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PHELPS SPACES AND FINITE DIMENSIONAL DECOMPOSITIONS

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We show that if X is a separable Banach space such that X^* fails the weak* convex point-of-continuity property (C^*PCP), then there is a subspace Y of X such that both Y^* and $(X/Y)^*$ fail C^*PCP and both Y and X/Y have finite dimensional Schauder decompositions.

It is still unknown whether for every separable Banach space X there exists a subspace Y of X such that both Y and X/Y have a basis. Luski [**12**] gave a positive answer to this question when X contains an isomorphic copy of c_0. Johnson and Rosenthal proved in [**10**] that for any separable Banach space X there exists $Y \subseteq X$ such that Y and X/Y have finite dimensional decompositions.

It is also an open problem whether for every separable space X with non-separable dual we can find a subspace Y such that both Y^* and $(X/Y)^*$ are non-separable (eventually with finite dimensional decompositions). The purpose of the note is to solve this problem when X has the stronger property that X^* does not have the weak* convex point of continuity property.

All Banach spaces considered here are real, and are infinite dimensional unless otherwise specified.

A dual Banach space X^* has the *Radon-Nikodým property* (RNP in short) if every w^*-compact subset C of X^* has a point at which the relative weak* and norm topologies coincide [**13**] and [**17**].

X^* has the C^*PCP if every w^*-compact convex subset C of X^* has a point at which the relative weak* and norm topologies coincide [**8**].

X contains no isomorphic copy of ℓ_1 if and only if for every w^*-compact subset A of X^* and every f in X^{**}, $f : (A, w^*) \to \mathbb{R}$ has a point of continuity [**14**] and [**16**].

From these characterisations it is clear that if X^* has the RNP then X^* has C^*PCP. Although there are several ways of proving it, it is not obvious that if X^* has

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C^*PCP, then X contains no isomorphic copy of ℓ_1. We cannot resist the temptation to give a simple proof of this statement using integral representation theory. An outline of these ideas appears in [**16**].

Let us recall some notation. If K is a topological space and f a real function on K, $\mathrm{cont}(f)$ is the set of points in K at which f is continuous. If K is a convex set, then $\mathrm{Ext}(K)$ denotes the set of extreme points of K. If A is a subset of a topological vector space E, $\overline{cv}(A)$ denotes the closed convex hull of A. The following lemma seems to be of independent interest:

LEMMA 1. *If K is convex and compact in a locally convex space E, and if f is a real valued, affine and bounded function on K, then* $\mathrm{cont}\, f \subseteq \overline{cv}(\mathrm{cont}\, f \cap \mathrm{Ext}\, K)$.

PROOF: Let $\varphi = \mathrm{osc}(f)$ be the oscillation of f. φ is concave, upper semi-continuous, positive and $\mathrm{cont}(f) = Z = \varphi^{-1}(0)$. Assume that $t_0 \in \mathrm{cont}(f)$ and $t_0 \notin \overline{cv}(\mathrm{cont}(f) \cap \mathrm{Ext}(K))$. Then there exists $u : K \to \mathbb{R}$ which is affine and continuous such that $u(t_0) > \alpha$ and $\varphi(v) > 0$ for every $v \in \mathrm{Ext}\, K$ with $u(v) > \alpha$. Let $S = K \cap \{u \geqslant \alpha\}$. We have $\mathrm{Ext}(S) \subseteq (K \cap \{u = \alpha\}) \cup \bigcup\limits_{n \geqslant 1} \{\varphi \geqslant \frac{1}{n}\} \equiv X$, and X is a K_σ, that is, a countable union of compact sets. Hence, using the integral representation theorem, there exist μ, a probability measure on X, such that $t_0 = r(\mu)$, the barycenter of μ. We have $\int \varphi d\mu \leqslant \varphi(t_0) = 0$, hence $\varphi = 0$ μ-almost everywhere and thus μ is supported by $K \cap \{u = \alpha\}$. But this is impossible since $t_0 = r(\mu)$ and $u(t_0) > \alpha$. ∎

A simple consequence of our lemma is the following result [**5**]:

COROLLARY. *With the same notation as above, if $\mathrm{cont}(f)$ is dense in K then $\mathrm{cont}(f) \cap \mathrm{Ext}\, K$ is dense in $\mathrm{Ext}\, K$.*

We can now show that if X is a Banach space and X^* has C^*PCP then X contains no isomorphic copy of ℓ_1. Indeed let A be a w^*-compact subset of X^* and $f \in X^{**}$. Let K be the weak* closure of the convex hull of A. By assumption K has a point x at which the relative weak* and norm topologies coincide, so x is a point of continuity of f on K. By the lemma, $\mathrm{cont}\, f \cap \mathrm{Ext}\, K \neq \emptyset$, and since $\mathrm{Ext}\, K \subseteq A$, the restriction of f to A has a point of continuity, and so X contains no isomorphic copy of ℓ_1.

