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COLLINEATIONS OF POLAR SPACES

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1. Introduction. The fundamental theorem of projective geometry describes the bijective collineations between two projective spaces $\mathbf{P}V$ and $\mathbf{P}V'$ of finite dimension (greater than one) over division rings k and k' in terms of an isomorphism $\varphi:k \rightarrow k'$ and a φ-semilinear bijective mapping between the underlying vector spaces V and V'. Tits [**9**, Theorem 8.6II] has given an extensive generalization of this theorem to embeddable polar spaces induced by polarities coming from either (σ, ϵ)-hermitian forms or from (σ, ϵ)-quadratic forms with Witt indices at least two. In another direction, Klingenberg [**7**] and later André [**1**] and Rado [**8**], have generalized the fundamental theorem by considering non-injective collineations. Now the isomorphism φ must be replaced by a place $\varphi:k \rightarrow k' \cup \infty$ and an integral structure over the valuation ring $A = \varphi^{-1}(k')$ is induced into the projective space $\mathbf{P}V$. In [**6**, XXII] and [**10**, p. 366], Weisfeiler asks for analogues of this to other Tits buildings. Recently, Faulkner and Ferrar [**3**] gave this for Moufang planes. In [**5**], generalizing work of Chow [**2**], we were also led to this type of theorem for polar spaces defined over symmetric and alternating forms with maximal Witt index. In the present work we will consider general embeddable polar spaces with Witt index at least three.

Let $S(\xi)$ be a polar space on the underlying finite dimensional k-vector space V with polarity ξ of trace type coming from either a non-degenerate trace-valued (σ, ϵ)-hermitian form f or a non-degenerate (σ, ϵ)-quadratic form q associated with a (σ, ϵ)-hermitian form f, with Witt index $i(\xi) \geqq 3$. Similarly, let $S(\xi')$ be a polar space on the k'-vector space V' with polarity ξ' of trace type having Witt index $i(\xi') \geqq 3$.

THEOREM. *Let $\pi:S(\xi) \rightarrow S(\xi')$ be a thick collineation between polar spaces with $i(\xi) = i(\xi') \geqq 3$. Then there exists a place $\varphi:k \rightarrow k' \cup \infty$ with valuation ring $A = \varphi^{-1}(k')$, an A-module M in V with $Mk = V$ and a φ-semilinear mapping $\beta:M \rightarrow V'$ such that*

$$\pi X = \beta(M \cap X)k' \text{ for all points } X \text{ in } S(\xi).$$

Moreover, the hermitian forms f and f' associated with the polarities ξ and ξ' can be chosen so that M has an orthogonal splitting $M = M(1) \perp M(2)$

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where $M(1)$ *is unimodular and free with rank equal to the dimension of* $\beta(M)$ *over* $\varphi(A)$, *while* $\beta(M(2)) = 0$, *and*

$$\varphi(f(x, y)) = f'(\beta(x), \beta(y)) \quad \textit{for all } x, y \in M.$$

The definition of a thick collineation is given in the following section. In the above theorem $M(2)$ need not be free. Also, $f(M(2), M(2)) \subseteq \mathfrak{m}$, the unique maximal two-sided ideal of A. If both the polar spaces $S(\xi)$ and $S(\xi')$ are associated with pseudo-quadratic forms then, with the appropriate interpretation, $q' \circ \beta = \varphi \circ q$. Conversely, it is fairly easily seen that any φ-semilinear mapping $\beta:M \to V'$, as in the theorem, induces a collineation. In the symplectic situation where σ is the identity and $\epsilon = -1$, the above theorem has been completely proved in [**5**, Theorem 2.1]. When π is bijective, the result reduces to Theorem 8.6II in [**9**]; however, the proof is entirely different and Tits also includes degenerate forms and the case $i(\xi) = i(\xi') = 2$ where there are exceptional situations. In the bijective case, $k = A \cong k'$, $V = M = M(1) \cong V'$ and the forms f and f' are isometric. See also the note added in proof.

If the collineation π is surjective, then $\varphi(A) = k'$ and $\beta(M) = V'$.

