb{C}$). We will also study a homogenized version of Gelfand's category $\mathcal{O}$ [**4, 25**].

Let M be a graded B-module. For a given $\lambda \in \mathbb{C}$, we consider the homogenized weight space V_λ defined by $V_\lambda = \{m \in M \mid hm = \lambda zm\}$.

We have the following:

LEMMA 5. *For any graded B-module M the $\mathbb{C}$-vector space $\sum_{\lambda\in\mathbb{C}} V_\lambda$ is a B-submodule of M.*

Proof. We claim that given m$\in V_\lambda$ and an integer $k \geq 0$, $e^k m \in V_{\lambda+2k}$, that $f^k m \in V_{\lambda-2k}$, $h^k m \in V_\lambda$ and $z^k m \in V_\lambda$.

We only prove the first claim, the others are similar.

We have proved above $he^k = e^k h + 2ke^k z$. Hence, $(h-\mu z)e^k = e^k(h-(\mu-2k)z)$.

Let μ be $\mu = \lambda + 2k$. Then, $(h-(\lambda+2k)z)e^k m = e^k(h-\lambda z)m = 0$ implies $e^k m \in V_{\lambda+2k}$. □

We are interested in the case the sum is direct.

LEMMA 6. *Let M be a graded z-torsion free B-module of the form $M = \sum_{\lambda\in\mathbb{C}} V_\lambda$. Then, $M = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$.*

Proof. Assume $0 = m_1 + m_2 + \cdots + m_k$ with m$_i \in V_{\lambda_i}$ and let h_i be $h_i = (h-\lambda_1 z)(h-\lambda_2 z)\ldots(h-\lambda_{i-1}z)(h-\lambda_{i+1}z)\ldots(h-\lambda_k z)$. Then, $h_i m_j = 0$ if $i \neq j$ and $h_i m_i = \prod_{i\neq j}(\lambda_i - \lambda_j)z^{k-1}m_i$.

It follows $h_i 0 = 0 = h_i m_1 + h_i m_2 + \cdots + h_i m_k = \prod_{i\neq j}(\lambda_i-\lambda_j)z^{k-1}m_i$. Since M is z-tosion free $m_i = 0$. □

DEFINITION 6. A homogenized weight B -module is a graded B-module of the form $M = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$. When there is no risk of confusion we will say a weight B-module, for short.

We also need the next lemma.

LEMMA 7. *Let M, N be graded B-modules, where M is of the form $M = \sum_{\lambda\in\mathbb{C}} V_\lambda$ and $\varphi: M \to N$ is a surjective homomorphism of B-modules. Then, $N = \sum_{\lambda\in\mathbb{C}} W_\lambda$ with $W_\lambda = \{n \in N \mid (h-\lambda z)n = 0\}$.*

Proof. Let $m \in V_\lambda$. Then, $\varphi((h-\lambda z)m) = \varphi(0) = 0 = (h-\lambda z)\varphi(m)$ and $\varphi(V_\lambda) \subset W_\lambda$.

Hence, $\varphi(M)$= $N = \sum_{\lambda\in\mathbb{C}}\varphi(V)_\lambda \subset \sum_{\lambda\in\mathbb{C}} W_\lambda \subset N$ implies $\sum_{\lambda\in\mathbb{C}} W_\lambda = N$. □

Corollary 6. *Assume* $M = \sum_{\lambda \in \mathbb{C}} V_\lambda$ *and let* $t_z(M)$ *be the* z*-torsion part of* M*. Then* $M/t_z(M)$ *is a weight* B*-module and* $M/t_z(M) = \bigoplus_{\lambda \in \mathbb{C}} W_\lambda$ *with* $W_\lambda = \varphi(V_\lambda)$.

Proof. We have a canonical epimorphism $M \to M/t_z(M) \to 0$. By the lemma $M/t_z(M) = \sum_{\lambda \in \mathbb{C}} W_\lambda$ and by lemma 6, the sum is direct.

We know $\varphi(V_\lambda) \subset W_\lambda$. Let $w \in W_\lambda$. Then, $w = \varphi(v_1) + \varphi(v_2) + \cdots \varphi(v_k)$ with $v_i \in V_{\lambda_i}$.

Hence, $\prod_{i=1}^{k}(h - \lambda_i z)w = \prod_{i=1}^{k}(\lambda - \lambda_i)zw = 0$ implies $\lambda = \lambda_i$ for some i. Therefore, $0 = \varphi(v_1) + \varphi(v_2) + \cdots \varphi(v_i) - w + \varphi(v_{i+1}) + \cdots \varphi(v_k)$ and the fact that the sum is direct implies $\varphi(v_j) = 0$ for $i \neq j$ and $w \in \varphi(V_\lambda)$. □

We have the following homogenized version of a result in [**25**].

Proposition 14. *Let* M *be a graded* B*-module of the form* $M = \sum_{\lambda \in \mathbb{C}} V_\lambda$*,* N *a graded submodule of* M *and* $W_\lambda = \{n \in N \mid (h - \lambda z)n = 0\}$*. Then* $N/\sum_{\lambda \in \mathbb{C}} W_\lambda$ *is of* z*-torsion.*

Proof. Let n be an element in N. Then $n = m_1 + m_2 + \cdots m_k$ with $m_i \in V_{\lambda_i}$

As above $h_i = (h - \lambda_1 z)(h - \lambda_2 z)\ldots(h - \lambda_{i-1} z)(h - \lambda_{i+1} z)\ldots(h - \lambda_k z)$. Then $h_i n = h_i m_i = \prod_{i \neq j}(\lambda_i - \lambda_j)z^{k-1} m_i \in N \cap V_{\lambda_i} = W_{\lambda_i}$.

It follows $z^{k-1} n \in \sum_{\lambda \in \mathbb{C}} W_\lambda$.

We have proved $N/\sum_{\lambda \in \mathbb{C}} W_\lambda$ is of z-torsion. □

We will consider the subcategory $\mathcal{C}$ of the category Gr_B of graded B-modules defined as follows:

A module M is in $\mathcal{C}$ if and only if $M/t_z(M)$ contains a homogeneous weight submodule $\bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ such that the cokernel of the inclusion map: $j : \bigoplus_{\lambda \in \mathbb{C}} V_\lambda \to M/\mathrm{t}_z(M)$ is of z-torsion.

Informally it means M can be "approximated" by a homogeneous weight module, or that up to z-torsion M is a weight module.

Definition 7. We will call $\mathcal{C}$ the category of generalized homogeneous weight modules, or just the generalized weight modules.

We have the following:

Proposition 15.

(*i*) *Every submodule of a generalized weight module is a generalized weight module.*

(*ii*) *Each quotient of a generalized weight module is a generalized weight module.*

(*iii*) *A sum of two generalized weight modules is a generalized weight module.*

(*iv*) *If* M, N *are two* B*-modules, then* $M \otimes_{\mathbb{C}[z]} N$ *has structure of* B*-module and if both* M, N *are generalized weight* B*-modules, then* $M \otimes_{\mathbb{C}[z]} N$ *is a generalized weight* B*-module.*

Proof.

(*i*) If N is a submodule of M, then $t_z(M) \cap N = t_z(N)$.

By hypothesis, there is an exact sequence $0 \to \bigoplus_{\lambda \in \mathbb{C}} V_\lambda \to M/t_z(M) \to X \to 0$, with X of z-torsion. Letting W be $W = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda \cap (N/t_z(N))$, we have an exact

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commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & W & \to & \underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda & \to & L' & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & N/t_z(N) & \to & M/t_z(M) & \to & L & \to 0 & . \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & X' & \to & X & \to & X'' & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here, X of z-torsion implies X' is of z-torsion.
Since W is contained in $\underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda$ and it is z-torsion free, Lemma 6, and Proposition 13 imply $Y = W/ \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda$ is of z-torsion .
We have an exact commutative diagram:

$$\begin{array}{cccccccc}
 & & & & & & 0 & \\
 & & & & & & \downarrow & \\
 & & 0 & & 0 & & Y & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda & \to & N/t_z(N) & \to & Y' & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & W & \to & N/t_z(N) & \to & X' & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & Y & & 0 & & 0 & \\
 & & \downarrow & & & & & \\
 & & 0 & & & & &
\end{array},$$

where Y and X' are of z-torsion. Therefore, Y' is of z-torsion.

(ii) Let M be a generalized weight module and f: $M \to N$ an epimorphism. Then, we have the following commutative exact diagram:

$$\begin{array}{cccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & t_z(L) & \to & t_z(M) & \to & N' & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & L & \to & M & \longrightarrow & N & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & L/t_z(L) & \to & M/t_z(M) & \to & N/N' & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}$$

The module N' is of z-torsion, hence $N' \subset t_z(N)$ and there is an epimorphism $N/N' \to N/t_z(N) \to 0$. It follows that there is an epimorphism: $M/t_z(M) \xrightarrow{\pi} N/t_z(N) \to 0$.
By hypothesis $M/t_z(M)$ has weight submodule $\underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda$ such the cokernel of the inclusion is of z-torsion.

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By lemma 6, $\pi(\underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda) = \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda$ is a weight submodule of $N/t_z(N)$ and we have an exact commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda & \to & M/t_z(\mathrm{M}) & \to & X & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda & \to & N/t_z(N) & \to & X'' & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array},$$

and X of z-torsion implies X'' is of z-torsion.

(*iii*) Let M, N be B-modules in $\mathcal{C}$. Then, there exists exact sequences $0 \to \underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda \to M/\mathrm{t}_z(M) \to X \to 0, 0 \to \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda \to N/\mathrm{t}_z(N) \to Y \to 0$, with X, Y of z-torsion.
Then the sequence:
$0 \to (\underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda \oplus \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda) \to (M/t_z(M) \oplus N/\mathrm{t}_z(N)) \to X \oplus Y \to 0$
is exact with $X \oplus Y$ of z-torsion $(\underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda \oplus \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda) \cong \underset{\lambda\in\mathbb{C}}{\oplus} (V \oplus W)_\lambda$, $\mathrm{t}_z(M) \oplus \mathrm{t}_z(N) \cong \mathrm{t}_z(M \oplus N)$, where $V = \underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda$, and $W = \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda$.
Therefore, $M \oplus N \in \mathcal{C}$.

□

To prove part (iv) of the proposition we need some preliminary results.

Since z is in the center B is a $\mathbb{C}[z]$-algebra and the inclusion $\mathbb{C}[z] \to B$ is a morphism of graded $\mathbb{C}$-algebras.

From the existence of a Poincare–Birkof–Witt basis of B, it follows B is a free $\mathbb{C}[z]$-module.

Let M, N be two left B-modules. Using the Poincare–Birkof–Witt basis we want to define a B-module structure on $M \otimes_{\mathbb{C}[z]} N$.

Let t be an element of $\{e, f, h\}$ and define $\varphi_t{:}M \times N \to M \otimes_{\mathbb{C}[z]} N$ as the morphism $\varphi_t(m, n) = (tm \otimes n + m \otimes tn)$.

The map φ_t is bilinear and $\mathbb{C}[z]$-balanced $\varphi_t(m_1 + m_2, n) = (t(m_1 + m_2) \otimes n + (m_1 + m_2) \otimes tn) = (tm_1 \otimes n + tm_2 \otimes n + m_1 \otimes tn + m_2 \otimes tn) = (tm_1 \otimes n + m_1 \otimes n) + (tm_2 \otimes n + m_2 \otimes tn) = \varphi_t(m_1, n) + \varphi_t(m_2, n)$.

Similarly, $\varphi_t(m, n_1 + n_2) = \varphi_t(m, n_1) + \varphi_t(m, n_2)$ and $\varphi_t(m, zn) = (tm \otimes zn + m \otimes tzn) = (tzm \otimes n + zm \otimes tn)$, since z is in the center of B.