REMARK: 1. It is shown in [**8**] that there are spaces X such that X^* has C^*PCP but not RNP and that there are spaces which contain no isomorphic copy of ℓ_1 and whose duals are not C^*PCP.

2. Consider a space X such that X^* is C^*PCP but not RNP. Then there exists a w^*-compact convex subset K of X^* which has points where the relative weak* and norm topologies coincide (the set of such points is even a dense $\mathcal{G}_\delta$ in K endowed with

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the weak* topology), but none of these points is extreme. This means that there is no analogue of Lemma 1 for the points where the relative weak* and norm topologies coincide.

We now turn to the study of spaces X such that X^* has C^*PCP. In [4], we studied Banach spaces X on which every convex, continuous, Gateaux differentiable function is Frechet differentiable on a dense subset of X. Because of his pioneering work in this area, we proposed calling such spaces *Phelps spaces.* We need here some further notation.

First of all, if $C \subseteq X$, $f \in X^*$, and $\alpha > 0$, then the set $S(C,f,\alpha) = \{x \in C : f(x) > \sup f(C) - \alpha\}$ is called a *slice* of C. If $C \subseteq X^*$ and $\delta > 0$, then C is *w^*-δ-dentable* if there exists a slice of C determined by an element of X and having diameter less then δ. C is *w^*-dentable* if C is w^*-δ-dentable for every $\delta > 0$.

If F is a subspace of X and H is a subspace of X^*, then $F^\perp$ denotes the annihilator of F in X^* and $H_\perp$ denotes the annihilator of H in X. For $x \in X$ and $\delta > 0$, $B(x,\delta) \equiv \{y \in X : \|y - x\| \leqslant \delta\}$, with the unit ball $B(0,1)$ being further abbreviated with the notation X_1.

If $A \subseteq X$, then $sp\ A$, $\overline{sp}\ A$, and w^*-$cl\ sp\ A$ denote the linear, closed linear, and w^*-closed linear hulls of A, respectively. Finally the set of positive integers is denoted by $\mathbb{N}$.

In [4], we obtained the following:

THEOREM 2. *Let X be a separable Banach space. The following are equivalent:*

(1) *X is not a Phelps space;*
(2) *There exists on X a norm such that the dual norm is strictly convex, but X_1^* is not w^*-dentable;*
(3) *X^* fails C^*PCP.*

We show that a refinement of Theorem IV.4 of [10] combined with Theorem 2 above yields the result mentioned in the Abstract, namely:

THEOREM 3. *Let X be a separable Banach space which is not a Phelps space. Then there exists a subspace Y of X such that neither Y nor X/Y is a Phelps space, and both Y and X/Y have finite dimensional Schauder decompositions.*

REMARK: The authors do not know if the corresponding version of Theorem 3 remains valid in the setting of non-Asplund spaces. The difficulty in this setting is noted in the remark preceding Lemma 5.

Now let X be a separable Banach space which is not a Phelps space. Choose a biorthogonal system $\{x_i, x_i^*\}_{i \in \mathbb{N}}$ so that $sp\{x_i\}$ is dense in X and $sp\{x_i^*\}$ is w^*-dense in X^* (see, for example, [11]). Also, since X is not a Phelps space, we can choose an

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equivalent norm $\|\cdot\|$ on X and a $\delta > 0$ such that the dual norm is strictly convex, but X_1^* is not w^*-3δ-dentable [**4**].

