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GROWTH OF FUNCTIONS IN H^p

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There are two Hardy and Littlewood theorems which describe the rate of growth of functions in H^p on the unit circle T. In this paper we first establish their analogues on Euclidean space $\mathbb{R}^n$ and then apply them to solve multiplier and factorization problems on $H^p(\mathbb{R}^n)$.

0. Introduction

Hardy and Littlewood [6], [7] gave two fundamental inequalities to describe the rate of growth of functions in H^p on the unit circle T. The inequalities can be stated as follows.

THEOREM 0.1. *If* $f \in H^p(T)$, $0 < p \le 2$, *then*

$$\sum_{n=1}^{\infty} n^{p-2}|a_n|^p < \infty \quad \text{and} \quad \left[\sum_{n=1}^{\infty} n^{p-2}|a_n|^p\right]^{1/p} \le c_p\|f\|_{H^p}$$

where c_p *depends only on* p, *and* $a_n = \hat{f}(n)$.

Theorem 0.1 is usually called the Hardy-Littlewood inequality for $0 < p \le 2$ and the Hardy inequality for $p = 1$.

THEOREM 0.2. *If* $f \in H^p(T)$, $0 < p \le 1$, *then, for* $n \ge 1$,

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$$|a_n| \leq cn^{(1/p)-1}\|f\|_{H^p}$$

where c *is independent of* n, f *and where* $a_n = \hat{f}(n)$.

Theorem 0.2 is due to Hardy and Littlewood [7] although Soviet Mathematicians ascribe it to G.A. Fridman.

The original proofs of Theorems 0.1 and 0.2 are complicated and use complex methods which cannot be extended to higher dimensions. In this article we use real methods to establish analogues on Euclidean space $\mathbb{R}^n$ of Theorems 0.1 and 0.2. Moreover our real methods could be applied to give a simple proof of Theorems 0.1 and 0.2. Finally we apply our results to solve multiplier and factorization problems on $H^p(\mathbb{R}^n)$. The main tools of our proofs are from Peetre [10], Fefferman and Stein [4] and Latter [8] (see Theorem 0.3 below).

For the later use we recall that, for $0 < p \leq 1$, a p-atom is a function a on $\mathbb{R}^n$ satisfying

(i) a is supported in a cube I in $\mathbb{R}^n$,

(ii) $\|a\|_{L^\infty} \leq |I|^{-1/p}$, where $|I|$ is the volume of I ,

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any multi-index α with $|\alpha| \leq n\big((1/p)-1\big)$.

THEOREM 0.3. *A distribution* f *is in* $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, *if and only if*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \quad \textit{with} \quad \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} < \infty ,$$

where a_j *are* p*-atoms and* λ_j *are complex numbers. Moreover, there are positive constants* c_1 *and* c_2 *which are independent of functions* f *with*

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$$c_1\|f\|_{H^p} \le \inf\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \le c_2\|f\|_{H^p} .$$

The "inf" runs over all representations

$$f = \sum_{j=1}^{\infty} \lambda_j a_j .$$

Theorem 0.3 was established by Latter [8] for $n \ge 1$ and $0 < p \le 1$, by Fefferman [3] for $p = n = 1$, and by Coifman [1] for $0 < p < 1$ and $n = 1$.

Throughout this article, we will use c to denote a constant which may be different in the same proof.

1. Hardy and Littlewood inequalities

In this section we prove analogous results on $\mathbb{R}^n$ of Hardy and Littlewood inequalities for Fourier coefficients.

THEOREM 1.1. *If $f \in H^p(\mathbb{R}^n)$, $0 < p \le 1$, then the Fourier transform $\hat{f}$ of f satisfies*

$$|\hat{f}(\xi)| \le c|\xi|^{n((1/p)-1)}\|f\|_{H^p} \quad \textit{for} \quad 0 \ne \xi \in \mathbb{R}^n$$

where c depends only on n and p.

We need the following lemma.