For t=z we define $\varphi_t(m, n) = tm \otimes n = m \otimes tn$, which is also bilinear an z-balanced.

Hence; φ_t induces a morphism $\overline{\varphi_t}$: $M \otimes_{\mathbb{C}[z]} N \to M \otimes_{\mathbb{C}[z]} N$ given by $\overline{\varphi_t}(m \otimes n) = tm \otimes n + m \otimes tn$ and $\overline{\varphi_z}(m \otimes n) = zm \otimes n = m \otimes zn$.

We can see that this morphism induces the structure of B-module on $M \otimes_{\mathbb{C}[z]} N$.

LEMMA 8. *Given two left graded B-modules M, N the $\mathbb{C}[z]$-module $M \otimes_{\mathbb{C}[z]} N$ has a structure of B-module.*

Proof. We prove the equivalent statement that there is a representation

$$\Phi{:}\mathrm{B} \to \mathrm{End}_{\mathbb{C}}(\mathrm{M} \otimes_{\mathbb{C}[z]} \mathrm{N}).$$

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We define first a ring homomorphism $\overline{\Phi}: \mathbb{C}\langle e, f, h, z >\to \mathrm{End}_{\mathbb{C}}(M \otimes_{\mathbb{C}[z]} N)$, as follows: by the universal property it is enough to give the map in words w. Let $w = t_1t_2 \ldots t_k$ with $t_i \in \{e, f, h, z\}$. Then $\overline{\Phi}(m \otimes n) = \varphi_{t_1}\varphi_{t_2} \ldots \varphi_{t_k}(m \otimes n)$.

To give the map Φ: $B \to \mathrm{End}_{\mathbb{C}}(M \otimes_{\mathbb{C}[z]} N)$ it is enough to check that the defining relations of B are in the kernel of $\overline{\Phi}$.

We check it for the relation $hf - fh + 2fz$ and the rest is left to the reader.

$hf(v \otimes w) = h(fv \otimes w + v \otimes fw) = hfv \otimes w + fv \otimes hw + hv \otimes fw + v \otimes hfw$

$fh(v \otimes w) = f(hv \otimes w + v \otimes hw) = fhv \otimes w + hv \otimes fw + fv \otimes hw + v \otimes fhw$

$2fz(v \otimes w) = 2z(fv \otimes w + v \otimes fw) = 2zfv \otimes w + v \otimes 2zfw$

Hence, $(hf - fh + 2fz)(v \otimes w) = (hf - fh + 2fz)(v) \otimes w + v \otimes (hf - fh + 2fz)(w) = 0$. □

LEMMA 9. *Let M be a z-torsion free graded B-module. Then M is torsion free as $\mathbb{C}[z]$-module.*

Proof. Let m be a non-zero element of M and $q \in \mathbb{C}[z]$, $q \neq 0$, with $qm = 0$. Since q is non-constant $q = (z - \lambda_1)(z - \lambda_2) \ldots (z - \lambda_n)$.

$m \neq 0$ implies there is an integer $1 < i \leq n$ such that $(z - \lambda_i) \ldots (z - \lambda_n)m = m' \neq 0$ and $(z - \lambda_{i-1})m' = 0$.

Changing m for m' and λ for λ_i, $m \neq 0$ and $(z - \lambda)m = 0$.

Since M is graded m has a decomposition in homogeneous components: $m = m_1 + m_2 + \cdots m_k$ with degree$(m_i) >$degree(m_{i+1}).

$0 = (z - \lambda)m = zm_1 + zm_2 + \cdots zm_k - \lambda m_1 - \lambda m_2 \ldots - \lambda m_k$ and $zm_1 \neq 0$ and it is of maximal degree, since it does not cancel with any other term in the sum, a contradiction. □

COROLLARY 7. *Let M be graded z-torsion free B-module. Then M is flat as $\mathbb{C}[z]$-module.*

Proof. The module M is a direct limit of finitely generated torsion free modules. In a principal ideal domain a finitely generated torsion free module is free **[17]** , hence M is a direct limit of flat modules, then flat. □

PROPOSITION 16. *Let M and N be graded B-modules, X and Y submodules of M and N, respectively, such that M/X and N/Y are z-torsion free. Then, there are monomorphisms: $X \otimes_{\mathbb{C}[z]} N \to M \otimes_{\mathbb{C}[z]} N$ and $M \otimes_{\mathbb{C}[z]} Y \to M \otimes_{\mathbb{C}[z]} N$ such that $X \otimes_{\mathbb{C}[z]} Y \cong X \otimes_{\mathbb{C}[z]} N \cap M \otimes_{\mathbb{C}[z]} Y$ and $M \otimes_{\mathbb{C}[z]} N/(X \otimes_{\mathbb{C}[z]} N + M \otimes_{\mathbb{C}[z]} Y) \cong (M/X) \otimes_{\mathbb{C}[z]} (N/Y)$.*

Proof. Since M/X and N/Y are flat as $\mathbb{C}$[z]-modules, $\mathrm{Tor}_1^{\mathbb{C}[z]}(\text{-}, N/Y)=0$ and $\mathrm{Tor}_1^{\mathbb{C}[z]}(M/X, \text{-})=0$, there is an exact commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & X \otimes_{\mathbb{C}[z]} Y & \to & X \otimes_{\mathbb{C}[z]} N & \to & X \otimes_{\mathbb{C}[z]} N/Y & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & M \otimes_{\mathbb{C}[z]} Y & \to & M \otimes_{\mathbb{C}[z]} N & \to & M \otimes_{\mathbb{C}[z]} N/Y & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & M/X \otimes_{\mathbb{C}[z]} Y & & M/X \otimes_{\mathbb{C}[z]} N & & M/X \otimes_{\mathbb{C}[z]} N/Y & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array},$$

which implies $X \otimes_{\mathbb{C}[z]} Y \cong X \otimes_{\mathbb{C}[z]} N \cap M \otimes_{\mathbb{C}[z]} Y$ and $M \otimes_{\mathbb{C}[z]} N/(X \otimes_{\mathbb{C}[z]} N) \cong M/X \otimes_{\mathbb{C}[z]} N$, $M/X \otimes_{\mathbb{C}[z]} N/(M/X \otimes_{\mathbb{C}[z]} Y) \cong M/X \otimes_{\mathbb{C}[z]} N/Y$.

Hence, we have the following chain of isomorphisms:

$$(M \otimes_{\mathbb{C}[z]} N)/(X \otimes_{\mathbb{C}[z]} N + M \otimes_{\mathbb{C}[z]} Y)$$

$$\cong (M \otimes_{\mathbb{C}[z]} N/X \otimes_{\mathbb{C}[z]} N)/((X \otimes_{\mathbb{C}[z]} N + M \otimes_{\mathbb{C}[z]} Y)/(X \otimes_{\mathbb{C}[z]} N))$$

$$\cong (M/X \otimes_{\mathbb{C}[z]} N)/((M \otimes_{\mathbb{C}[z]} Y)/X \otimes_{\mathbb{C}[z]} Y))$$

$$\cong (M/X \otimes_{\mathbb{C}[z]} N)/(M/X \otimes_{\mathbb{C}[z]} Y) \cong M/X \otimes_{\mathbb{C}[z]} N/Y.$$

□

We will also need the following:

PROPOSITION 17. *Let M, N be graded z-torsion free B-modules. Then, $M \otimes_{\mathbb{C}[z]} N$ is z-torsion free.*

Proof. Let $\sum_{i=1}^{k} m_i \otimes n_i$ be a non-zero element of $M \otimes_{\mathbb{C}[z]} N$ and $\ell > 0$ such that $z^{\ell} \sum_{i=1}^{k} m_i \otimes n_i = 0$. We may assume m_i, n_i are homogeneous.

$\sum^{k} \mathbb{C}[z] n_i$ is a finitely generated z-torsion free submodule of N, hence it is a free $\mathbb{C}$[z]-module.

Let $\varphi: \overset{t}{\underset{j=1}{\oplus}} \mathbb{C}[z] \to \overset{k}{\sum} \mathbb{C}[z] n_i$ be an isomorphism and $e_j = (0, 0 \ldots 0, 1, 0 \ldots 0)$ the canonical basis of $\overset{t}{\underset{j=1}{\oplus}} \mathbb{C}[z]$.

The morphism φ induces an isomorphism:

$1 \otimes \varphi: M \otimes_{\mathbb{C}[z]} \overset{t}{\underset{j=1}{\oplus}} \mathbb{C}[z] \to M \otimes_{\mathbb{C}[z]} \sum_{i=1}^{k} \mathbb{C}[z] n_i$.

Since M is a flat $\mathbb{C}[z]$-module, the inclusion $j: \sum_{i=1}^{k} \mathbb{C}[z] n_i \to N$ induces a monomorphism $1 \otimes j: M \otimes_{\mathbb{C}[z]} \sum_{i=1}^{k} \mathbb{C}[z] n_i \to M \otimes_{\mathbb{C}[z]} N$.

We have $\varphi^{-1}(n_j) = \sum_{i=1}^{t} c_{ij} e_i$ with $c_{ij} \in \mathbb{C}[z]$.

Then $(1 \otimes j)(1 \otimes \varphi)(\sum_{i=1}^{k} m_i \otimes (\sum_{i=1}^{t} c_{ij} e_i) = \sum_{i=1}^{k} m_i \otimes n_i$.

$\sum_{i=1}^{k} m_i \otimes (\sum_{i=1}^{t} c_{ij} e_i) = \sum_{i=1}^{k} \sum_{i=1}^{t} m_i c_{ij} \otimes e_i$.

Then $z^{\ell} \sum_{i=1}^{k} m_i \otimes n_i = 0 = z^{\ell}(1 \otimes j)(1 \otimes \varphi)(\sum_{i=1}^{k} m_i \otimes (\sum_{i=1}^{t} c_{ij} e_i))$, $1 \otimes j\varphi$ a monomorphism implies $\sum_{i=1}^{k} z^{\ell}(\sum_{i=1}^{t} m_i c_{ij}) \otimes e_i = 0$. Hence; $z^{\ell}(\sum_{i=1}^{t} m_i c_{ij}) = 0$, and M z-torsion free implies $\sum_{i=1}^{t} m_i c_{ij} = 0$.

Therefore, $\sum_{i=1}^{k} m_i \otimes n_i = (1 \otimes j)(1 \otimes \varphi)(\sum_{i=1}^{k} \sum_{i=1}^{t} m_i c_{ij} \otimes e_i) = 0$.

We have proved $M \otimes_{\mathbb{C}[z]} N$ is z-torsion free. □

Applying the previous results we get:

LEMMA 10. *Let M and N be graded B-modules and denote by $t_z(\text{-})$ the radical corresponding to z- torsion. Then, $t_z(M \otimes_{\mathbb{C}[z]} N) = t_z(M) \otimes_{\mathbb{C}[z]} N + M \otimes_{\mathbb{C}[z]} t_z(N)$.*

Proof. Since $N/t_z(N)$ and $M/t_z(M)$ are flat $\mathbb{C}$[z]-modules, the exact sequences:

$$0 \to t_z(M) \to M \to M/t_z(M) \to 0\ ,$$

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$$0 \to \quad \mathrm{t}_z(N) \quad \to \quad N \quad \to \quad N/\mathrm{t}_z(N) \quad \to 0.$$

Induce by tensoring exact sequences:

$$0 \to t_z(M) \otimes_{\mathbb{C}[z]} N \to M \otimes_{\mathbb{C}[z]} N \to M/t_z(M) \otimes_{\mathbb{C}[z]} N \to 0,$$

$$0 \to M \otimes_{\mathbb{C}[z]} \mathrm{t}_z(N) \to M \otimes_{\mathbb{C}[z]} N \to M \otimes_{\mathbb{C}[z]} N/t_z(N) \to 0.$$

Since $\mathrm{t}_z(M)\otimes_{\mathbb{C}[z]}N$, $M\otimes_{\mathbb{C}[z]}\mathrm{t}_z(N)$ are z-torsion modules, $\mathrm{t}_z(M)\otimes_{\mathbb{C}[z]}N+M\otimes_{\mathbb{C}[z]}\mathrm{t}_z(N)$ is a z-torsion submodule of $M \otimes_{\mathbb{C}[z]} N$, hence it is contained in $\mathrm{t}_z(M \otimes_{\mathbb{C}[z]} N)$.