2. Pseudo-quadratic forms and polar spaces. We give now the definitions of (σ, ϵ)-hermitian forms and pseudo-quadratic forms and the connection with polar spaces; further details can be found in [**9**, Section 8]. Let k be a division ring, V a finite dimensional right k-vector space and $\sigma:k \to k$ an antiautomorphism, that is, an additive automorphism of k such that

$$(ab)^\sigma = b^\sigma a^\sigma \quad \text{for all } a, b \in k.$$

A function $f:V \times V \to k$ is called a σ-sesquilinear form if it is biadditive and if

$$f(xa, yb) = a^\sigma f(x, y)b \quad \text{for all } x, y \in V \text{ and } a, b \in k.$$

The form f is reflexive if the relation $f(x, y) = 0$ is symmetric for $x, y \in V$. This condition is equivalent to the existence of a nonzero $\epsilon \in k$ such that

$$f(y, x) = f(x, y)^\sigma \epsilon \quad \text{for all } x, y \in V.$$

Necessarily,

$$\epsilon^\sigma = \epsilon^{-1} \quad \text{and} \quad t^{\sigma^2} = \epsilon t \epsilon^{-1} \quad \text{for all } t \in k.$$

A form f satisfying these conditions is said to be (σ, ϵ)-*hermitian*.

Now assume $\epsilon \neq -1$ when σ is the identity and the characteristic of k is not two. Set

$$k_{\sigma \epsilon} = \{t - t^\sigma \epsilon | t \in k\},$$

an additive subgroup of k, and denote by $k(\sigma, \epsilon)$ the quotient group $k/k_{\sigma,\epsilon}$. A function $q:V \rightarrow k(\sigma, \epsilon)$ is called a (σ, ϵ)-*quadratic form* or a *pseudo-quadratic form relative to* σ *and* ϵ, if there exists a σ-sesquilinear form $g:V \times V \rightarrow k$ such that

$$q(x) = g(x, x) + k_{\sigma,\epsilon} \quad \text{for all } x \in V.$$

Then

$$q(xa) = a^{\sigma}q(x)a \quad \text{for } a \in k \text{ and } x \in V.$$

Also,

$$q(x + y) = q(x) + q(y) + (f(x, y) + k_{\sigma,\epsilon})$$

for all $x, y \in V$, where $f:V \times V \rightarrow k$ is the trace-valued (σ, ϵ)-hermitian form defined by

$$f(x, y) = g(x, y) + g(y, x)^{\sigma}\epsilon.$$

The form f is uniquely determined by q. The pseudo-quadratic form q is determined by the associated form f and the values taken by q on the elements of a basis of V. A pseudo-quadratic form is called non-degenerate when the associated hermitian form f is non-degenerate, that is, $f(x, V) = 0$ only when $x = 0$.

A subspace U of V is called *totally singular* with respect to the pseudo-quadratic form q if q vanishes on U. If U is totally singular for q, then U is also *totally isotropic* with respect to the associated hermitian form f, that is, $f(U, U) = 0$. All maximal totally singular (respectively, totally isotropic) subspaces of V have the same dimension called the Witt index of q (respectively, f). If the characteristic of k is not two, all totally isotropic subspaces of V are also totally singular.

The projective space $\mathbf{P}V$ of V is the set of all one-dimensional subspaces of V. Let f be a non-degenerate trace-valued (σ, ϵ)-hermitian form on V with Witt index $i(f) \geqq 2$. Then f determines a polarity ξ of trace type for the space $\mathbf{P}V$. Denote by $S(\xi)$ the set of all isotropic points X in $\mathbf{P}V$. Thus $f(X, X) = 0$. Then $S(\xi)$ is the *polar space relative to the polarity* ξ (or form f). More strictly, $S(\xi)$ should be defined relative to an equivalence class of proportional forms, rather than to a representative of the class, as we have done. Let q be a non-degenerate (σ, ϵ)-quadratic form on V with Witt index $i(q) \geqq 2$ and f the associated hermitian form. Again, this determines a polarity ξ of trace type. Denote by $S(\xi)$ the set of all singular points X in $\mathbf{P}V$. Thus $q(X) = 0$. Then $S(\xi)$ is the *polar space relative to the proportionality class of the pseudo-quadratic form* q. The linear subspaces of $S(\xi)$ are the subspaces of V which are totally isotropic, respectively totally singular, with respect to the hermitian form f, respectively pseudo-quadratic form q, associated with ξ. In particular, a *line* of $S(\xi)$ is a totally isotropic, respectively totally singular, two-dimensional subspace of V. If X and Y are points in a polar space with $f(X, Y) = 0$, the line joining X and Y is denoted by $X + Y$.