LEMMA 1.2. *Let a be a p-atom, $0 < p \le 1$, and ψ a rapidly decreasing function on $\mathbb{R}^n$ such that $\hat{\psi}$ does not vanish anywhere on the unit disk. Let $\psi_t(x) = t^{-n}\psi(t^{-1}x)$ for $t > 0$ and $u(x, t) = a * \psi_t(x)$. Then*

$$\sup_{x\in\mathbb{R}^n} |u(x, t)| \le ct^{-n/p}\|a\|_{H^p} , \quad t > 0 .$$

Proof. Denote the nontangential maximal function u^* of u by

$$u^*(x) = \sup_{|y-x|\le t} |u(y, t)| .$$

By Fefferman and Stein [4], $u^* \in L^p(\mathbb{R}^n)$ and there are constants c_1 and c_2 with $c_1\|a\|_{H^p} \le \|u^*\|_{L^p} \le c_2\|a\|_{H^p}$. Fix $t > 0$, $x \in \mathbb{R}^n$; we have $|u(y, t)| \le u^*(x)$ for $|y-x| \le t$. Thus

$$\begin{aligned}|u(x, t)|^p\Omega_n t^n &= \int_{|y-x|\le t} |u(x, t)|^p dy \\ &\le \int_{|y-x|\le t} |u^*(y)|^p dy \\ &\le \|u^*\|^p_{L^p} \\ &\le c\|a\|^p_{H^p}\end{aligned}$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. We obtain

$$\sup_{x\in\mathbb{R}^n} |u(x, t)| \le ct^{-n/p}\|a\|_{H^p} .$$

This proves our assertion.

Proof of Theorem 1.1. We divide the proof into several steps.

(1) Let a be a p-atom and ψ as in Lemma 1.2. Let $M = \min_{|\xi|=1} |\hat{\psi}(\xi)|$; then $M > 0$. For $t > 0$ we have

$$\begin{aligned}|\hat{a}(\xi)\hat{\psi}(t\xi)| &= \left|\left(a*\psi_t\right)^\wedge(\xi)\right| \\ &\le \int_{\mathbb{R}^n} |u(x, t)|dx \\ &\le \left(\int_{\mathbb{R}^n} |u(x, t)|^p dx\right)\left(\sup_{x\in\mathbb{R}^n} |u(x, t)|\right)^{1-p} \\ &\le c\|a\|^p_{H^p} t^{-(n/p)(1-p)}\|a\|^{1-p}_{H^p} \quad \text{by Lemma 1.2} \\ &= ct^{-n\left((1/p)-1\right)}\|a\|_{H^p} \\ &= ct^{-n\left((1/p)-1\right)} .\end{aligned}$$

The last equality is due to the fact $\|a\|_{H^p} \le c$ for some $c > 0$ and all

p-atoms a . Choose $t > 0$ with $|t\xi| = 1$, since $|\hat{\psi}(t\xi)| \geq M$,

$$|\hat{a}(\xi)| \leq c|\xi|^{n((1/p)-1)} .$$

(2) Let $f \in H^p(\mathbb{R}^n)$ with $f = \sum_{j=1}^{N} \lambda_j a_j$, where a_j are p-atoms and $\left(\sum_{j=1}^{N} |\lambda_j|^p\right)^{1/p} \leq c\|f\|_{H^p}$. By (1), the inequality $(a+b)^p \leq a^p + b^p$ for $0 < p \leq 1$, $a \geq 0$, $b \geq 0$, and Theorem 0.3, we have

$$\begin{aligned} |\hat{f}(\xi)| &\leq \sum_{j=1}^{N} |\lambda_j||\hat{a}_j(\xi)| \\ &\leq \left(\sum_{j=1}^{N} |\lambda_j|^p\right)^{1/p} c|\xi|^{n((1/p)-1)} \\ &\leq c|\xi|^{n((1/p)-1)}\|f\|_{H^p} . \end{aligned} \tag{1.3}$$

Set

$$D = \left\{f \in H^p(\mathbb{R}^n) \quad f = \sum_{j=1}^{k} \lambda_j a_j,\ \left(\sum_{j=1}^{k} |\lambda_j|^p\right)^{1/p} \leq c\|f\|_{H^p},\ k = 1, 2, \ldots\right\} .$$