In the other hand,

$$M \otimes_{\mathbb{C}[z]} N/(\mathrm{t}_z(M) \otimes_{\mathbb{C}[z]} N + M \otimes_{\mathbb{C}[z]} \mathrm{t}_z(N)) \cong (M/\mathrm{t}_z(M)) \otimes_{\mathbb{C}[z]} (N/\mathrm{t}_z(N)).$$

By Proposition 17, $(M/t_z(M))\otimes_{\mathbb{C}[z]}(N/\mathrm{t}_z(N))$ is z-torsion free.

It follows, $\mathrm{t}_z(M \otimes_{\mathbb{C}[z]} N)=\mathrm{t}_z(M)\otimes_{\mathbb{C}[z]}N + M\otimes_{\mathbb{C}[z]}\mathrm{t}_z(N)$. □

After these results, we can finally prove:

Claim 1. *Let M and N be generalized weight B-modules. Then $M \otimes_{\mathbb{C}[z]} N$ is a generalized weight B-modules.*

Proof. By hypothesis, we have exact sequences:

$$0 \to \bigoplus_{\lambda\in\mathbb{C}} \mathrm{V}_\lambda \to \mathrm{M}/\mathrm{t}_z(\mathrm{M}) \to \mathrm{Z} \to 0,$$

$$0 \to \bigoplus_{\lambda\in\mathbb{C}} \mathrm{W}_\lambda \to \mathrm{N}/\mathrm{t}_z(\mathrm{N}) \to \mathrm{R} \to 0,$$

with Z, R, z-torsion modules.

Since $\bigoplus_{\lambda\in\mathbb{C}} V_\lambda$, $\bigoplus_{\lambda\in\mathbb{C}} W_\lambda$, $M/t_z(M)$, $N/t_z(N)$ are flat $\mathbb{C}[z]$-modules, we have the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \bigoplus_{\lambda\in\mathbb{C}} \mathrm{V}_\lambda \otimes_{\mathbb{C}[z]} \bigoplus_{\lambda\in\mathbb{C}} \mathrm{W}_\lambda & \to & \mathrm{M}/\mathrm{t}_z(\mathrm{M}) \otimes_{\mathbb{C}[z]} \bigoplus_{\lambda\in\mathbb{C}} \mathrm{W}_\lambda & \to & \mathrm{Z} \otimes_{\mathbb{C}[z]} \bigoplus_{\lambda\in\mathbb{C}} \mathrm{W}_\lambda \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \bigoplus_{\lambda\in\mathbb{C}} \mathrm{V}_\lambda \otimes_{\mathbb{C}[z]} \bigoplus_{\lambda\in\mathbb{C}} \mathrm{W}_\lambda & \to & \mathrm{M}/\mathrm{t}_z(\mathrm{M}) \otimes_{\mathbb{C}[z]} \mathrm{N}/\mathrm{t}_z(\mathrm{N}) & \to & \mathrm{L} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \to & \mathrm{M}/\mathrm{t}_z(\mathrm{M}) \otimes_{\mathbb{C}[z]} \mathrm{R} & \to & \mathrm{M}/\mathrm{t}_z(\mathrm{M}) \otimes_{\mathbb{C}[z]} \mathrm{R} \to 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}.$$

Since $Z \otimes_{\mathbb{C}[z]} \bigoplus_{\lambda\in\mathbb{C}} W_\lambda$ and $M/t_z(M)\otimes_{\mathbb{C}[z]}R$ are of z-torsion, L is of z-torsion.

Letting V and W be the weight spaces $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$ and $W = \bigoplus_{\lambda\in\mathbb{C}} W_\lambda$ we only need to prove that $V \otimes_{\mathbb{C}[z]} W = \bigoplus_{\lambda\in\mathbb{C}} (V \otimes_{\mathbb{C}[z]} W)_\lambda$.

We will actually prove $(V \otimes_{\mathbb{C}[z]} W)_\lambda = \bigoplus_{\lambda=\mu+\sigma} V_\mu \otimes_{\mathbb{C}[z]} W_\sigma$.

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We have a decomposition as $\mathbb{C}$[z]-module: $V \otimes_{\mathbb{C}[z]} W = \bigoplus_{\mu,\sigma\in\mathbb{C}} V_\mu \otimes_{\mathbb{C}[z]} W_\sigma$.

Let v be an element of $(V \otimes_{\mathbb{C}[z]} W)_\lambda$.Then, $v = v_1 + v_2 + \cdots v_k$ with $v_i \in V_{\mu_i} \otimes_{\mathbb{C}[z]} W_{\sigma_i}$, this is: $v_i = \sum_{j=1}^{t_i} v_j^{(i)} \otimes w_j^{(i)}$, $v_j^{(i)} \in V_{\mu_i}$ and $w_j^{(i)} \in W_{\sigma_i}$.

Using the structure of $V \otimes_{\mathbb{C}[z]} W$ as B-module, $hv_i = (\sum_{j=1}^{t_i} hv_j^{(i)} \otimes w_j^{(i)} + \sum_{j=1}^{t_i} v_j^{(i)} \otimes hw_j^{(i)})$

$= z(\sum_{j=1}^{t_i} \mu_i v_j^{(i)} \otimes w_j^{(i)} + \sum_{j=1}^{t_i} v_j^{(i)} \otimes \sigma_j w_j^{(i)}) = (\mu_i + \sigma_j) z \sum_{j=1}^{t_i} v_j^{(i)} \otimes w_j^{(i)} = (\mu_i + \sigma_j) z v_i$.

Hence, $hv = hv_1 + hv_2 + \cdots hv_k = z((\mu_1 + \sigma_1)v_1 + (\mu_2 + \sigma_2)v_2 + \cdots (\mu_k + \sigma_k)v_k) = \lambda z v = z(\lambda v_1 + \lambda v_2 + \cdots \lambda v_k)$.

Therefore, $z(\lambda - (\mu_i + \sigma_i))v_i = 0$, $V \otimes_{\mathbb{C}[z]} W$ is z-torsion free. It follows $\lambda = \mu_i + \sigma_i$.

We have proved $(V \otimes_{\mathbb{C}[z]} W)_\lambda \subset \bigoplus_{\lambda=\mu+\sigma} V_\mu \otimes_{\mathbb{C}[z]} W_\sigma$.

Now, let $v \in V_\mu$ and $w \in W_\sigma$ with $\lambda = \mu + \sigma$.

Then, $h(v \otimes w) = (hv \otimes w + v \otimes hw) = (\mu + \sigma) z (v \otimes w) = \lambda z (v \otimes w)$.

Then, $v \otimes w \in (V \otimes_{\mathbb{C}[z]} W)_\lambda$.

We have proved $(V \otimes_{\mathbb{C}[z]} W)_\lambda = \bigoplus_{\lambda=\mu+\sigma} V_\mu \otimes_{\mathbb{C}[z]} W_\sigma$. □

DEFINITION 8. The homogenized category $\mathcal{O}_B$ is the subcategory of the category of generalized weight B-modules $\mathcal{C}$ satisfying the following two conditions:

(i) The module M in $\mathcal{O}_B$ is finitely generated.
(ii) For all $v \in M/t_z(M)$, $\dim \mathbb{C}[e]v < \infty$.

We will see below that the homogenized Verma modules form a subcategory of $\mathcal{O}_B$.

THEOREM 5. *The category $\mathcal{O}_B$ is closed under submodules, quotients and finite direct sums. In particular it is an abelian Krull–Schmidt category.*

Proof. By proposition 15 we only need to prove that condition (ii) is satisfied by submodules and quotients.

If N and M are finitely generated graded B-modules and N is a submodule of M, then $N/t_z(N)$ is a submodule of $M/t_z(M)$ it is clear that $N/t_z(N)$ satisfies condition (ii) whenever $M/t_z(M)$ does.

Assume there is an epimorphism $\pi: M \to N$. Then, $\ker\pi \cap t_z(M) = t_z(\ker\pi)$ implies there is an epimorphism $M/t_z(M) \to N/\pi(t_z(M)) \to 0$ and $\pi(t_z(M)) \subset t_z(N)$. Therefore, there is an epimorphism $N/\pi(t_z(M)) \to N/t_z(N)$ and, hence an epimorphism $M/t_z(M) \to N/t_z(N) \to 0$. It is clear that $N/t_z(N)$ satisfies condition (ii) whenever $M/t_z(M)$ does. □

REMARK 1. Since B is a Koszul algebra of finite global dimension, for a given finitely generated B-module M there is a truncation $M_{\geq k}[k]$ which is Koszul in particular if M is in $\mathcal{O}_B$ the Koszul module $M_{\geq k}[k]$ is in $\mathcal{O}_B$.

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Moreover, there is a commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & t_z(M_{\geq k}) & \to & t_z(M) & \to & X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_{\geq k} & \to & M & \to & M/M_{\geq k} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_{\geq k}/t_z(M_{\geq k}) & \to & M/t_z(M) & \to & Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},$$

with both X, Y finite dimensional $\mathbb{C}$-vector spaces.

PROPOSITION 18. *Every homogenized Verma module $V(\lambda)$ is in $\mathcal{O}_B$.*

Proof. We proved above that the graded B-module $V(\lambda)$ is defined as $V(\lambda)= B/(B\mathrm{e}+B(h\text{-}\lambda z))$ and that as $\mathbb{C}$-module it decomposes $V(\lambda)= \oplus_{k\geq 0}(\oplus_{i+j=k}\mathbb{C}\mathrm{f}^i z^j)$.

An element b of $V(\lambda)$ has a decomposition in homogeneous components: $\mathrm{b}=\mathrm{b}_{i_1}+\mathrm{b}_{i_2}+\cdots b_{i_k}$ with b_{i_j} homogeneous of degree m_j.

We have: $(h\text{-}\mu z)$ b=0=$(h\text{-}\mu z)\mathrm{b}_{i_1}+(h\text{-}\mu z)\mathrm{b}_{i_2}+\cdots(h-\mu z)b_{i_k}$ if and only if for all j, $(h\text{-}\mu z)\mathrm{b}_{ij}=0$.

We may assume b is homogeneous of degree k. Then, $\mathrm{b}=\sum_{i+j=k}\mathrm{c}_{ij}\mathrm{f}^i\mathrm{z}^j$, with $\mathrm{c}_{ij}\in\mathbb{C}$.

$(h\text{-}\mu z)\mathrm{f}^i=h\mathrm{f}^i\text{-}\mu z\mathrm{f}^i=\mathrm{f}^i h\text{-}2i\mathrm{f}^i z\text{-}\mu z\mathrm{f}^i=\mathrm{f}^i(h\text{-}(2\mathrm{i}+\mu)z)$.

It follows $(h-\mu z)f^i=0$ in $V(\lambda)$ if and only if $\lambda=2i+\mu$.

Therefore, $(\mathrm{h}-(\lambda-2\mathrm{i})z)f^i z^j=0$ for all i, j, $\mathrm{i}+\mathrm{j}=\mathrm{k}$.

Since $V(\lambda)$ is z-torsion free it follows $V(\lambda)=\oplus_{i\geq 0}V_{\lambda-2i}$ is a weight module.

By induction we have the following equality: $\mathrm{ef}^n=\mathrm{f}^n\ \mathrm{e}+n\ \mathrm{f}^{n-1}z\ (h\text{-}\ \lambda z)+n\,(\lambda-(n-1))\ \mathrm{f}^{n-1}z^2$.