The proof of Theorem 3 results from a strategic partitioning of the positive integers into two sets, σ and Δ, from which we set $Y = \overline{sp}\{x_i : i \in \sigma\}$ and obtain $(X/Y)^* \cong w^*\text{-}cl\ sp\{x_i^* : i \in \Delta\}$. This partitioning is accomplished by the following main lemma, whose proof we defer until after the proof of the theorem:

LEMMA 4. *With the notation described above, there exist increasing sequences* $\{\sigma_n\}$ *and* $\{\Delta_n\}$ *of finite subsets of* $\mathbb{N}$ *such that:*

(a) $\sigma_n \cap \Delta_n = \emptyset$ *and* $\{1, 2, \ldots, n\} \subset \sigma_n \cup \Delta_{n-1}$ *for each* $n \in \mathbb{N}$.

(b) *For each* n, σ_n *satisfies:*

(i) *For every non-zero* $x^* \in sp\{x_i^* : i \in \Delta_{n-1}\}$ *there exists* $x \in sp\{x_i : i \in \sigma_n \cup \Delta_{n-1}\}$ *such that* $\|x\| = 1$ *and* $|x^*(x)| > \left(1 - \frac{1}{n}\right)\|x^*\|$.

(ii) *For every* $x \in sp\{x_i : i \in \sigma_{n-1}\}$, *with* $\|x\| = 1$, *there exist* $y^*, z^* \in \left(sp\{x_i : i \in \sigma_n\}\right)^*$ *of norm 1 such that* $y^*(x) > 1 - \frac{1}{n}$, $z^*(x) > 1 - \frac{1}{n}$ *and* $\|y^* - z^*\| > \delta$.

(c) *For each* n, Δ_n *satisfies:*

(i) *For every non-zero* $x \in sp\{x_i : i \in \sigma_n\}$ *there exists* $x^* \in sp\{x_i^* : i \in \sigma_n \cup \Delta_n\}$ *such that* $\|x^*\| = 1$ *and* $|x^*(x)| > \left(1 - \frac{1}{n}\right)\|x\|$.

(ii) *For every* $x \in sp\{x_i : i \in \sigma_n \cup \Delta_{n-1}\}$ *of norm 1 satisfying*

$$\left\{y^* \in sp\{x_i^* : i \in \Delta_{n-1}\} : \|y^*\| \leqslant 1 \text{ and } y^*(x) \geqslant 1 - \frac{1}{2n}\right\} \neq \emptyset,$$

there exists $y^* \in sp\{x_i^* : i \in \Delta_n\}$ *of norm 1 such that* dist $\left(y^*,\ sp\{x_i^* : i \in \Delta_{n-1}\}\right) > \delta$ *and* $y^*(x) > 1 - \frac{1}{n}$.

PROOF OF THEOREM 3: Let $\sigma = \bigcup_{n=1}^{\infty} \sigma_n$, $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$, with $\{\sigma_n\}$ and $\{\Delta_n\}$ as constructed in Lemma 4, and let

$$Y = \overline{sp}\{x_i : i \in \sigma\}.$$

We will show that Y has the desired properties.

First, it was shown in [**10**, Theorem IV.4] that the conditions (b.i) and (c.i) of Lemma 4 imply that Y and X/Y both have finite dimensional Schauder decompositions.

To establish that neither Y nor X/Y is a Phelps space, it suffices, by Theorem 2, to show that both Y^* and $(X/Y)^*$ are strictly convex, but neither Y_1^* nor $(X/Y)_1^*$ is w^*-dentable (recall that X itself has been renormed with these properties).

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The fact that $(X/Y)^* \cong Y^\perp$ is strictly convex is trivial. The well known fact that $Y^* \cong X^*/Y^\perp$ is strictly convex is a result of the fact that $Y^\perp$ is w^*-closed. Indeed, suppose $\overline{f}, \overline{g} \in X^*/Y^\perp$ with $\|\overline{f}\| = \|\overline{g}\| = 1$ and $\|\overline{f} + \overline{g}\| = 2$. Since $Y^\perp$ is w^*-closed, we can choose $f, g \in X^*$ such that $f \in \overline{f}$, $g \in \overline{g}$, and $\|f\| = \|g\| = 1$. Then

$$2 = \|\overline{f}\| + \|\overline{g}\| = \|f\| + \|g\| \geqslant \|f + g\| \geqslant \|\overline{f} + \overline{g}\| = 2,$$

so by the strict convexity of X^*, $f = g$, hence $\overline{f} = \overline{g}$.