Fix $\xi \in \mathbb{R}^n$. By inequality (1.3), the mapping $U : f \to \hat{f}(\xi)$ is a continuous linear functional on D . By Theorem 0.3, D is dense in H^p . For general $f \in H^p(\mathbb{R}^n)$, take a sequence $\{f_k\}_{k=1}^{\infty}$ in D with $f_k \to f$ in H^p . Therefore $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in H^p . As a consequence of inequality (1.3), $\{\hat{f}_k(\xi)\}_{k=1}^{\infty}$ is a Cauchy sequence in the complex numbers $\mathbb{C}$. Take $l \in \mathbb{C}$ with $l = \lim_{k\to\infty} \hat{f}_k(\xi)$. Define

$$\hat{f}(\xi) = l .$$

We have

$$\begin{aligned}|\hat{f}(\xi)| &= \lim_{k\to\infty} |\hat{f}_k(\xi)| \\ &\le \lim_{k\to\infty} c|\xi|^{n((1/p)-1)}\|f_k\|_{H^p} \quad \text{by inequality (1.3)} \\ &= c|\xi|^{n((1/p)-1)}\|f\|_{H^p} .\end{aligned}$$

Note that since $f_k \to f$ in H^p implies that $f_k \to f$ and $\hat{f}_k \to \hat{f}$ in the sense of distribution, and $\hat{f}$ is locally integrable, the Fourier transform $\hat{f}$ defined here and the Fourier transform $\hat{f}$ in the sense of distribution coincide.

Next we come to another description of the rate of growth of Fourier transforms of functions on H^p.

THEOREM 1.4. *Let* $f \in H^p(\mathbf{R}^n)$, $0 < p \le 2$. *Then*

$$\left(\int_{\mathbf{R}^n} |\xi|^{n(p-2)}|\hat{f}(\xi)|^p d\xi\right)^{1/p} \le c\|f\|_{H^p} .$$

Proof. (1) $0 < p \le 1$. Let a be a p-atom supported in a cube Q in $\mathbf{R}^n$ with sides parallel to the axes centered at 0 and with side length l. Then

$$\begin{aligned}|\hat{a}(\xi)| &\le \int_Q |a(x)||e^{-2\pi i x\cdot\xi}-1|dx \\ &\le c\int_Q |x||\xi||Q|^{-1/p}dx \\ &= cl^{1+n-(n/p)}|\xi| ,\end{aligned}$$

$$\begin{aligned}\int_{|\xi|\le l^{-1}} |\xi|^{n(p-2)}|\hat{a}(\xi)|^p d\xi &\le c\int_{|\xi|\le l^{-1}} |\xi|^{n(p-2)}l^{p+np-n}|\xi|^p d\xi \\ &= c ,\end{aligned}$$

$$\int_{|\xi|>l^{-1}} |\xi|^{n(p-2)}|\hat{a}(\xi)|^p d\xi \le \left(\int |\hat{a}(\xi)|^2 d\xi\right)^{p/2}\left(\int_{|\xi|\ge l^{-1}} |\xi|^{-2n} d\xi\right)^{(2-p)/2}$$
$$\le \left(\int |a(x)|^2 dx\right)^{p/2}\left(\int_{|\xi|\ge l^{-1}} |\xi|^{-2n} d\xi\right)^{(2-p)/2}$$
$$= cl^{(np/2)-n}l^{n-(np/2)}$$
$$= c .$$

Combining the above inequalities we have

$$\left(\int_{\mathbf{R}^n} |\xi|^{n(p-2)}|\hat{a}(\xi)|^p d\xi\right)^{1/p} \le c . \tag{1.5}$$

Note that the translation of a p-atom does not change inequality (1.5).

Let $f \in D$, where D is as in the proof of Theorem 1.1. From (1.5) we get

$$\left(\int_{\mathbf{R}^n} |\xi|^{n(p-2)}|\hat{f}(\xi)|^p d\xi\right)^{1/p} \le c\|f\|_{H^p} .$$

For general $f \in H^p(\mathbf{R}^n)$, since D is dense in $H^p(\mathbf{R}^n)$, if we apply the Fatou Lemma, we have the desired inequality.

(2) $1 < p \le 2$. The operator

$$B : f \to \left(\int_{\mathbf{R}^n} |\xi|^{n(p-2)}|\hat{f}(\xi)|^p d\xi\right)^{1/p}$$

is a bounded sublinear operator of type (H^1, L^1) and (L^2, L^2) . By a result of Macias (see Coifman and Weiss [2, p. 597])

$$\left(\int_{\mathbf{R}^n} |\xi|^{n(p-2)}|\hat{f}(\xi)|^p d\xi\right)^{1/p} \le c\|f\|_{H^p} .$$

This completes the proof.