Hence, $\mathrm{ef}^n+(B\mathrm{e}+B(h-\lambda z))=n(\lambda-(n-1))\mathrm{f}^{n-1}z^2+(B\mathrm{e}+B(h-\lambda z))$.

By induction we have for $k\leq n$, $\mathrm{e}^k\ \mathrm{f}^n+(B\mathrm{e}+B(\ h-\lambda z))=n(n\text{-}1)\ldots(n\text{-}(k\text{-}1))\ (\lambda\text{-}(n\text{-}1))\ ((\lambda\text{-}(n\text{-}2))\ldots(\lambda\text{-}(n\text{-}\mathrm{k}))\ \mathrm{f}^{n-k}z^{2k}+(B\mathrm{e}+B\ (h-\lambda z))$.

In particular for $k=n$, $\mathrm{e}^n\ \mathrm{f}^n\ +(B\mathrm{e}\ +B(h-\lambda z))=\ n!(\lambda\text{-}(n-1))((\lambda-(n-2))\ldots(\lambda-1)\ \lambda z^{2n}+(B\mathrm{e}+B(h-\lambda z))$.

Therefore, $\mathrm{e}^{n+1}\mathrm{f}^n=0$ in $V(\lambda)$.

It follows that for any element $\mathrm{f}^i z^j$ of $V(\lambda)$, $\mathrm{e}^{i+1}\mathrm{f}^i z^j=0$.

From this it follows $V(\lambda)$ is in $\mathcal{O}_B$. □

3.2. Homogeneous weight modules and the category $\mathcal{O}_B$. In this subsection, we continue the study of weight B-modules, and concentrate in those that are z-torsion free and belong to the category $\mathcal{O}_B$. For the full subcategory of $\mathcal{O}_B$ containing such modules we define a duality.

Let $V=\oplus_{\lambda\in\mathbb{C}}V_\lambda$ be a graded weight B-module. Then, V decomposes in homogeneous components as $V=\oplus_{i\in\mathbb{Z}}V_i$. We compare both decompositions. Let $\mathrm{v}\in V_\lambda$ and $v=v_{i_1}+v_{i_2}+\cdots v_{i_k}$ be a decomposition in homogeneous components $\mathrm{v}_{ij}\in V_{i_j}$ with $\mathrm{i}_1<\mathrm{i}_2<\ldots<\mathrm{i}_k$.

$(h-\lambda z)v=0=(h-\lambda z)v_{i_1}+(h-\lambda z)v_{i_2}+\cdots(h-\lambda z)v_{i_k}$ and $(h-\lambda\mathrm{z})\mathrm{v}_{ij}\in V_{i_j+1}$ implies $(h-\lambda\mathrm{z})\mathrm{v}_{ij}=0$ for $1\leq\mathrm{j}\leq\mathrm{k}$. and $\mathrm{v}_{ij}\in V_{ij}\cap V_\lambda$.

We have $\sum_i V_\lambda\cap V_i\subset V_\lambda\subset\sum_i V_\lambda\cap V_i$, therefore: $\oplus_{i\in\mathbb{Z}}V_\lambda\cap V_i=V_\lambda$.

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For the other decomposition $\sum_{\lambda} V_\lambda \cap V_i \subset V_i$.

Let v be an element of V_ℓ and take its decomposition $\mathrm{v} = \mathrm{v}_1 + v_2 + \cdots v_k$, with $\mathrm{v}_i \in V_{\lambda_i} = \oplus_{j\in\mathbb{Z}} V_{\lambda_i} \cap V_j$.

Then each $\mathrm{v}_i = \sum_{j=1}^{t_i} \mathrm{w}_j^{(i}$ and $\mathrm{w}_j^{(i)} \in V_{\lambda_i} \cap V_j$ and degree$(\mathrm{w}_j^{(i)}) = \mathrm{j}$.

We write v as $\mathrm{v} = \sum_{i=1}^{k} \sum_{j=1}^{t_i} \mathrm{w}_j^{(i)} = \sum_{j=1}^{t_i} \sum_{i=1}^{k} \mathrm{w}_j^{(i)}$. It follows $\sum_{i=1}^{k} \mathrm{w}_j^{(i)}$ is homogeneous of degree j .

Since v is homogeneous of degree ℓ the element $\sum_{i=1}^{k} \mathrm{w}_j^{(i)} = 0$ for j$\neq \ell$, and $\mathrm{v} = \sum_{i=1}^{k} \mathrm{w}_\ell^{(i)}$ with $\mathrm{w}_\ell^{(i)} \in V_\ell \cap V_{\lambda_i}$.

We proved $V_\ell \subset \sum_{\lambda\in\mathbb{C}} V_\lambda \cap V_\ell$. It follows $V_\ell = \oplus_{\lambda_\ell\in\mathbb{C}} V_{\lambda_\ell} \cap V_\ell$.

Then $V = \oplus_{\lambda\in\mathbb{C}} \oplus_{i\geq 0} (V_\lambda \cap V_i)$.

As a consequence we have, $V_{\geq k} = \oplus_{i\geq k} V_i = \oplus_{i\geq k}(\oplus_{\lambda\in\mathbb{C}}(V_\lambda \cap V_i))= \oplus_{\lambda\in\mathbb{C}}(\oplus_{i\geq k}(V_\lambda \cap V_i))$ and $(V_{\geq k})_\lambda = \oplus_{i\geq k}(V_\lambda \cap V_i)$.

It follows $V_{\geq k} = \oplus_{\lambda\in\mathbb{C}}(V_{\geq k})_\lambda$, this is, any truncation of a weight module is a weight module.

We have actually proved:

THEOREM 6. *Let $V = \oplus_{\lambda\in\mathbb{C}} V_\lambda$ be a graded weight B-module. Then there is a truncation $V_{\geq k}$ which is Koszul and $V_{\geq k}$ is a weight module.*

COROLLARY 8. *Let M be a finitely generated generalized weight B-module. Then, there exists a weight submodule $V = \oplus_{\lambda\in\mathbb{C}} V_\lambda$ of $M/t_z(M)$ such that for some integer k, $V[k]$ is Koszul and $(M/t_z(M))/V$ is of z-torsion.*

Proof. By definition, there is an exact sequence: $0 \to \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda \to M/\mathrm{t}_z(M) \to R \to 0$ with R of z-torsion. Since $M/t_z(M)$ is finitely generated and B-noetherian, $W = \underset{\lambda\in\mathbb{C}}{\oplus} W_\lambda$ is finitely generated and there is a truncation $V = W_{\geq k}$ with V[k] Koszul and $V = \underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda$ a weight module,

We have an exact commutative diagram,

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & L & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & V & \to & M/t_z(M) & \to & X & \to & 0 \\
 & & \downarrow & & 1\downarrow & & \downarrow & & \\
0 & \to & W & \to & M/t_z(M) & \to & R & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & L & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array},
$$

with L a finite dimensional $\mathbb{C}$-vector space and R of z-torsion.

Therefore: X is of z-torsion. □

THEOREM 7. *Let $V = \underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda$ be a z-torsion free weight module and assume V is in the category $\mathcal{O}_B$. Then, for each $\lambda \in \mathbb{C}$, V_λ is a finitely generated $\mathbb{C}[z]$-module, in particular each V_λ is a free $\mathbb{C}[z]$-module of finite rank.*

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Proof. We have proved the following equalities: $he^n = e^n h + 2ne^n z$ and $hf^n = f^n h - 2nf^n z$.

By hypothesis V is a finitely generated B-module and for each $v \in V$ there is a positive integer ℓ such that $e^\ell v = 0$.

We have decompositions $V = \bigoplus_{i \geq k_0} V_i$ in homogeneous components and $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ decomposes in weight spaces and $V = \bigoplus_{\lambda \in \mathbb{C}} \bigoplus_{i \geq k_0} V_\lambda \cap V_i$.

Hence, we can assume V is generated by homogeneous elements $v_{i_1}, v_{i_2}, \dots v_{i_k}$ of degree $v_{i_j} = i_j$ and $(h - \lambda_j z)v_{i_j} = 0$.

For each j there is an ℓ_j such that $e^{\ell_j} v_{i_j} = 0$, then for $\ell = \max\{\ell_j\}$, $e^\ell v_{i_j} = 0$ for all j.

Let v be an element of V_λ. Then, $v = w_{k_0} + w_{k_0+1} + \cdots w_{k_0+n}$ is a decomposition in homogeneous components $w_{k_0+j} \in V_{k_0+j} \cap V_\lambda$.

We write $w_{k_0+t} = \sum_{j=1}^k b_{tj} v_{i_j}$ with $b_{tj} = \sum_{s+r+n+m=t-i_j} c^{(j)}_{s,r,n,m} f^s e^r h^n z^m$ and $c^{(j)}_{s,r,n,m} \in \mathbb{C}$.

Then, $w_{k_0+t} = \sum_{j=1}^k \sum_{s+r+n+m=t-i_j} c^{(j)}_{s,r,n,m} f^s e^r h^n z^m v_{i_j}$.

Hence each w_{k_0+t} is a linear combination of the vectors$\{ f^s e^r h^n z^m v_{i_j} \}$ and changing for a smaller set if necessary we my assume they are linearly independent and w_{k_0+t} is still a linear combination of such vectors.

We have $h^n v_{i_j} = \lambda_j^n z^n v_{i_j}$, hence $w_{k_0+t} = \sum_{j=1}^k \sum_{s+r+n+m=t-i_j} \lambda_j^n c^{(j)}_{s,r,n,m} f^s e^r z^{m+n} v_{i_j}$.

From the equalities, $(h - \lambda z) f^s e^r = f^s e^r (h\text{-}(\lambda+2(s-r))z)$ and $(h - (\lambda + 2(s\text{-}r))z)v_{i_j} = (\lambda_j - (\lambda+2(s\text{-}r)))z v_{i_j}$ it follows:

$$0 = (h - \lambda z) w_{k_0+t} = z \sum_{j=1}^{k} \sum_{s+r+n+m=t-i_j} (\lambda_j\text{-}(\lambda+2(s\text{-}r)))\lambda_j^n c^{(j)}_{s,r,n,m} f^s e^r z^{m+n} v_{i_j}.$$

Since V is z-torsion free $\sum_{j=1}^k \sum_{s+r+n+m=t-i_j} (\lambda_j\text{-}(\lambda+2(s\text{-}r)))\lambda_j^n c^{(j)}_{s,r,n,m} f^s e^r z^{m+n} v_{i_j} = 0$. By the hypothesis that they are linearly independent $\lambda_j = \lambda+2(s\text{-}r)$ and $v_{i_j} \in V_{\lambda+2(s-r)}$. Therefore, $f^s e^r v_{i_j} \in V_\lambda$ with $0 \leq r \leq \ell\text{-}1$, $s \geq 0$.

If $f^{s'} e^{r'} v_{i_j}$ is another element of V_λ, then $\lambda_j = \lambda+2(s'\text{-}r')$. It follows $s' = s+r'\text{-}r$ with $0 \leq r' \leq \ell\text{-}1$ and $f^{s'} e^{r'} v_{i_j} = f^{s+r'-r} e^{r'} v_{i_j}$. For any pair of fixed elements (s,r), $0 \leq r \leq \ell\text{-}1$, $s \geq 0$, there are only a finite number of elements of the form $f^{s+r'-r} e^{r'} v_{i_j} = f^{s'} e^{r'} v_{i_j}$ in V_λ.

This implies w_{k_0+t} is a polynomial combination $w_{k_0+t} = \sum q(z)_{s,r,i_j} f^s e^r v_{i_j}$ with $q(z)_{s,r,i_j} \in \mathbb{C}[z]$.