Now let k' be a second division ring and $S(\xi')$ a polar space with polarity ξ' associated with either a non-degenerate trace-valued (σ', ϵ')-hermitian form $f':V' \times V' \rightarrow k'$ on the finite dimensional k'-vector space V', or with a non-degenerate (σ', ϵ')-quadratic form

$$q':V' \rightarrow k'(\sigma', \epsilon')$$

with associated (σ', ϵ')-hermitian form f'. Assume $i(\xi') \geqq 2$. A *collineation* between the polar spaces $S(\xi)$ and $S(\xi')$ is a mapping

$$\pi:S(\xi) \rightarrow S(\xi')$$

with the following properties. Let $X, Y \in S(\xi)$ with $X + Y$ a line (so $f(X, Y) = 0$). Then

$$f'(\pi X, \pi Y) = 0.$$

Moreover, if $\pi X \neq \pi Y$, then for any point Z on the line $X + Y$ of $S(\xi)$, the point πZ is on the line $\pi X + \pi Y$ of $S(\xi')$. In particular, it follows that any line of $S(\xi)$ is carried by π into a line of $S(\xi')$ (usually not surjectively). It is possible for π to carry all the points of a line of $S(\xi)$ into a single point in $S(\xi')$.

Let $S(\xi)$ be a polar space with Witt index $i(\xi) = n \geqq 2$. A *polar frame* for $S(\xi)$ is a set of points $F = \{X_1, Y_1, \dots, X_n, Y_n\}$ in $S(\xi)$ with

$$f(X_i, X_j) = f(Y_i, Y_j) = 0 \text{ for } 1 \leqq i, j \leqq n,$$

and

$$f(X_i, X_j) = 0 \text{ for } i \neq j,$$

but

$$f(X_i, Y_i) \neq 0 \text{ for } 1 \leqq i \leqq n.$$

Since $i(\xi) = n$ and f is trace-valued and non-degenerate, it follows that $S(\xi)$ has a polar frame. Let span F be the set of points in $S(\xi)$ that are also in the subspace of $\mathbf{P}V$ spanned by the points in F.

A collineation $\pi:S(\xi) \rightarrow S(\xi')$ is called *thick* if there exists a polar frame F of $S(\xi)$ such that πF is a polar frame of $S(\xi')$ (so necessarily $i(\xi) = i(\xi')$) and, moreover, for each line L of $S(\xi)$ in span F the cardinality of the set $\{\pi X | X \text{ a point on } L\}$ is at least three. Thus, in particular, each line L' of span πF coming (via π) from a line $L \subseteq$ span F contains at least three points coming from points on L (in general, L' will also contain many points not coming from $S(\xi)$).

As a consequence of our theorem, the image $\pi S(\xi)$ of a thick collineation π is a polar space defined over the subring $\varphi(A)$ of k'. If the

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mapping π is surjective, then $\varphi(A) = k'$. However, in general, the image $\pi S(\xi)$ will be properly inside a polar space defined over the larger division ring k'.

3. Thick collineations. In this section we prove the theorem in the special case where $S(\xi)$ is spanned by any of its polar frames, that is, when $\dim V = 2n$ where $n = i(\xi) \geqq 3$, by generalizing the ideas of Theorem 2.1 in [**5**] to our present situation. The polar space $S(\xi)$ is associated with either a non-degenerate trace-valued (σ, ϵ)-hermitian form f, or with a (σ, ϵ)-quadratic form q with non-degenerate trace-valued (σ, ϵ)-hermitian form f. In the first case the linear subspaces of $S(\xi)$ are totally isotropic and in the second case they are totally singular. Likewise for $S(\xi')$.

Let $\pi:S(\xi) \to S(\xi')$ be a thick collineation. It is possible for $S(\xi)$ to be associated with a pseudo-quadratic form while $S(\xi')$ is associated with an hermitian form; for example, let π be the identity mapping but in the image space forget the pseudo-quadratic form and consider the larger space determined by the totally isotropic points (both k and k' will have characteristic two). It is also possible for $S(\xi)$ to be associated with an hermitian form and $S(\xi')$ with a pseudo-quadratic form; for example, f a symmetric form over the 2-adic number field $\mathbf{Q}_2$ and q' a quadratic form over the finite field $\mathbf{F}_2$.

Now assume $\dim V = 2i(\xi) \geqq 6$. Since π is a thick collineation, there exists a polar frame $\{X_1, Y_1, \ldots, X_n, Y_n\}$ for $S(\xi)$ with $\{\pi X_1, \pi Y_1, \ldots, \pi X_n, \pi Y_n\}$ a polar frame for $S(\xi')$. Let S be the totally isotropic or totally singular subspace of V spanned by $X_1, \ldots, X_n$ and T the totally isotropic or totally singular subspace spanned by $Y_1, \ldots, Y_n$. Then

$$S \cap T = 0 \quad \text{and} \quad V = S + T.$$

Likewise, if $\pi X_1, \ldots, \pi X_n$ spans S' in V' and $\pi Y_1, \ldots, \pi Y_n$ spans T', then