We now use condition (b.ii) of Lemma 4 to show that Y_1^* is not w^*-δ-dentable. Let $\varepsilon > 0$ and let $x \in Y$ with $\|x\| = 1$. By the norm density of $sp\{x_i : i \in \sigma\}$ in Y, we may assume that $x \in sp\{x_i : i \in \sigma_{n_0}\}$ for some $n_0 \in \mathbb{N}$, and that $\varepsilon \geqslant \frac{1}{n_0}$. Condition (b.ii) then implies the existence of elements $y^*, z^* \in \left(sp\{x_i : i \in \sigma_{n_0}\}\right)^*$ of norm 1 satisfying $y^*(x) > 1 - \varepsilon$, $z^*(x) > 1 - \varepsilon$, and $\|y^* - z^*\| > \delta$.

By the Hahn–Banach Theorem, we may assume $y^*, z^* \in Y^*$. Then the above estimates show that $y^*, z^* \in S(Y_1^*, x, \varepsilon)$ and diam $S(Y_1^*, x, \varepsilon) > \delta$. Since ε and x were arbitrary, Y_1^* is not w^*-δ-dentable.

Lastly, we show that condition (c.ii) implies that $(X/Y)_1^* \cong Y_1^\perp$ is not w^*-δ-dentable. Let $x \in X$, $\|x\| = 1$, and let

$$S(Y_1^\perp, x, \varepsilon) \equiv \left\{z^* \in w^*\text{-}cl\ sp\{x_i^* : i \in \Delta\} : \|z^*\| \leqslant 1,\ z^*(x) > 1 - \varepsilon\right\}$$

be a (non-empty) w^*-slice of $Y_1^\perp$. By the density of $sp\{x_i : i \in \sigma \cup \Delta\}$ in X, we may assume that $x \in sp\{x_i : i \in \sigma_{n_0} \cup \Delta_{n_0}\}$ for some $n_0 \in \mathbb{N}$. We can choose n_0 so large that $\varepsilon > \frac{1}{n_0}$ and

$$\left\{z^* \in sp\{x_i^* : i \in \Delta_{n_0 - 1}\} : \|z^*\| \leqslant 1,\ z^*(x) > 1 - \varepsilon\right\}$$

is non-empty. By (c.ii), diam $S(Y_1^\perp, x, \varepsilon) > \delta$.

The theorem is proved. ∎

It remains to prove Lemma 4. To do this, we first establish some permanence properties of strictly convex nondentable dual unit balls. The assumption of strict convexity is essential here. Indeed if X is a separable Banach space whose dual is C^*PCP but not RNP, we can choose an equivalent unit ball of X such that its dual unit ball is not w^*-dentable. However, it is shown in [**9**] that for every $\varepsilon > 0$, there exists a finite dimensional subspace H of X such that $H_1^\perp$ is ε-w^*-dentable. So the conclusion of Lemma 5 fails in this setting and therefore X_1^* is necessarily non-strictly convex.

Note that Lemma 5 is a variant of Lemma 9 of [**1**].

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LEMMA 5. *Let X be a Banach space such that X^* is strictly convex and X_1^* is not w^*-δ-dentable, for some $\delta > 0$. Then for every finite dimensional subspace Y of X, $Y_1^\perp$ is not w^*-δ-dentable.*

PROOF: By induction and contraposition, it is clearly enough to show that if Y is a w^*-closed subspace of X^*, $z \in X$, and $Z = Y \cap \ker z$, then w^*-δ-dentability of Z_1 implies the same for Y_1. Thus, let S be a w^*-slice of X_1^* such that $S \cap Z_1 \neq \emptyset$ and diam $S \cap Z_1 < \delta$. Given $\varepsilon > 0$, consider the set

$$U_\varepsilon = S \cap Y \cap z^{-1}(-\varepsilon, \varepsilon).$$

U_ε is a w^*-open set in the relative w^*-topology of Y_1. We claim that ε can be chosen sufficiently small so that diam $U_\varepsilon < \delta$. The result then follows from the strict convexity of $Y_1^\perp$ since w^* slices form a base of w^* neighbourhoods of an extreme point in a dual unit ball (see, for example, [**3**]).