It follows V_λ is a finitely generated $\mathbb{C}[z]$-module and V torsion free implies V_λ is a free $\mathbb{C}[z]$-module of finite rank. $\square$

Let M be an object in $\mathcal{O}_B$. Then, there is a weight B-module V and an exact sequence:

$$0 \to V \xrightarrow{i} M/t_z(M) \xrightarrow{p} R \to 0,$$

with R of z-torsion, which induces an exact sequence:

$$0 \to V_\lambda \xrightarrow{i} (M/t_z(M))_\lambda \xrightarrow{p} p((M/t_z(M))_\lambda) \to 0$$

with $p((M/t_z(M))_\lambda)$a $\mathbb{C}[z]$-submodule of R, hence of z-torsion. There is an integer $k \geq 0$ such that $z^k p((M/t_z(M)_\lambda))=0$.It follows $z^k (M/t_z(M))_\lambda) \subset V_\lambda$

But $(M/t_z(M))_\lambda$ z-torsion free, implies the map given by multiplication $z^k:(M/t_z(M))_\lambda \to z^k((M/t_z(M))_\lambda$ is an isomorphism and there is a monomorphism $z^k:(M/t_z(M))_\lambda \to V_\lambda$ hence, $(M/t_z(M))_\lambda$ is a finitely generated torsion free $\mathbb{C}[z]$-module. We have proved the first part of the following:

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PROPOSITION 19. *Let M be an object in $\mathcal{O}_B$. Then,*

(*i*) *For any $\lambda \in \mathbb{C}$ the $\mathbb{C}[z]$-module $(M/t_z(M))_\lambda$ is either zero or it is a free $\mathbb{C}[z]$-module of finite rank.*

(*ii*) *There is a $\mathbb{C}[z]$-module decomposition $M_\lambda = t_z(M_\lambda) \oplus M_\lambda/t_z(M_\lambda)$ where $M_\lambda/t_z(M_\lambda)$ is either zero or it is a free $\mathbb{C}[z]$-module of finite rank.*

Proof. (ii) We have the following commutative exact diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathrm{t}_z(M)_\lambda & \to & M_\lambda & \to & \mathrm{p}(M_\lambda) & \to 0 & , \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathrm{t}_z(M) & \xrightarrow{i} & M & \xrightarrow{p} & M/t_z(M) & \to 0 & \end{array}$$

where $\mathrm{p}(M_\lambda) \subset (M/t_z(M))_\lambda$. It is clear that $\mathrm{t}_z(M_\lambda) = M_\lambda \cap \mathrm{t}_z(M) = \mathrm{t}_z(M)_\lambda$. Therefore: $\mathrm{p}(M_\lambda) = M_\lambda/\mathrm{t}_z(M_\lambda)$

By the first part, $(M/t_z(M))_\lambda$ is either zero or a free $\mathbb{C}[z]$-module of finite rank. In the first case, $\mathrm{p}(M_\lambda) = M_\lambda/\mathrm{t}_z(M_\lambda) = 0$ and the decomposition holds.

In the second case, $\mathrm{p}(M_\lambda) = M_\lambda/\mathrm{t}_z(M_\lambda)$ is a free $\mathbb{C}[z]$-module of finite rank and we have again the decomposition $M_\lambda = \mathrm{t}_z(M_\lambda) \oplus M_\lambda/\mathrm{t}_z(M_\lambda)$. □

Assume now $M \in \mathcal{O}_B$ is z-torsion free. By hypothesis there is a weight submodule V and an exact sequence: $0 \to V \to M \to R \to 0$ with $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ and R of z-torsion. Since $V_\lambda \subset M_\lambda$ implies $\sum_{\lambda \in \mathbb{C}} V_\lambda \subset \sum_{\lambda \in \mathbb{C}} M_\lambda$ and M z-torsion free implies $\sum_{\lambda \in \mathbb{C}} M_\lambda = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$.

There is an epimorphism: $M/V \to M/(\bigoplus_{\lambda \in \mathbb{C}} M_\lambda) \to 0$ and $M/(\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)$ is also of z-torsion. Hence, $\bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ is another weight module approximating M and we may assume $V = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ where each M_λ is either zero or a free $\mathbb{C}[z]$-module of finite rank.

We saw that for $\mathrm{m} \in M_\lambda$, $\mathrm{e}^k\mathrm{m} \in M_{\lambda+2k}$, $\mathrm{f}^k m \in M_{\lambda-2k}$, $\mathrm{h}^k m \in M_\lambda$, $\mathrm{z}^k m \in M_\lambda$. Let M_λ^* be $M_\lambda^* = \mathrm{Hom}_{\mathbb{C}[z]}(M_\lambda, \mathbb{C}[\mathrm{z}])$ and define $(\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)^* = \bigoplus_{\lambda \in \mathbb{C}} \mathrm{Hom}_{\mathbb{C}[z]}(M_\lambda, \mathbb{C}[\mathrm{z}]) = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda^*$.

Given a map $\varphi = (0,0 \ldots \varphi_{\lambda_1}, \varphi_{\lambda_2}, \ldots \varphi_{\lambda_r}, 0,0, \ldots) \in \bigoplus_{\lambda \in \mathbb{C}} (M_\lambda)^*$.

We define $\mathrm{e}^k\varphi = (0,0,..\mathrm{e}^k\varphi_{\lambda_1}, \mathrm{e}^k\varphi_{\lambda_2}, \ldots \mathrm{e}^k\varphi_{\lambda_r}, 0,0, \ldots)$, where $\mathrm{e}^k\varphi_{\lambda_i}: M_{\lambda_i - 2k} \to \mathbb{C}[\mathrm{z}]$ given by $\mathrm{e}^k\varphi_{\lambda_i}(\mathrm{m}) = \varphi_{\lambda_i}(\mathrm{e}^k\mathrm{m})$.

Similarly $\mathrm{f}^k\varphi = (0,0,..\mathrm{f}^k\varphi_{\lambda_1}, \mathrm{f}^k\varphi_{\lambda_2}, \ldots \mathrm{f}^k\varphi_{\lambda_r}, 0,0, \ldots)$, with $\mathrm{f}^k\varphi_{\lambda_i}: M_{\lambda_i+2k} \to \mathbb{C}[\mathrm{z}]$ given by $\mathrm{f}^k\varphi_{\lambda_i}(\mathrm{m}) = \varphi_{\lambda_i}(\mathrm{f}^k\mathrm{m})$.

$\mathrm{h}^k\varphi = (0,0,..\mathrm{h}^k\varphi_{\lambda_1}, \mathrm{h}^k\varphi_{\lambda_2}, \ldots \mathrm{h}^k\varphi_{\lambda_r}, 0,0, \ldots)$, with $\mathrm{h}^k\varphi_{\lambda_i}: M_{\lambda_i} \to \mathbb{C}[z]$ given by $\mathrm{h}^k\varphi_{\lambda_i}(\mathrm{m}) = \varphi_{\lambda_i}(\mathrm{h}^k\mathrm{m})$.

$z^k\varphi = (0,0,..z^k\varphi_{\lambda_1}, z^k\varphi_{\lambda_2}, \ldots z^k\varphi_{\lambda_r}, 0,0, \ldots)$, with $z^k\varphi_{\lambda_i}: M_{\lambda_i} \to \mathbb{C}[z]$ given by $z^k\varphi_{\lambda_i}(\mathrm{m}) = \varphi_{\lambda_i}(z^k\mathrm{m})$.

With these operations $(\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)^*$ becomes a B-module.

We claim $(\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)^*$ is a weight B-module with weight space $(\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)_\mu^* = (M_\mu)^*$

Given a map $\varphi: M_\mu \to \mathbb{C}[\mathrm{z}]$, $(\mathrm{h} - \mu\mathrm{z})\varphi(\mathrm{m}) = \varphi((\mathrm{h} - \mu\mathrm{z})\mathrm{m}) = 0$, hence $(M_\mu)^* \subset (\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)_\mu^*$.

Therefore, $\sum_{\lambda \in \mathbb{C}} (M_\mu)^* \subset \sum_{\mu \in \mathbb{C}} (\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)_\mu^* \subset (\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)^*$.

By definition $\sum_{\mu \in \mathbb{C}} (M_\mu)^* = (\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)^*$. Therefore, $\sum_{\mu \in \mathbb{C}} (\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)_\mu^* = (\bigoplus_{\lambda \in \mathbb{C}} M_\lambda)^*$.

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Each $M_\lambda \neq 0$ is free of finite rank, hence $(M_\lambda)^*$ is free of finite rank and $(\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^*$ is a free $\mathbb{C}$[z]-module, hence z-torsion free.

It follows $\sum_{\mu\in\mathbb{C}}(\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^*_\mu = \underset{\mu\in\mathbb{C}}{\oplus}(\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^*_\mu$.

We have proved $(\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^*$is a weight space with $(\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^{**} \cong \underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda$.

We have proved the following:

Theorem 8. *Let $\mathcal{W}_B$ be the full subcategory of $\mathcal{O}_B$ consisting of the z-torsion free weight B-modules. Then there is a duality $(-)^* : \mathcal{W}_B \to \mathcal{W}_B$ given by: $(\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^* = \underset{\lambda\in\mathbb{C}}{\oplus}(M_\lambda)^*$, where $(\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^*_\mu = (M_\mu)^*$ and $\underset{\mu\in\mathbb{C}}{\oplus}(\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^*_\mu = (\underset{\lambda\in\mathbb{C}}{\oplus} M_\lambda)^*$.*

We end the section with the following.

Proposition 20. *Let M be a finitely generated graded B-module. Then $M \in \mathcal{O}_B$ if and only if its deshomogenization $M/(z\text{-}1)M$ is in $\mathcal{O}$.*

Proof. If M is in $\mathcal{O}_B$, then it is a generalized weight module, this means that there is an exact sequence $0 \to V \to M/t_z(M)$, where $V = \underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda$ and $V_\lambda = \{m \in V \mid hm = \lambda zm\}$. It is clear that V_λ is a $\mathbb{C}[z]$-module and $V_\lambda/(z-1)V_\lambda = \{\overline{m} \in V/(z\text{-}1)V \mid h\overline{m} = \lambda\overline{m}\}$ is a weight U-module. It is also clear that $V/(z\text{-}1)V \cong M/(z\text{-}1)M$.

If $v \in M/t_z(M)$ has $\dim\mathbb{C}[e]v$ finite, then $\overline{v} \in M/(z\text{-}1)M$ has $\dim\mathbb{C}[e]\overline{v}$ finite.

Let's assume that $M/(z\text{-}1)M \in \mathcal{O}$. We know that if $M/t_z(M) = N$, then $N/(z\text{-}1)N = M/(z\text{-}1)M$ is a weight module $N/(z\text{-}1)N = \underset{\lambda\in\mathbb{C}}{\oplus}(N/(z\text{-}1)N)_\lambda$.

Let N_λ be the $\mathbb{C}[z]$-submodule of N, $N_\lambda = \{n \in N \mid (h-\lambda z)n = 0\}$ and $\pi : N \to N/(z\text{-}1)N$ is the natural projection, then for $n \in N$, $(h-\lambda)n + \lambda(z-1)n = (h-\lambda z)n$ implies $\pi(n) \in (N/(z-1)N)_\lambda$.

Given an element $n+(z\text{-}1)N$ of $(N/(z\text{-}1)N)_\lambda$, we decompose n in its homogeneous components $n = n_1 + n_2 + \cdots n_k$, with $\deg(n_i) > \deg(i_{i+1})$ and $t_i = \deg(n_1)\text{-}\deg(n_i)$. Hence, $m = n_1 + z^{t_2}n_2 + \cdots z^{t_k}n_k$is an homogeneous element of N with $\pi(m) = \pi(n)$.