2. Multiplier and factorization problems

The following multiplier theorem is a consequence of Theorem 1.1.

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THEOREM 2.1. *For* $0 < p \le 1$, m *is a multiplier of* $H^p(\mathbb{R}^n)$ *into* $L^\infty(\mathbb{R}^n)$ *if and only if*

$$|m(\xi)| \le c|\xi|^{-n((1/p)-1)} \quad \textit{for} \quad \xi \ne 0 .$$

Proof. Let $|m(\xi)| \le c|\xi|^{-n((1/p)-1)}$ for $\xi \ne 0$. By Theorem 1.1,

$$|\hat{f}(\xi)| \le c|\xi|^{n((1/p)-1)}\|f\|_{H^p} \quad \text{for} \quad f \in H^p \quad \text{and all} \quad \xi \ne 0 .$$

We obtain

$$|\hat{f}(\xi)m(\xi)| \le c\|f\|_{H^p} , \text{ for all } \xi \ne 0 .$$

Therefore $m(\xi)$ is a multiplier of H^p into L^∞ .

Conversely, suppose m is a multiplier of H^p into L^∞ . We have

(2.2) $$|m(\xi)\hat{f}(\xi)| \le c\|f\|_{H^p} \quad \text{for} \quad f \in H^p(\mathbb{R}^n) \quad \text{and} \quad \xi \in \mathbb{R}^n .$$

Let $\psi \in L^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \psi \subset B_1(0)$, where $B_1(0)$ is the unit ball centered at 0 , $\int \psi(x)x^\alpha dx = 0$ for $|\alpha| \le N = n((1/p)-1)$, $\|\psi\|_{L^\infty} = 1$ and $\hat{\psi}$ never vanishes on $B_1(0)$. Let $\psi_t(x) = t^{-n}\psi(t^{-1}x)$ for $t > 0$. Routine arguments reveal that we can decompose ψ_t into

$$\psi_t(x) = a(t)b(x)$$

where $b(x)$ is a p-atom and $|a(t)| \le ct^{n((1/p)-1)}$. By Theorem 0.3, $\psi_t \in H^p(\mathbb{R}^n)$ and

$$\|\psi_t\|_{H^p} \le ct^{n((1/p)-1)} .$$

Choose $f = \psi_t$ in inequality (2.2). We have

$$|m(\xi)\hat{\psi}(t\xi)| \le ct^{n((1/p)-1)} .$$

Take $t > 0$ with $|t\xi| = 1$ for $\xi \ne 0$; then

$$|m(\xi)| \leq c|\xi|^{-n((1/p)-1)} .$$

This proves the theorem.

We next give an application of Hardy inequality to factorization problem of $H^1(\mathbb{R}^n)$.

DEFINITION 2.3. A homogeneous Banach algebra on $\mathbb{R}^n$ is a Banach algebra $(B, \| \|)$ of integrable functions on $\mathbb{R}^n$ satisfying $\| \| \geq \| \|_{L^1}$, and

H1. for all $f \in B$, $x \in \mathbb{R}^n$ then $f_x \in B$ and $\|f_x\| = \|f\|$ where $f_x(y) = f(y-x)$ for all $y \in \mathbb{R}^n$,

H2. the mapping $x \to f_x$ is continuous from $\mathbb{R}^n$ into $(B, \| \|)$.

$L^1(\mathbb{R}^n)$ is the largest homogeneous Banach algebra on $\mathbb{R}^n$. Moreover $H^1(\mathbb{R}^n)$ is a homogeneous Banach algebra on $\mathbb{R}^n$ (see Neri [9], Wigley [13], Wang [11], [12]). Therefore we have $H^1(\mathbb{R}^n) * H^1(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$. Wigley [13] asked if $H^1(\mathbb{R}^n) * H^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$. The answer is negative just by Theorem 1.4 for $p = 1$ and the following result of Feichtinger, Graham and Lakien [5].

THEOREM 2.4. *Let B be a homogeneous Banach algebra on $\mathbb{R}^n$, and $d\mu$ an unbounded regular measure on $\mathbb{R}^n$. If $B * B = B$, then there is $f \in B$ with $\hat{f} \in L^p(d\mu)$ for all p , $0 < p < \infty$.*

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