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ON DUNFORD–PETTIS OPERATORS

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ABSTRACT. Let X be a complemented subspace of a Banach lattice E. It is shown that if every Dunford–Pettis operator from $L_1[0, 1]$ into X is Pettis-representable then X has the Radon–Nikodym property.

In [1] Bourgain showed that if every Dunford–Pettis operator from $L_1[0, 1]$ to a Banach space X is Bochner-representable then X has the Radon–Nikodym property. In this paper we show that if X is complemented in a Banach lattice E and if every Dunford–Pettis operator from $L_1[0, 1]$ into X is Pettis-representable then X has the Radon–Nikodym property.

All the notions used in this paper and not defined can be found in ([2], [4], [6]). Let E be a Banach space and let T be an operator from $L_1[0, 1]$ into E.

DEFINITION 1. (i) The operator T is said to be Dunford–Pettis if the set $\{T(1_A);\ A$ is a measurable subset of $[0, 1]\}$ is relatively compact in E.

(ii) The operator T is said to be Bochner- (resp. Pettis) representable if there exists $g:[0, 1]\to E$ Bochner integrable and essentially bounded (resp. Pettis integrable and scalarly essentially bounded) such that for every f in $L_1[0, 1]$, $T(f) = \text{Bochner-}\int_0^1 fg\, d\lambda$ (resp., $T(f) = \text{Pettis-}\int_0^1 fg\, d\lambda$).

It is well known that the Dunford–Pettis operators are precisely those which map weakly convergent sequences into norm convergent sequences, it is also known that a Pettis representable operator is Dunford–Pettis and that a Dunford–Pettis operator is not in general Pettis-representable.

Bourgain showed in [1] that a Banach space E has the Radon–Nikodym property if and only if every Dunford–Pettis operator is Bochner-representable; he also constructed an operator $T: L_1[0, 1]\to c_0$ such that T is Dunford–Pettis but T is not even Pettis-representable in l_∞.

Because this operator is useful in the sequel we are going to describe it quickly. First construct a sequence $(A_n)_{n\geq 1}$ of measurable subsets of $[0, 1]$ such

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that

(i) $\lim_n \lambda(A_n)=0$

(ii) $\{1_{A_n}\}_{n\ge 1}$ is dense in $\{0,1\}^{[0,1]}$ endowed with the product topology.

It is clear that $T: L_1[0,1]\to c_0$ defined by $T(f)=(\int_{A_n} f\,d\lambda)_{n\ge 1}$ is Dunford–Pettis and by using a non measurable cluster point of the sequence $\{1_{A_n}\}_{n\ge 1}$ one can show that T is not Pettis-representable in l_∞.

Definition 2. A Banach space E has the separable complementation property if every separable subspace Y of E is contained in a complemented separable subspace Z of E.

Lemma 3. *Let E be a Banach space having the separable complementation property and let F be a subspace of E. If $T: L_1[0,1]\to F$ is Pettis-representable in F, then it is Bochner-representable in F.*

Proof. Let Y = the closure of $T(L_1[0,1])$. Let Z be a separable subspace of E containing Y and complemented in E by a projection $V: E\to Z$. Let g be the Pettis derivative of T. The map $t\to V(g(t))$ from $[0,1]\to Z$ is strongly measurable and essentially bounded and hence is Bochner integrable. Therefore for every f in $L_1[0,1]$

$$T(f)=V(T(f))=V\left(\text{Pettis-}\int_0^1 fg\,d\lambda\right)$$

$$=\text{Pettis-}\int_0^1 fV(g)\,d\lambda=\text{Bochner-}\int_0^1 fV(g(t))\,d\lambda.$$

This implies that the map $t\to V(g(t)$ takes its values λ-almost everywhere in F and it is the Bochner derivative of T in F.

Proposition 4. *Let E be a Banach space such that every Dunford–Pettis operator from $L_1[0,1]$ into E is Pettis-representable, then E does not contain a subspace isomorphic to c_0.*

Proof. Suppose that c_0 is isomorphic to a subspace of E, let $S: c_0\to E$ be this isomorphism, the double adjoint S^{**} of S embeds l_∞ in E^{**}. Let U be a projection from E^{**} to $S^{**}(l_\infty)$ and consider the following diagram:

$$c_0 \xrightarrow{S} E \xrightarrow{Q} E^{**} \xrightarrow{U} S^{**}(l_\infty).$$

Let $T: L_1[0,1]\to c_0$ the Dunford–Pettis operator constructed by Bourgain. By hypothesis SoT is Pettis-representable in E and hence by the above diagram T will be Pettis-representable in l_∞, a contradiction that finishes the proof.

Corollary 5. *If a Banach space has the weak-Radon Nikodym property then E does not contain any isomorphic copy of c_0.*

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The only known proofs of the above Corollary 5 ([4], [5]) rely heavily either on a deep result of Fremlin [3] or of Sierpinski [7].

The following fact was shown in [4]. We are going to give a slightly different proof of it.

PROPOSITION 6. *Let E be an order continuous Banach lattice. Then E has the separable complementation property.*

Proof. Let Y be a separable subspace of E. Let $(x_n)_{n\geq 1}$ be a dense subset of the positive unit ball of Y. Consider

$$u=\sum_{n=1}^{\infty}\frac{x_n}{2^n}.$$

Let F be the closed ideal generated by u, i.e.

$$F=\overline{\bigcup_{n=1}^{\infty}[-nu, nu]}.$$

It is clear that F contains Y. Since E is order continuous, the space F is weakly compactly generated and F is complemented in E. Choose Z a separable subspace of F which is complemented in F and containing Y. It is clear that Z is complemented in E.

Combining the results of [1] and ([6], p. 36) with Lemma 3, Proposition 4, and Proposition 6 we get

THEOREM 7. *Let X be a complemented subspace of a Banach lattice E. If every Dunford–Pettis operator from L_1 to X is Pettis-representable then X has the Radon–Nikodym property.*

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