$(h-\lambda z)m + (z\text{-}1)N = (h-\lambda)n + (z\text{-}1)N$. Therefore, $(h-\lambda z)m \in (z\text{-}1)N$ and, as above, $(h-\lambda z)m$ homogeneous implies $(h-\lambda z)m = 0$. We have proved that π induces an exact sequence: $0 \to \ker\pi \cap N_\lambda \to N_\lambda \to (N/(z\text{-}1)N)_\lambda \to 0$, where $(z\text{-}1)N_\lambda \subset \ker\pi$. Hence, there is an epimorphism: $N_\lambda/(z\text{-}1)N_\lambda \to (N/(z\text{-}1)N)_\lambda \to 0$.

Since N is z-torsion free there is an exact sequence: $0 \to \underset{\lambda\in\mathbb{C}}{\oplus} N_\lambda \to N \to K \to 0$ which induces an exact sequence: $0 \to \underset{\lambda\in\mathbb{C}}{\oplus} N_\lambda/(z\text{-}1)N_\lambda \to \underset{\lambda\in\mathbb{C}}{\oplus}(N/(z-1)N)_\lambda \to K/(z\text{-}1)K \to 0$.

It follows $K/(z\text{-}1)K = 0$, and by Corollary 4, K is of z-torsion and M is a generalized weight module.

Let $\overline{v} \in N/(z\text{-}1)N$ be such that $\dim\mathbb{C}[e]\overline{v} < \infty$, this is equivalent to say that there is a polynomial f(e) in e such that $f(e)\overline{v} = 0$, by **[25]** page 136, there is an integer k such that $e^k\overline{v} = 0$.

As above, we may assume v homogeneous, therefore e^k v an homogeneous element in (z-1) N implies e^k v=0.

We have proved $\dim\mathbb{C}[e]v$ is finite. □

We obtain as a corollary the following:

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THEOREM 9. *The full subcategory* $(\mathcal{O}_B)_z$ *of* mod_{B_z} *consisting of all modules* M_z *such that* M *is a module in* $\mathcal{O}_B$ *and the category* $\mathcal{O}$ *are equivalent.*

Proof. By Proposition 3 any finitely generated B_z-module is of the form M_z with M a finitely generated B-module. We proved in Proposition 13 that there is an isomorphism of $(Bz)_0$-modules $M/(z\text{-}1)M \cong M_z/(z\text{-}1)M_z$ which in turn by Proposition 12, isomorphic to the degree zero part of M_z. Since $(Bz)_0 \cong U$, by the previous proposition, the equivalence res:$gr_{B_z} \to \mathrm{mod}_U$ induces an equivalence of categories $(\mathcal{O}_B)_z \cong \mathcal{O}$. □

Let $V = \underset{\lambda\in\mathbb{C}}{\oplus} V_\lambda$ with $V_\lambda = \{\mathrm{v}\in V \mid (\mathrm{h}-\lambda z)\mathrm{v}=0\}$ be a weight B-module and assume $V_\lambda \neq 0$. Then, for v$\in V_\lambda$ ev$\in V_{\lambda-2}$, fv$\in V_{\lambda+2}$, hv$\in V_\lambda$, define ξ by $\xi = \lambda+2\mathbb{Z}$. Then, $V^\xi = \underset{\mu\in\xi}{\oplus} V_\mu$ is a submodule of V, moreover V decomposes as $V = \underset{\xi\in\mathbb{C}/2\mathbb{Z}}{\oplus} V^\xi$.

LEMMA 11. *Consider the homogeneous Casimir element C of B, defined as* $C=(h+z)^2+4fe$. *Then, C is an element of the center of B.*

Proof. hC=$h(h+z)^2$+4$h$fe=$(h+z)^2h$+4(fh-2fz)e=$(h+z)^2h$+4f(eh+2ez-2ez)= k($(h+z)^2$+4fe)h=Ch.

In a similar way, we check eC=Ce and fC=Cf, C commutes with the generators then C commutes with any element of B. □

Let M be a finitely generated graded z-torsion free B-module. Then, $M_\lambda = \{\mathrm{m}\in M \mid (h-\lambda z)\mathrm{m}=0\}$ is a $\mathbb{C}[z]$-module, and $(z\text{-}1)\, M_\lambda$ is a submodule of M_λ.

We claim $(z\text{-}1)\, M_\lambda = ((z\text{-}1)\, M)_\lambda$. It is clear $(z\text{-}1)\, M_\lambda \subset ((z\text{-}1)\, M)_\lambda$.

Let $(z\text{-}1)$m be an element of $((z\text{-}1)\, M)_\lambda$. Then, $(h-\lambda z)(z\text{-}1)\mathrm{m}=(z\text{-}1)(h-\lambda z)\mathrm{m}=0$. Then, $n=(h-\lambda z)$m $n\neq 0$ decomposes in homogeneous components $n=n_1+n_2+\cdots n_k$ with $\deg(n_i) > \deg(n_{i+1})$ and $(z\text{-}1)(n_1+n_2+\cdots n_k) = zn_1+zn_2+\cdots zn_k\text{-}n_1\text{-}n_2\text{-}\ldots\text{-}n_k = 0$ where $\deg(zn_1)>\deg(n_i)$ for all i and $\deg(zn_1)>\deg(zn_i)$ for i$\neq$1. It follows $zn_1 = 0$, but this is contradicts the assumption M is z-torsion free. Therefore, m$\in M_\lambda$.

Given $\tau\in\mathbb{C}$, define $M_\lambda(\tau)=\{\mathrm{m}\in M_\lambda \mid \text{there is } \mathrm{k}\geq 0 \text{ with } (\mathrm{C}\text{-}\mathrm{z}^2\tau)^k\mathrm{m}=0\}$.

$M_\lambda(\tau)$ is a $\mathbb{C}[z]$-submodule of M_λ.

Since C is in the center of B then C $M_\lambda \subset M_\lambda$ and $\mathrm{C}(z\text{-}1)\, M_\lambda \subset (z\text{-}1)M_\lambda$.

For any integer k$\geq$0 and $\tau\in\mathbb{C}$, we have an exact commutative diagram:

$$\begin{array}{ccccccc}
0\to & (z\text{-}1)M_\lambda & \to & M_\lambda & \to & M_\lambda/(z\text{-}1)M_\lambda & \to 0\\
 & \downarrow (\mathrm{C}\text{-}\tau \mathrm{z}^2)^k & & \downarrow (\mathrm{C}\text{-}\tau z^2)^k & & \downarrow \overline{(\mathrm{C}\text{-}\tau z^2)}^k & \\
0\to & (z\text{-}1)M_\lambda & \to & M_\lambda & \to & M_\lambda/(z\text{-}1)M_\lambda & \to 0
\end{array}$$

Assume M is in $\mathcal{O}_B$. Then, the des homogenized module $M/(\mathrm{z}\text{-}1)M$ is in $\mathcal{O}$ and by [25 pag 137] $M_\lambda/(\mathrm{z}\text{-}1)M_\lambda = (M/(z\text{-}1)\ M)_\lambda$ has finite dimension as $\mathbb{C}$-vector space. Then, it has a Jordan form decomposition $(M/(z\text{-}1)\ M)_\lambda = \underset{\tau\in\mathbb{C}}{\oplus} (M/(\mathrm{z}\text{-}1)\ M)_\lambda(\tau)$.

Denote by $M_\lambda(\tau)_k$ the kernel of $(\mathrm{C}-\tau z^2)^k$. Then by the snake lemma we have an exact sequence: $0\to(z\text{-}1)M_\lambda(\tau)_k \to M_\lambda(\tau)_k \xrightarrow{\pi} (M(z\text{-}1)M)_\lambda(\tau))_k$.

We want to prove that π is an epimorphism. Let $m\in M_\lambda$ be such that $\pi(\mathrm{m})\in (M(z\text{-}1)M)_\lambda(\tau))_k$. As above, we may choose m homogeneous. $\overline{(\mathrm{C}\text{-}\tau z^2)}^k(\pi(\mathrm{m}))=0$ implies there is n$\in(z\text{-}1)M_\lambda$ such that $(\mathrm{C}-\tau z^2)^k(\mathrm{m})$=n. Since we are assuming m homogeneous, n is

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homogeneous, then n should be zero, and m$\in M_\lambda(\tau)_k$. If k$\leq \ell$, then $M_\lambda(\tau)_k \subset M_\lambda(\tau)_\ell$ and $M_\lambda(\tau)= \bigcup_{k\geq 0} M_\lambda(\tau)_k$. We have proved that the sequence:

$$0 \to (z\text{-}1)M_\lambda(\tau) \to M_\lambda(\tau) \xrightarrow{\pi} (M/(z-1)M)_\lambda(\tau)) \to 0$$

is exact and it induces the exact commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & (z\text{-}1)(\bigoplus_{\tau\in\mathbb{C}} M_\lambda(\tau)) & \to & \bigoplus_{\tau\in\mathbb{C}} M_\lambda(\tau) & \to & \bigoplus_{\tau\in\mathbb{C}} (M/(z\text{-}1)M)_\lambda(\tau)) & \to 0 \\ & \downarrow u & & \downarrow v & & \downarrow \sigma & \\ 0 \to & (z\text{-}1)M_\lambda & \to & M_\lambda & \to & M_\lambda/(z\text{-}1)M_\lambda & \to 0 \end{array}$$

where the vertical maps u,v, σ are the maps induced by the inclusion and the last column σ is an isomorphism, since $M/(z\text{-}1)M$ is in $\mathcal{O}$.

Restricting to the images of, u,v, σ, we have an exact sequence:

$$0 \to (z\text{-}1)\sum_{\tau\in\mathbb{C}} M_\lambda(\tau) \to \sum_{\tau\in\mathbb{C}} M_\lambda(\tau) \xrightarrow{\pi} \bigoplus_{\tau\in\mathbb{C}} (M(z-1)M)_\lambda(\tau)) \to 0$$

Since M is graded B-module, it is also graded as $\mathbb{C}[z]$-module. and $\sum_{\tau\in\mathbb{C}} M_\lambda(\tau)$ is a graded z-torsion free $\mathbb{C}[z]$-module. We prove now that the sum is direct.

Assume $m_1+m_2+\cdots m_k=0$ with $m_i\in M_\lambda(\tau_i)$ and decompose each m_i in its homogeneous components: $m_i=\sum_j m_{ij}$. Then, for each j, $m_{1j}+m_{2j}+\cdots m_{kj}=0$ and $\pi(m_{1j})+\pi(m_{2j})+\cdots\pi(m_{kj})=0$.

But in $M_\lambda/(z\text{-}1)M_\lambda$ the sum is direct. Therefore, $\pi(m_{ij})=0$ and $m_{ij}\in (z\text{-}1)M_\lambda$. By the argument used above, $m_{ij}=0$, for all i, j. It follows each $m_i=0$.

We have proved there is an exact commutative diagram:

$$\begin{array}{ccccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & (z\text{-}1)(\bigoplus_{\tau\in\mathbb{C}} M_\lambda(\tau)) & \to & \bigoplus_{\tau\in\mathbb{C}} M_\lambda(\tau) & \to & \bigoplus_{\tau\in\mathbb{C}} (M/(z\text{-}1)M)_\lambda(\tau)) & \to 0 \\ & \downarrow u & & \downarrow v & & \downarrow \sigma & \\ 0 \to & (z\text{-}1)M_\lambda & \to & M_\lambda & \to & M_\lambda/(z\text{-}1)M_\lambda & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & K^\lambda & \to & K^\lambda & \to & 0 & \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array} .$$

Applying the functor $B/(z\text{-}1)B\otimes_B$- to the exact sequence:

$$0 \to \bigoplus_{\tau\in\mathbb{C}} M_\lambda(\tau) \to M_\lambda \to K^\lambda \to 0,$$

we obtain the exact sequence:

$$0 \to \bigoplus_{\tau\in\mathbb{C}} (M/(z\text{-}1)M)_\lambda(\tau)) \xrightarrow{\sigma} M_\lambda/(z-1)M_\lambda \to 0.$$

Therefore, $K^\lambda/(z-1)K^\lambda=0$, and by Corollary 4, K^λ is of z-torsion.

