On a Tauberian Theorem of G. Ricci.[1]

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1. I prove in this note some theorems on Rieszian and Dirichlet summabilities involving a Tauberian hypothesis with gaps. One of the theorems (§ 2, Theorem A) has been proved by Ricci [4, § 6][2] in a slightly less general form. Another theorem (§ 3) contains a Riesz version of a (C, k)-summability problem studied by Meyer-König [1, Satz 1].

The principal difference between Ricci's work and mine is that the former is based on a Tauberian technique of Vijayaraghavan while the latter employs a technique of Bosanquet as embodied in one of my Theorems [2, Theorem A] combined with an extension of a theorem of Szász [3, Theorem 1].

In what follows, $\sum_{n=0}^{\infty} a_n$ represents a real series and $\{\lambda_n\}$ a sequence such that

$$0 \leqq \lambda_0 < \lambda_1 < \ldots, \qquad \lambda_n \to \infty ;$$

$$\sigma_0(x) = A(x) \equiv \sum_{\lambda_\nu < x} a_\nu,$$

$$\sigma_k(x) = \frac{k}{x^k} \int_0^x (x-t)^{k-1} A(t)\, dt \qquad (k > 0);$$

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s} \qquad \text{is convergent for } s > 0.$$

2. In this notation my main theorem assumes the form:

Theorem A.[3] *If there is a constant* $\tau \geqq 0$ *and a set of points* E *in* $(0, \infty)$ *such that*

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[1] I wish to express my gratitude to Dr L. S. Bosanquet who drew my attention to an error in my original statement of Theorems A, a.

[2] Numbers in bold face type within [ ] indicate the references given at the end of this note.

[3] This result and the others which follow supplement the concluding remarks of my paper "On some extensions of Ananda Rau's converse of Abel's theorem," *Journal London Math. Soc.*, 23 (1948), 38-44.

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A (ia) $\lim_{\eta \to 0} \underline{\lim}_{x \to \infty \text{ over } E} \underline{\text{bound}}_{x < y < x(1+\eta)} \{A(y) - A(x)\} \geqq -\tau$

[*or alternatively,* $\lim_{\eta \to 0} \underline{\lim}_{x \to \infty \text{ over } E} \underline{\text{bound}}_{x(1-\eta) < y < x} \{A(x) - A(y)\} \geqq -\tau$],

A (ib) *for some* $k \geqq 0$, $\sigma_k(x) - \sigma_{k+1}(x) = O_L(1)$, $x \to \infty$,

A (ii) $\lim_{s \to +0} f(s) = S$,

then $\overline{\lim}_{x \to \infty \text{ over } E} A(x) \leqq S + \tau$

[*or alternatively* $\underline{\lim}_{x \to \infty \text{ over } E} A(x) \geqq S - \tau$].

Ricci has proved this theorem with the hypothesis A (ia) more particularised and A (ib) replaced by the hypothesis:

$$A(y) - A(x) > -K, \qquad x < y \leqq x(1+H), \qquad K > 0, \qquad H > 0.$$

The last hypothesis implies

$$\sigma_0(x) - \sigma_1(x) > -K(1 + H^{-1}),$$

as we can see from a lemma of Szász [5, Hilfssatz 1, (14) with $\beta = 0$] and is therefore included in A (ib).

Theorem A can be proved by the use of

THEOREM a. *If there is a constant* $\tau \geqq 0$ *and a set of points* E *in* $(0, \infty)$ *such that*

a (i) $\lim_{\eta \to 0} \underline{\lim}_{x \to \infty \text{ over } E} \underline{\text{bound}}_{x < y < x(1+\eta)} \{A(y) - A(x)\} \geqq -\tau$

[*or alternatively,* $\lim_{\eta \to 0} \underline{\lim}_{x \to \infty \text{ over } E} \underline{\text{bound}}_{x(1-\eta) < y < x} \{A(x) - A(y)\} \geqq -\tau$],

a (ii) Σa_n *is summable* $R(\lambda_n, k)$ *to* S,

then

$$\overline{\lim}_{x \to \infty \text{ over } E} A(x) \leqq S + \tau$$

[*or alternatively* $\underline{\lim}_{x \to \infty \text{ over } E} A(x) \geqq S - \tau$].

This theorem which need only be proved for integral k is deducible from a combination of the following lemmas which appear elsewhere [2, Theorem A; (3), (4)] with some minor differences.

LEMMA 1. *Let*

$$\underline{\overline{\sigma}}_p = \underline{\overline{\lim}}_{x \to \infty} \sigma_p(x) \qquad (p = 1, 2, \ldots).$$

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Then, in the notation set forth in § 1,

$$\overline{\lim_{x \to \infty \text{ over } E}} \; A(x) \leqq \frac{\mathscr{A}_p(\eta)\,\bar{\sigma}_p + \mathscr{B}_p(\eta)\,\underline{\sigma}_p}{\mathscr{A}_p(\eta) + \mathscr{B}_p(\eta)}$$

$$-\frac{p}{\eta}\int_{1+\eta-\eta/p}^{1+\eta} \; \overline{\lim_{x \to \infty \text{ over } E}} \; \overline{\operatorname{bound}_{x < y < tx}} \; [\{A(y) - A(x)\}\,dt,$$

where $\mathscr{A}_p(\eta)$, $\mathscr{B}_p(\eta)$ *are polynomials of degree* p *in* η.