$$S' \cap T' = 0 \quad \text{and} \quad V' = (S' + T') \perp W'$$

with W' a subspace of V'. Since $n \geqq 3$, the restriction

$$\pi:\mathbf{P}S \to \mathbf{P}S'$$

satisfies the conditions of Theorem 3.1 in [**4**] (the proof remains valid over division rings). Hence there exists a place

$$\varphi_S:k \to k' \cup \infty$$

with valuation ring

$$A_S = \varphi_S^{-1}(k'),$$

a free A_S-module $M_S = u_1A_S + \ldots + u_nA_S$ in S with rank n and a φ_S-semilinear mapping $\beta_S:M_S \to S'$ defined by

$$\beta_S\left(\sum u_i a_i\right) = \sum u_i' \varphi_S(a_i)$$

such that

$$\pi X = \beta_S(M_s \cap X)k'$$

for all points X in $\mathbf{P}S$. Here u_i is a nonzero element from the one-dimensional subspace X_i of V, and u_i' is a nonzero element from πX_i, $1 \leqq i \leqq n$. The module M_S and the mapping β_S are not uniquely determined but can be changed by multiplication by scalars. Likewise, considering the restriction

$$\pi:\mathbf{P}T \rightarrow \mathbf{P}T',$$

there exist corresponding φ_T, A_T, $M_T = v_1 A_T + \ldots + v_n A_T$ in T with $v_i \in Y_i$, and $\beta_T:M_T \rightarrow T'$ where

$$\beta_T\left(\sum v_i b_i\right) = \sum v_i' \varphi_T(b_i) \quad \text{and}$$

$$\pi Y = \beta_T(M_T \cap Y)k'$$

for points Y in $\mathbf{P}T$.

Since π is a thick collineation, there exists a point $(u_1 + v_n b)k$ on the line $u_1 k + v_n k$ with image

$$(u_1' + v_n' b')k' \quad \text{where } b' \neq 0.$$

Replacing M_T by $M_T b$ and adjusting β_T we may assume $b = 1$. Likewise, by changing the choice of the v_i', we may assume $b' = 1$. Let $0 \neq a \in A_S$. Since π is a collineation, the image of $(u_1 a + u_2 + v_n a)k$ must be

$$(u_1' \varphi_S(a) + u_2' + v_n' \varphi_S(a))k'.$$

Hence the image of $(u_2 + v_n a)k$ must be

$$(u_2' + v_n' \varphi_S(a))k'.$$

Assume first that $a^{-1} \in A_T$. Then, by a similar argument, the image of $(u_2 + v_n a)k$ is also

$$(u_2' \varphi_T(a^{-1}) + v_n')k'.$$

Hence $\varphi_S(a) = \varphi_T(a)$. On the other hand, if $a^{-1} \notin A_T$, then $\varphi_T(a) = 0$. Now the image of $(u_2 + v_1 + v_n a)k$ is in $u_2' k' + v_1' k'$ and hence the image of $(u_2 + v_n a)k$ is $u_2' k'$. Thus $\varphi_S(a) = 0$ also. Hence

$$\varphi_S(a) = \varphi_T(a) \quad \text{for all } a \in A_S.$$

By symmetry $A_S = A_T = A$ and $\varphi_S = \varphi_T = \varphi$, say. Now $M = M_S + M_T$ is a free A-module of rank $2n$.

From the definition of a polar frame,

$$f(u_i, v_j) = \delta_{ij} c_j$$

where $0 \neq c_i \in k$. Assume there exist c_i, c_j with

$$\varphi(c_j^{-1}c_i) = 0.$$

Since $u_i + u_j$ and $v_i - v_j c_j^{-1} c_i$ are orthogonal, it follows from the properties of π that $u_i' + u_j'$ and $v_i' - v_j'\varphi(c_j^{-1}c_i)$ are orthogonal. This is a contradiction if

$$\varphi(c_j^{-1}c_i) = 0.$$

Hence $c_j^{-1}c_i$ is a unit in A and M is a modular A-module. Changing the choice of each $u_i \in X_i$ by a unit, we may assume

$$f(u_i, v_i) = c \neq 0, \quad 1 \leqq i \leqq n.$$

Replacing f by the proportional form $c^{-1}f$ we may assume M is unimodular and $c = 1$. Moreover, now

$$f'(u_i', v_i') = c' \neq 0, \quad 1 \leqq i \leqq n.$$

For any $a \in A$ the elements $u_1 + u_2 a$ and $v_1 a^\sigma - v_2$ are orthogonal. Hence $u_1' + u_2'\varphi(a)$ and $v_1'\varphi(a^\sigma) - v_2'$ are orthogonal so that