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Adding over all λ, we get an exact sequence:

$$0 \to \bigoplus_{\lambda\in\mathbb{C}} \bigoplus_{\tau\in\mathbb{C}} M_\lambda(\tau) \to \bigoplus_{\lambda\in\mathbb{C}} M_\lambda \to \bigoplus_{\lambda\in\mathbb{C}} K^\lambda \to 0.$$

Then, $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$ is a weight module and $K = \bigoplus_{\lambda\in\mathbb{C}} K^\lambda$is a module of z-torsion. Interchanging sums we have an exact sequence:

$$0 \to \bigoplus_{\tau\in\mathbb{C}} \Big(\bigoplus_{\lambda\in\mathbb{C}} M_\lambda(\tau)\Big) \to M \to K \to 0,$$

where $\bigoplus_{\lambda\in\mathbb{C}} M_\lambda(\tau)$ is a submodule of M.

We showed in Section 2 that the usual category $\mathcal{O}$ has a decomposition in blocks $\mathcal{O} = \bigoplus_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} \mathcal{O}^{\xi,\tau}$ and in Theorem 4 we gave the structure of the blocks.

It follows by the above remarks that the homogenized category $\mathcal{O}_B$ decomposes as a union of subcategories $\mathcal{O}_B = \bigcup_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} \mathcal{O}_B^{\xi,\tau}$, where $M \in \mathcal{O}_B^{\xi,\tau}$ if and only if, its des homogenization $M/(z-1)M \in \mathcal{O}^{\xi,\tau}$. We call to the categories $\mathcal{O}_B^{\xi,\tau}$the blocks of $\mathcal{O}_B$.

The categories $\mathcal{O}_B^{\xi,\tau}$are abelian and it is clear that if M and N are in different blocks, any map φ: $M \to N$ factors through a module of z-torsion.

By definition, each block $\mathcal{O}_B^{\xi,\tau}$ of $\mathcal{O}_B$ contains the full subcategory of gr_B of all B-modules of z-torsion, in particular, it contains the finite dimensional B-modules.

By Theorem 9, $(\mathcal{O}_B)_z$ is equivalent to $\mathcal{O}$ and $(\mathcal{O}_B)_z = \prod_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} (\mathcal{O}_B^{\xi,\tau})_z$, where each $(\mathcal{O}_B^{\xi,\tau})_z$ is equivalent to $\mathcal{O}^{\xi,\tau}$.

4. The categories $(\mathcal{O}_B)_z$, $\mathbf{Q}(\mathcal{O}_B)$, $\mathcal{O}$. We start recalling the construction of the categories of "tails" QGr_B and Qgr_B.

Let M be a graded B-module, $t(M) = \sum_{L\in J} L$, and $J = \{L \mid L \subseteq M \dim_{\Bbbk} L < \infty\}$.

Claim: $t(M/t(M)) = 0$.

Let N be a finitely generated sub module of M such that $N + t(M)/t(M) = N/N \cap t(M)$ is finite dimensional over $\Bbbk$. Since B_n is noetherian $N \cap t(M)$ is a finitely generated submodule of $t(M)$, hence of finite dimension over $\Bbbk$. It follows N is finite dimensional, so $N \subset t(M)$.

Let N be an arbitrary sub module of M, with $N + t(M)/t(M)$ finite dimensional over $\Bbbk$, then $N = \sum N_i$, with N_i finitely generated, each $N_i + t(M)/t(M)$ is finite dimensional, therefore $N_i \subset t(M)$. It follows $N \subset t(M)$, and t is an idempotent radical.

If we denote by gr_B, and $\mathrm{gr}_{(B)_Z}$ the categories of finitely generated graded B, and $(B)_Z$-modules, respectively, then the localization functor Q restricts to a functor Q: $\mathrm{gr}_B \to \mathrm{gr}_{(B)_Z}$.

DEFINITION 9. We say that a (graded) B-module is torsion, if $t(M) = M$, and torsion free if $t(M) = 0$.

It is clear $t(M)$ is Z-torsion and $t(M) \subset t_Z(M)$. Therefore if M is torsion, then it is Z -torsion, and if M is Z-torsion free, then it is torsion free.

The torsion free modules form a Serre (or thick) subcategory of Gr_B, we localize with respect to this subcategory, as explained in [**8, 28**]. Denote by QGr_B

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the quotient category, and let π:Gr$_B \to$QGr$_B$ be the quotient functor, QGr$_B$ = Gr$_B$/Torsion, is an abelian category with enough injective objects and π is an exact functor. When taking this quotient we are inverting the maps of B-graded modules, φ: M$\to$N such that Kerφ and Cokerφ are torsion.

The category QGr$_B$ has the same objects as Gr$_B$ and maps:

$$\mathrm{Hom}_{\mathrm{QGr}_B}(\pi(\mathrm{M}),\pi(\mathrm{N})) = \varinjlim \mathrm{Hom}_{\mathrm{Gr}_B}(\mathrm{M}',\mathrm{N}/\mathrm{t}(\mathrm{N})),$$

the limit running through all the sub modules M$'$of M such that M/M$'$ is torsion.

If M is a finitely generated module, then the limit has a simpler form:

$$\mathrm{Hom}_{\mathrm{QGr}_B}(\pi(\mathrm{M}),\pi(\mathrm{N})) = \varinjlim_{\mathrm{k}} \mathrm{Hom}_{\mathrm{Gr}_B}(\mathrm{M}_{\geq k},\mathrm{N}/\mathrm{t}(\mathrm{N})).$$

In case N is torsion free, $\mathrm{Hom}_{\mathrm{QGr}_B}(\pi(\mathrm{M}),\pi(\mathrm{N})) = \varinjlim_{\mathrm{k}} \mathrm{Hom}_{\mathrm{Gr}_B}(\mathrm{M}_{\geq k},\mathrm{N})$.

The functor π, Gr$_B \to$QGr$_B$ has a right adjoint: ϖ: QGr$_B \to$Gr$_B$ such that $\pi\varpi \cong 1$. [**28**].

If we denote by gr$_B$ the category of finitely generated graded B-modules, and by Qgr$_B$ the full subcategory of QGr$_B$ consisting of the objects π(N) with N finitely generated, then the functor π induces by restriction a functor: π:gr$_B \to$Qgr$_B$. The kernel of π is, Kerπ = $\{M \in \mathrm{gr}_B \mid \pi(M) = 0\} = \{M \in \mathrm{gr}_B \mid t(M) = M\}$.

In the other hand, the functor

$Q = (B)_Z \underset{B}{\otimes}$-: gr$_B \to$gr$_{(B)_Z}$ has kernel $\{\mathrm{M} \in \mathrm{gr}_B \mid M_Z = 0\} = \{M \in \mathrm{gr}_B \mid t_Z(M) = M\}$.

It follows: Ker$\pi \subset$Ker($(B)_Z \underset{B}{\otimes}$-).

According to [**28**] (pag. 173 Corollary 3.11) there exists a unique functor ψ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{gr}_B \arrow[rr, "\pi"] \arrow[dr, "(B)_Z \underset{B}{\otimes} -"'] & & \mathrm{Qgr}_B \arrow[dl, "\psi"] \\ & \mathrm{gr}_{(B)_Z} & \end{tikzcd}.$$

This is $\psi\pi = (B)_Z \underset{B}{\otimes}$-.

Proposition 21. *The functor ψ $Qgr_B \to gr_{(B)_Z}$ is exact.*

By definition the category $\mathcal{O}_B$ contains all z-torsion modules, in particular the torsion modules and we can consider the quotient category $Q(\mathcal{O}_B)$ of $\mathcal{O}_B$ and identify it with the full subcategory of Qgr$_B$ consisting of objects in $\mathcal{O}_B$. The functors π, $(B)_Z \underset{B}{\otimes}$- and ψ induce by restriction a commutative diagram of exact functors:

$$\begin{tikzcd} \mathcal{O}_B \arrow[rr, "\pi"] \arrow[dr, "(B)_Z \underset{B}{\otimes} -"'] & & \mathrm{Q}(\mathcal{O}_B) \arrow[dl, "\psi"] \\ & (\mathcal{O}_B)_z & \end{tikzcd}.$$

We proved in [**21**] the following:

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PROPOSITION 22. *Denote by Q the localization functor $Q=(B_n)_Z\underset{B}{\otimes}$- and by C^b(-), the category of bounded complexes. The induced functor $C^b(Q){:}C^b(gr_{B_n})\to C^b(gr_{(B_n)_Z})$ is dense.*

COROLLARY 9. *The functor $C^b(\psi){:}C^b(Qgr_{B_n})\to C^b(gr_{(B_n)_Z})$ is dense.*

Proof. There are functors

$$C^b(\pi):C^b(gr_{B_n})\to C^b(Qgr_{B_n}) \text{ and } C^b(\psi):C^b(Qgr_{B_n})\to C^b(gr_{(B_n)_Z})$$

such that $C^b(\psi)\,C^b(\pi)=C^b(Q)$, and $C^b(Q)$ dense implies $C^b(\psi)$ is dense. □

COROLLARY 10. *The induced functors*

$$K^b(Q){:}K^b(gr_{B_n})\to K^b(gr_{(B_n)_Z}),\ \text{and}\ K^b(\psi){:}K^b(Qgr_{B_n})\to K^b(gr_{(B_n)_Z})$$

are dense.

COROLLARY 11. *The induced functors*

$$D^b(Q){:}D^b(gr_{B_n})\to D^b(gr_{(B_n)_Z})\ \text{and}\ D^b(\psi){:}D^b(Qgr_{B_n})\to D^b(gr_{(B_n)_Z})$$

are dense.

We will describe next he kernel of the functor $D^b(\psi)$. By definition, $\mathrm{Ker}D^b(\psi)=\{\hat{M}^\circ \mid D^b(\psi)(\hat{M}^\circ)$ is acyclic$\}$.

PROPOSITION 23. *There is the following description of $\mathcal{T}=KerD^b(\psi)$:*
$\mathcal{T}=\{\pi M^\circ \mid M^\circ\in D^b(gr_{B_n})$ such that for all i, $H^i(M^\circ)$ is of z-torsion$\}$.

Let's recall some further results on G-algebras from [**21**].

One of the main theorems of the paper is that there exists an equivalence of triangulated categories $D^b(Qgr_{B_n})\,/\mathcal{T}\cong D^b(gr_{(B_n)Z})$, where the category $\mathcal{T}$ is an "épaisse" subcategory of $D^b(Qgr_{B_n})$, and $D^b(Qgr_{B_n})\,/\mathcal{T}$ is the Verdier quotient. [**27**]

We then consider a full embedding of a subcategory $\mathcal{F}$ of $D^b(Qgr_{B_n})$ in $D^b(gr_{(B_n)Z})$. Here $\mathcal{F}$ is the full subcategory of $D^b(Qgr_{B_n})$ consisting of the $\mathcal{T}$-local objects [**27**], this is: $\mathcal{F}=\{\,X^\circ \mid \mathrm{Hom}_{D^b(Qgr_{B_n})}(\mathcal{T},X^\circ)=0\}$.