Choose and fix $x_1, x_2 \in S \cap Y$, so that $z(x_1) < 0 < z(x_2)$. Denote by K_i the (positive) cone generated by x_i and $S \cap Z_1$, $i = 1, 2$. By convexity, $U_\varepsilon \subset (K_1 \cup K_2) \cap z^{-1}(-\varepsilon, \varepsilon)$. A homothety argument then shows that, since diam $S \cap Z_1 < \delta$, we have diam $U_\varepsilon < \delta$, for ε sufficiently small. ∎

LEMMA 6. *Let K be a separable subset of a dual space X^*, $x^* \in X^*$ and $A \subset K + B(x^*, \delta)$ be a Baire space in the relative w^* topology. Then A has a relative w^*-open subset of diameter $\leqslant 2\delta$.*

PROOF: This is an easy consequence of the Baire Category Theorem. ∎

LEMMA 7. *Let X be as in Lemma 5. Then for every finite dimensional subspace H of X^*, $(X^*/H)_1 \cong (H_\perp)_1^*$ is not w^*-$\delta/2$-dentable.*

PROOF: As in the proof of Lemma 5, the strict convexity of X^* gives that it is enough to show that if $(H_\perp)_1^*$ were w^*-$\delta/2$-dentable, then X_1^* would have a relatively w^*-open subset of diameter less than δ.

Thus, let $U \subset (H_\perp)_1^*$ be a relatively w^*-open subset with diam $U < \delta/2$. Let $\pi : X_1^* \to (H_\perp)_1^*$ denote the natural restriction map. Then π is a w^*-w^* continuous surjection, so $\pi^{-1}(U)$ is w^*-open in X_1^*. Since $\pi^{-1}(U) \subset H + B(y, \delta/2)$, the result follows from Lemma 6. ∎

PROOF OF LEMMA 4: : We define σ_n and Δ_n by induction. Let $\sigma_1 = \{1\}$ and $\Delta_1 = \emptyset$, and suppose that for some $n \geqslant 2$, σ_{n-1} and Δ_{n-1} have been constructed satisfying (a), (b), and (c) of the statement of the lemma.

First we explain the construction of σ_n:

Since Δ_{n-1} is finite, the unit sphere of $sp\{x_i : i \in \Delta_{n-1}\}$ is compact, so we can choose a finite subset σ'_n of $N \backslash \Delta_{n-1}$ satisfying $\{1, 2, \dots, n\} \subset \sigma'_n \cup \Delta_{n-1}$ and also (b.i) (with σ'_n in place of σ_n).

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Next observe that since evidently $sp\{x_i^* : i \in \Delta_{n-1}\} \subset \left(sp\{x_i : i \notin \Delta_{n-1}\}\right)^{\perp}$ and since these two spaces have the same dimension, namely $\operatorname{card}(\Delta_{n-1})$, they must in fact be equal. Letting $H = sp\{x_i^* : i \in \Delta_{n-1}\}$, it follows that $H_{\perp} = \overline{sp}\{x_i : i \notin \Delta_{n-1}\}$.

Now let $u \in sp\{x_i : i \in \sigma_{n-1}\}$, with $\|u\| = 1$. Since $u \in H_{\perp}$, Lemma 7 says that

$$\operatorname{diam}\{x^* \in (H_{\perp})_1^* : x^*(u) > 1 - \frac{1}{n}\} > \delta$$

(recall X_1^* is not w^*-3δ-dentable), so there exist $y_u^*, z_u^* \in (H_{\perp})_1^*$ satisfying $y_u^*(u) > 1 - \frac{1}{n}$, $z_u^*(u) > 1 - \frac{1}{n}$, and $\|y_u^* - z_u^*\| > \delta$. Choose $t_u \in sp\{x_i : i \notin \Delta_{n-1}\}$ of norm 1 such that $|y_u^*(t_u) - z_u^*(t_u)| > \delta$, and denote by $\sigma_{n,u}$ the support of t_u (that is, if $t_u = \sum_{j \in K} \alpha_j x_j$, with $\alpha_j \neq 0$ for all $j \in K$, then $\sigma_{n,u} = K$). Note $\sigma_{n,u}$ is a finite subset of $N \backslash \Delta_{n-1}$.