LEMMA 2. *In the notation of Lemma* 1,

$$\underline{\lim_{x \to \infty \text{ over } E}} \; A(x) \geqq \frac{\mathscr{C}_p(\eta)\,\bar{\sigma}_p + \mathscr{D}_p(\eta)\,\underline{\sigma}_p}{\mathscr{C}_p(\eta) + \mathscr{D}_p(\eta)}$$

$$+\frac{p}{\eta}\int_{1-\eta}^{1-\eta+\eta/p} \; \overline{\lim_{x \to \infty \text{ over } E}} \; \overline{\operatorname{bound}_{tx < y < x}} \; \{A(x) - A(y)\}\,dt,$$

where $\mathscr{C}_p(\eta)$, $\mathscr{D}_p(\eta)$ *are polynomials of degree* p *in* η.

In Theorem a, $\bar{\sigma}_p = \underline{\sigma}_p = S$ for some p, so that the conclusion of the theorem follows at once from the conclusions of Lemmas 1, 2 when we let $\eta \to +0$ in the latter, using the hypothesis **a** (i).

After this it is obvious that to prove Theorem A we have merely to appeal to the lemma stated below and proved in another note of mine [3, Corollary 1 under Theorem 1].

LEMMA 3. *The conditions* $\lim_{s \to +0} f(s) = S$ *and* $\sigma_k(x) - \sigma_{k+1}(x) = O_L(1)$, $x \to \infty$, *for some* $k \geqq 0$, *together ensure* $\lim_{x \to \infty} \sigma_{k+1}(x) = S$.

3. The Tauberian hypothesis A (ia) or a (i) may be presented in the classical Hardy-Landau form as in the following

DEDUCTION FROM THEOREMS A, a. *Replace* A (ia) *in Theorem* A *and* a (i) *in Theorem* a *by the supposition that there is a constant* $\omega > 0$ *and two sequences of integers* h_r, k_r $(r = 1, 2 \ldots)$, $h_r < k_r < h_{r+1}$ $(h_r \to \infty)$ *such that*

(1) $\lambda_{k_r} > \lambda_{h_r}(1+\omega)$, $a_n \geqq -K\dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n}$ *for* $h_r < n \leqq k_r$ $(r = 1, 2 \ldots)$;

(2) *either* $\underline{\lim}_{n \to \infty} a_n \geqq 0$ *or* $\lim_{n \to \infty} (\lambda_n/\lambda_{n-1}) = 1$ *for* $h_r < n \leqq k_r$ $(r = 1, 2, \ldots)$.

Then, provided there is no other change in the hypotheses of Theorems A *and* a, *the conclusion of either theorem takes the form*

$$\overline{\lim_{r \to \infty}} \; A(\lambda_{h_r}) \leqq S = \lim_{r \to \infty} A\left(\frac{\lambda_{h_r} + \lambda_{k_r}}{2}\right) \leqq \underline{\lim_{r \to \infty}} \; A(\lambda_{k_r}),$$

the part of the conclusion $\overline{\lim}_{r \to \infty} A(\lambda_{h_r}) \leqq S$ *being independent of* (2).

PROOF. Since (1) gives $A(\lambda_{h_r}\overline{1+\eta}) - A(\lambda_{h_r}) \geqq -K\eta$,[1] the first alternative of A (ia) or a (i) is satisfied with $\tau = 0$ when E is $\{\lambda_{h_1}, \lambda_{h_2}, \ldots\}$. Therefore $\overline{\lim}_{r\to\infty} A(\lambda_{h_r}) \leqq S$.

Next, since (1) and (2) together give
$A(\lambda_{k_r}) - A(\lambda_{k_r}\overline{1-\eta}) > -K\eta/(1-\eta) + O_L(1)$, $r \to \infty$,[2] the alternative within [ ] of A (ia) or a (i) holds with $\tau = 0$ and E: $\{\lambda_{k_1}, \lambda_{k_2}, \ldots\}$. Consequently $\underline{\lim}_{r\to\infty} A(\lambda_{k_r}) \geqq S$.

Finally, since (1) and (2) together make both alternatives of A (ia) or a (i) true with $\tau = 0$ and E: $\left\{\dfrac{\lambda_{h_1} + \lambda_{k_1}}{2}, \dfrac{\lambda_{h_2} + \lambda_{k_2}}{2} \ldots\right\}$, we have

$$\overline{\lim_{r\to\infty}} A\left(\frac{\lambda_{h_r} + \lambda_{k_r}}{2}\right) \leqq S \leqq \underline{\lim_{r\to\infty}} A\left(\frac{\lambda_{h_r} + \lambda_{k_r}}{2}\right) \text{ or } S = \lim_{r\to\infty} A\left(\frac{\lambda_{h_r} + \lambda_{k_r}}{2}\right).$$

The last conclusion, under the hypothesis of Rieszian summability, belongs to the same order of ideas as the theorem of Meyer-König referred to at the outset.

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[1] $$A(\lambda_{h_r}\overline{1+\eta}) - A(\lambda_{h_r}) = \sum_{\lambda_{h_r} < \lambda_n \leqslant \lambda_{h_r}(1+\eta)} a_n \geqslant -K \sum \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}$$
$$\geqslant -\frac{K}{\lambda_{h_r}} \sum (\lambda_n - \lambda_{n-1}) \geqslant -K\eta.$$

[2] $$A(\lambda_{k_r}) - A(\lambda_{k_r}\overline{1-\eta}) = \sum_{\lambda_{k_r}(1-\eta) < \lambda_n \leqslant \lambda_{k_r}} a_n = a_{l_r} + \sum_{l_r+1 \leqslant n \leqslant k_r} \ldots$$
$$\geqslant a_{l_r} - \frac{K}{\lambda_{l_r}} \sum (\lambda_n - \lambda_{n-1}) > a_{l_r} - K\frac{\eta}{1-\eta}$$

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