J. Austral. Math. Soc. (*Series A*) **52** (1992), 141–142

A NOTE ON RIGHT EQUIVALENCE OF MODULE PRESENTATIONS

M. J. CANFELL

(Received 28 July 1989; revised 27 November 1990)

Communicated by B. J. Gardner

Abstract

If R has 1 in the stable range, then any two presentations $f, g: P \to M$ of an R-module M by a finitely generated projective P are right equivalent, that is, $f = gh$ for some automorphism h of P. The converse is also true.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*) 16 A 48.

Let R be an associative ring with 1. All modules considered will be unital right R-modules, and all homomorphisms will act on the left.

Two module epimorphisms $f, g: P \to M$ are *right equivalent* if $f = gh$ for some automorphism h of P.

After Bass, we say that R *has* 1 *in the stable range* if $a, b \in R$ and $aR + bR = R$ imply $a + bt$ is a unit of R for some $t \in R$.

For $P = R^n$, the implication (a) $\Rightarrow$ (b) below is due to Warfield [2], [3]. The methods used there are different. In fact the proof in [2] is incorrect, while the proof in [3] involves the substitution property. We give a simple direct proof and also show that the converse implication holds.

THEOREM. *The following conditions are equivalent.*

(a) *R has 1 in the stable range.*

(b) *Each pair of epimorphisms $f, g: P \to M$ from a finitely generated projective R-module P to an R-module M are right equivalent.*

https://doi.org/10.1017/S1446788700032924 Published online by Cambridge University Press

Proof. (a) $\Rightarrow$ (b) Assume (a) and let P be a finitely generated projective R-module. By [1, Corollary 2.9], the endomorphism ring $\mathrm{End}(P)$ also has 1 in the stable range. Given epimorphisms $f, g: P \to M$, choose $h_1, h_2 \in \mathrm{End}(P)$ such that $gh_1 = f$ and $fh_2 = g$. Thus $g(1 - h_1h_2) = 0$. Since $h_1h_2 + (1 - h_1h_2) = 1$ and $\mathrm{End}(P)$ has 1 in the stable range, there is a $k \in \mathrm{End}(P)$ such that $h = h_1 + (1 - h_1h_2)k$ is an automorphism of P. Also $gh = f$ and so f, g are right equivalent.

(b) $\Rightarrow$ (a) Assume (b) holds and suppose $a, b \in R$ satisfy $aR + bR = R$. Choose $s, t \in R$ such that $as + bt = 1$. Let $M = R/bR$ and let $g: R \to M$ be the canonical map. Define $f: R \to M$ by $f(r) = g(ar)$. Since $g(1) = g(as + bt) = g(as) = f(s)$ we see that f is an epimorphism. By assumption there is an automorphism h of R such that $gh = f$. Now $h(1)$ is a unit of R and $gh(1) = f(1) = g(a)$. Thus $h(1) - a \in \mathrm{Ker}\, g = bR$ so $a + bt$ is a unit for some $t \in R$.

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Department of Mathematics, Statistics and Computing Science
University of New England
Armidale, N.S.W. 2351
Australia