The compactness of the unit sphere S_1 of $sp\{x_i : i \in \sigma_{n-1}\}$ allows us to work with a finite set of such elements u. Namely, for each $u \in sp\{x_i : i \in \sigma_{n-1}\}$ of norm 1, let

$$U_u = \left\{x \in sp\{x_i : i \in \sigma_{n-1}\} : \|x\| = 1,\ y_u^*(x) > 1 - \frac{1}{n} \text{ and } z_u^*(x) > 1 - \frac{1}{n}\right\}.$$

Since for all u chosen as above the sets U_u are open in S_1, the compactness of S_1 gives that there exists a finite set $\{u_j\}_1^k \subset S_1$ such that $\bigcup U_{u_j} = S_1$. If we denote the corresponding objects chosen as above by $\{y_j^*\}_1^k$, $\{z_j^*\}_1^k$, $\{t_j\}_1^k$, and $\{\sigma_{n,j}\}_1^k$, we have that for each $x \in sp\{x_i : i \in \sigma_{n-1}\}$ of norm 1, there is a $1 \leqslant j \leqslant k$ such that

$$y_j^*(x) > 1 - \frac{1}{n}, \quad z_j^*(x) > 1 - \frac{1}{n}, \quad \text{and} \quad |y_j^*(t_j) - z_j^*(t_j)| > \delta.$$

Now set $\sigma_n = \sigma_n' \cup \left(\bigcup_{j=1}^{k} \sigma_{n,j}\right)$.

If $S_n : \left(\overline{sp}\{x_i : i \notin \Delta_{n-1}\}\right)^* \to \left(sp\{x_i : i \in \sigma_n\}\right)^*$ is the restriction map, we have by the above estimates that $\|S_n y_j^* - S_n z_j^*\| > \delta$, and so condition (b.ii) is fulfilled.

We now proceed to the construction of Δ_n.

By the compactness of the unit ball of $sp\{x_i : i \in \sigma_n\}$, we can choose Δ_n' such that condition (c.i) is satisfied (with Δ_n' in place of Δ_n).

We claim that for each $u \in sp\{x_i : i \in \sigma_n \cup \Delta_{n-1}\}$ with $\|u\| = 1$ and which satisfies

$$(1) \qquad \left\{y^* \in sp\{x_i^* : i \in \Delta_{n-1}\} : \|y^*\| \leqslant 1 \quad \text{and} \quad y^*(u) \geqslant 1 - \frac{1}{2n}\right\} \neq \emptyset,$$

we can choose, by Lemma 5, a $y_u^* \in sp\{x_i^* : i \notin \sigma_n\}$ of norm 1 and satisfying $y_u^*(u) > 1 - \frac{1}{n}$ and $\operatorname{dist}\left(y_u^*,\ sp\{x_i^* : i \in \Delta_{n-1}\}\right) > \delta$.

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Indeed, otherwise

$$U = \left\{ y^* \in sp\{x_i^* : i \notin \sigma_n\} : \|y^*\| = 1 \quad \text{and} \quad y^*(u) > 1 - \frac{1}{n} \right\}$$

would be non-empty (by the choice of u) and would be included in

$$\left(U \cap sp\{x_i^* : i \in \Delta_{n-1}\}\right) + B(0, \delta).$$

But this would imply by Lemma 6 (since $U \cap sp\{x_1^* : i \in \Delta_{n-1}\}$ is separable) that U contains a w^* open subset of diameter less than or equal to 2δ, and, by strict convexity, it would contain a w^* slice of diameter $\leqslant 2\delta$, which contradicts Lemma 5 since X_1^* is supposed to be not w^*-3δ-dentable. The claim is proved.

Now let

$$K = \left\{ u \in sp\{x_i : i \in \sigma_n \cup \Delta_{n-1}\} : \|u\| = 1, u \text{ satisfies } (1) \right\}.$$

For each $u \in K$, consider $\Delta_{n,u}$ the support of y_u^* and $V_u = \left\{x \in K : y_u^*(x) > 1 - \frac{1}{n}\right\}$. By compactness of K, choose finitely many $u_1, \dots, u_p$ such that $\bigcup_{k=1}^{p} V_{u_k} = K$ and set

$$\Delta_n = \Delta_n' \cup \left(\bigcup_{k=1}^{p} \Delta_{n,u_k} \right).$$

Condition (c.ii) is fulfilled and Lemma 4 is proved. ∎

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