We then go to the Yoneda algebra $B_n^!$ of B_n and use the duality

$$\overline{\phi}{:}\ \underline{\mathrm{gr}}_{B_n^!}\to D^b(Qgr_{B_n}),$$

given in [**19**] and [**23**]. We obtain a pair of triangulated subcategories $(\mathcal{F}',\mathcal{T}')$ of $\underline{\mathrm{gr}}_{B_n^!}$ such that $\mathcal{F}'\to\mathcal{F}$ and $\mathcal{T}'\to\mathcal{T}$ under the duality ϕ. We obtain the following characterization of the subcategories $\mathcal{T}'$ and $\mathcal{F}'$ of $\underline{\mathrm{gr}}_{B^!}$: $\mathcal{T}'$ is the smallest triangulated subcategories of $\underline{\mathrm{gr}}_{B_n^!}$ containing the induced modules $B_n^!\otimes_{C_n^!}M$ and closed under the Nakayama automorphism, $\mathcal{T}'$ has Auslander–Reiten triangles, and they are of the type $\mathbb{Z}A_\infty$. For the category $\mathcal{F}'$ we obtain the following characterization: $\mathcal{F}'$ consists of the graded finitely generated $B_n^!$-modules, whose restriction to $C_n^!$ is injective. Furthermore, the category $\mathcal{F}'$ is closed under the Nakayama automorphism, it has Auslander–Reiten triangles, and they are of type $\mathbb{Z}A_\infty$.

In order to obtain an equivalence, instead of a duality, we apply the usual duality $D{:}\underline{\mathrm{gr}}_{B_n^!}\to\underline{\mathrm{gr}}_{B_n^{!op}}$ to obtain subcategories T and F of $\underline{\mathrm{gr}}_{B_n^{!op}}$, with $T=D(\mathcal{T}')$, and $F=$

D($\mathcal{F}'$), where T is the smallest triangulated subcategory of $\underline{\mathrm{gr}}_{B_n^{!op}}$ containing the induced modules, and closed under the Nakayama automorphism, and F is the full subcategory of $\underline{\mathrm{gr}}_{B_n^{!op}}$ consisting of those modules, whose restriction to $C_n^!$ is projective.

Finally, we obtain the main results of the paper: there is an equivalence of triangulated categories $\underline{\mathrm{gr}}_{B_n^{!op}}/\ T \cong \mathrm{D}^b(\mathrm{mod}_{A_n})$, and there is a full embedding of triangulated categories $F \to \mathrm{D}^b(\mathrm{mod}_{A_n})$.

We will apply here these results to the particular case of the homogenized enveloping algebra B of sℓ(2,$\mathbb{C}$) its Yoneda algebra $B^!$ and the enveloping algebra U of sℓ(2,$\mathbb{C}$) and the corresponding Gelfand categories $\mathcal{O}_B$, $\mathcal{O}_{B^!}$and $\mathcal{O}$.

Using the restriction of the functor $(B)_Z\underset{B}{\otimes}$- to $\mathcal{O}_B$ and the restriction of ψ to Q($\mathcal{O}_B$) we obtain the following:

Proposition 24. *The functors* $(B)_Z\underset{B}{\otimes}\text{-}\ :\mathcal{O}_B \to (\mathcal{O}_B)_z$ *and* $\psi\ :Q(\mathcal{O}_B)\to(\mathcal{O}_B)_z$*, induce by restriction dense functors in the categories of bounded complexes:*

$$C^b(Q){:}C^b(\mathcal{O}_B) \to C^b((\mathcal{O}_B)_z) \text{ and}$$

$$C^b(\psi){:}C^b(Q(\mathcal{O}_B)) \to C^b((\mathcal{O}_B)_z).$$

Using the exactness of the functors $(B)_Z\underset{B}{\otimes}$- and ψ we obtain induced dense functors in the corresponding homotopy categories.

Proposition 25. *The functors* $(B)_Z\underset{B}{\otimes}$- *and* ψ *induce dense functors of triangulated categories in the corresponding homotopy and derived categories, respectively:*

$$K^b(Q){:}K^b(\mathcal{O}_B) \to K^b((\mathcal{O}_B)_z) \text{ and } K^b(\psi){:}K^b(Q(\mathcal{O}_B)) \to K^b((\mathcal{O}_B)_z).$$

$$D^b(Q){:}D^b(\mathcal{O}_B) \to D^b((\mathcal{O}_B)_z) \text{ and } D^b(\psi){:}D^b(Q(\mathcal{O}_B)) \to D^b((\mathcal{O}_B)_z).$$

Denote the kernel of D^b(Q) by $\mathcal{T}_{\mathcal{O}_B}$=$\{\hat{\mathrm{M}}^\circ \in \mathrm{D}^b(\mathrm{Q}(\mathcal{O}_B))|D^b(\psi)(\hat{\mathrm{M}}^\circ)$ is acyclic$\}$.

Proposition 26. *There is the following description of* $\mathcal{T} = KerD^b(\psi)$*:*
$\mathcal{T}_{\mathcal{O}_B} = \{\pi M^\circ\ |M^\circ \in D^b(\mathcal{O}_B)$ *such that for all i,* $H^i(M^\circ)$ *is of z-torsion*$\}$.

We remarked in Section 1 that the Nakayama automorphism σ: $B \to B$ is of the form $\sigma(z) = u_0 z$, $\sigma(e) = u_1 e$, $\sigma(f) = u_2 f$, $\sigma(h) = u_3 h$, with $u_i \in \mathbb{C} - \{0\}$. Hence, it is clear that Nakyama's automorphism send weight modules to weight modules an induces an automorphism in the full subcategory of gr_B of the modules of z-torsion, hence on the full subcategory of z-torsion free modules. It follows σ induces an automorphism σ: $\mathcal{O}_B \to \mathcal{O}_B$, where $\sigma(M)$ is the $\mathbb{C}$-vector space M with twisted multiplication b $*$m= σ(b)m.

As a consequence we have that there is an auto equivalence σ: $\mathrm{D}^b(\mathcal{O}_B)\to\mathrm{D}^b(\mathcal{O}_B)$, inducing by restriction an equivalence $\sigma{:}\mathcal{T}_{\mathcal{O}_B} \to \mathcal{T}_{\mathcal{O}_B}$.

The subcategory $\mathcal{T}_{\mathcal{O}_B}$ is a thick (épaisse) subcategory of $\mathrm{D}^b(\mathcal{O}_B)$ and we can take the Verdier quotient $\mathrm{D}^b(\mathcal{O}_B)/\mathcal{T}_B$ [**27**].

It follows by Proposition 26 and [**21**] the following:

Theorem 10. *There are equivalences of triangulated categories:*

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(*i*) $D^b(\mathcal{O}_B)/\mathcal{T}_{\mathcal{O}_B} \cong D^b((\mathcal{O}_B)_z) \cong D^b(\mathcal{O})$

(*ii*) $D^b((\mathcal{O}_B)_z) \cong \prod_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} D^b((\mathcal{O}_B^{\xi,\tau})_z) \cong \prod_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} D^b(\mathcal{O}^{\xi,\tau})$.

Proof. (i) Is obtained by restriction of the equivalence given in **[21]** and by Theorem 9. (ii) is the block decomposition given above, plus Dade´s theorem. □

We can use know the equivalence $D\overline{\phi}$: $\underline{\mathrm{gr}}_{B^{!op}} \to D^b(\mathrm{Qgr}_B)$ to obtain a triangulated full subcategory $\underline{\mathcal{O}}_{B^{!op}}$ of $\underline{\mathrm{gr}}_{B_n^{!op}}$ which is by restriction equivalent to $D^b(\mathcal{O}_B)$. We will call $\underline{\mathcal{O}}_{B^{!op}}$ the Gelfand category $\mathcal{O}$ of the algebra $B^!$.

REMARK 2. The category $\underline{\mathcal{O}}_{B^{!op}}$ has Auslander–Reiten triangles and they are of type $\mathbb{Z}A_\infty$.

Proof. Let M be an indecomposable finitely generated $B^{!op}$-modules in $\underline{\mathcal{O}}_{B^{!op}}$. There is an almost split sequence: $0\to \Omega^2\sigma M\to E\to M\to 0$, which induces an Auslander–Reiten triangle $\Omega^2\sigma M\to E\to M\to \Omega\sigma M$ in $\underline{\mathrm{gr}}_{B_n^{!op}}$. Applying the equivalence $D\overline{\phi}$, and observing that the equivalence sends the shift to the shift, we obtain an Auslander–Reiten triangle $D\overline{\phi}(\sigma M)[2]\to D\overline{\phi}(E)\to D\overline{\phi}(M)\to D\overline{\phi}(\sigma M)[1]$. Since $D^b(\mathcal{O}_B)$ is invariant under the Nakayama permutation σ both $D\overline{\phi}(M)$ and $D\overline{\phi}(\sigma M)[1]$ are in $D^b(\mathcal{O}_B)$ the map $D\overline{\phi}(M)\to D\overline{\phi}(\sigma M)[1]$ induces a triangle in $D^b(\mathcal{O}_B)$ of the form $D\overline{\phi}(\sigma M)[2]\to X\to D\overline{\phi}(M)\to D\overline{\phi}(\sigma M)[1]$. It follows that X is isomorphic to $D\overline{\phi}(E)$ in $D^b(\mathcal{O}_B)$ and $D\overline{\phi}(\sigma M)[2]\to D\overline{\phi}(E)\to D\overline{\phi}(M) \to D\overline{\phi}(\sigma M)[1]$ is a triangle in $D^b(\mathcal{O}_B)$. By definition of $\underline{\mathcal{O}}_{B^{!op}}$ the triangle $\Omega^2\sigma M\to E\to M\to \Omega\sigma M$ is a triangle in $\underline{\mathcal{O}}_{B^{!op}}$. □

The pair of triangulated sub categories (T,F) corresponding to the pair $(\mathcal{T},\mathcal{F})$ under the equivalence $D\overline{\phi}$, where $\mathcal{T}$ is the subcategory of $D^b(\mathrm{Qgr}_{B_n})$ of all complexes with homology of z-torsion and $\mathcal{F}$ the corresponding category of $\mathcal{T}$-local objects, was described in **[21]** as follows:

(*i*) $\mathcal{T}$ is the smallest triangulated sub category of $\underline{\mathrm{gr}}_{B_n^{!op}}$ containing the induced modules $M \otimes_{C_n^!} B_n^!$.

(*ii*) $\mathcal{T}$ has Auslander–Reiten triangles and they are of type $\mathbb{Z}A_\infty$.

(*iii*) The category $\mathcal{F}$ consists of the objects M in $\underline{\mathrm{gr}}_{B_n^{!op}}$ whose restriction to $C_n^!$ is projective.

(*iv*) $\mathcal{F}$ has Auslander–Reiten triangles and they are of type $\mathbb{Z}A_\infty$

We can define now the corresponding categories in $\underline{\mathcal{O}}_{B^{!op}}$.

The category $\mathcal{T}_{\mathcal{O}_{B^!}}$ is $\mathcal{T}\cap\underline{\mathcal{O}}_{B^{!op}}$ and the category of $\mathcal{T}_{\mathcal{O}_{B^!}}$-local objects is $\mathcal{F}_{\mathcal{O}_{B^!}} = \mathcal{F}\cap\underline{\mathcal{O}}_{B^{!op}}$.

Hence, $\mathcal{T}_{\mathcal{O}_{B^!}}$ is an épaisse subcategory of $\underline{\mathcal{O}}_{B^{!op}}$ and the Verdier quotient $\underline{\mathcal{O}}_{B^{!op}}/\mathcal{T}_{\mathcal{O}_{B^!}}$ is equivalent to $\prod_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} D^b((\mathcal{O}^{\xi,\tau})$.

There is a full embedding $\mathcal{F}_{\mathcal{O}_{B^!}}$ in $\prod_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} D^b((\mathcal{O}^{\xi,\tau})$.

Since the triangles in $\underline{\mathcal{O}}_{B^{!op}}$ are the same as in $\underline{\mathrm{gr}}_{B_n^{!op}}$, the categories $\mathcal{T}_{\mathcal{O}_{B^!}}$ and $\mathcal{F}_{\mathcal{O}_{B^!}}$ have Auslander–Reiten triangles and they are of type $\mathbb{Z}A_\infty$.

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