$$\varphi(a^\sigma) = c'^{-1}\varphi(a)^{\sigma'}c'.$$

Again, since $u_1 + v_2$ is orthogonal to $v_1\epsilon - u_2$, it follows that

$$c'\varphi(\epsilon) = c'^{\sigma'}\epsilon'.$$

If we now replace f' by the proportional (σ'', ϵ'')-hermitian form $c'^{-1}f'$ where

$$\epsilon'' = c'^{-1}c'^{\sigma'}\epsilon' \quad \text{and} \quad b^{\sigma''} = c'^{-1}b^{\sigma'}c' \quad \text{for } b \in k',$$

then

$$\varphi(\epsilon) = \epsilon'' \quad \text{and} \quad \varphi(a^\sigma) = \varphi(a)^{\sigma''} \quad \text{for all } a \in A.$$

(If ξ' is associated with a pseudo-quadratic form q', this will also be changed to a proportional form.) Now we may assume

$$f'(u_i', v_i') = 1, \quad 1 \leqq i \leqq n,$$

$$\varphi(\epsilon) = \epsilon' \quad \text{and}$$

$$\varphi(a^\sigma) = \varphi(a)^{\sigma'} \quad \text{for } a \in A.$$

Define $\beta: M \to V$ by

$$\beta(x + y) = \beta_S(x) + \beta_T(y) \quad \text{for } x \in M_S \text{ and } y \in M_T.$$

We must prove

$$\pi X = \beta(M \cap X)k'$$

for all points X in $S(\xi)$. This has already been done for X in $\mathbf{P}S$ or $\mathbf{P}T$, and for

$$X = (u_i + v_j a)k \quad \text{with } i \neq j \text{ and } a \in A.$$

Now consider $X = xk$ where

$$x = \sum (u_i a_i + v_i b_i) \quad \text{with } a_i, b_i \in A.$$

Without loss in generality, we may assume $a_1 = 1$. Let $\pi X = x'k'$ where

$$x' = \sum (u_i' a_i' + v_i' b_i').$$

Since, for $i \geqq 2$, x is orthogonal to $v_1 a_i^\sigma - v_i$, it follows that x' is orthogonal to $v_1' \varphi(a_i^\sigma) - v_i'$ and hence that

$$a_i' = \varphi(a_i) a_1'.$$

Similarly,

$$b_i' = \varphi(b_i) a_1' \quad \text{for } i \geqq 2.$$

Now assume that a_2, say, is a unit in A. Then x is orthogonal to $u_1 - v_2(b_1 a_2^{-1})^\sigma \epsilon$, from which it follows that

$$b_1' = \varphi(b_1) a_1'.$$

Thus, in this situation,

$$x' = \beta(x) a_1'$$

and the proof is complete. It remains to consider the case where $a_2, b_2, \ldots, a_n, b_n$ are all nonunits in A. Then

$$x' \in u_1' k' + v_1' k'.$$

We can now find $y \in u_2 k + u_3 k$ such that

$$f(x, y) = 0, \ \pi(yk) = \beta(y)k' \quad \text{and}$$

$$\pi((x + y)k) = \beta(x + y)k'.$$

It follows from collinearity that

$$\pi X = \beta(x)k'.$$

Also, it is clear that

$$\varphi(f(x, y)) = f'(\beta(x), \beta(y)) \quad \text{for all } x, y \in M.$$

This completes the proof when $\dim V = 2n$. In particular, in the symplectic situation where σ is the identity mapping and $\epsilon = -1$, the theorem has been established.

4. The general case. We now complete the proof of the theorem in the general situation. Since $\pi : S(\xi) \to S(\xi')$ is a thick collineation, by the results of the previous section, there exists a place $\varphi : k \to k' \cup \infty$ and a free unimodular A-module

$$M_0 = \perp_{i=1}^{n}(u_iA + v_iA)$$

where

$$n = i(\xi) = i(\xi') \geqq 3 \text{ and } f(u_i, v_j) = \delta_{ij}, \quad 1 \leqq i, j \leqq n$$

(after normalizing the form f). Moreover, there is a φ-semilinear mapping $\beta_0:M_0 \to V'$ such that

$$\pi X = \beta_0(M_0 \cap X)k' \quad \text{for all points } X \text{ in } S(\xi) \cap \mathbf{P}(M_0k).$$

We must extend M_0 to an A-module M in V with $Mk = V$, and β_0 to a φ-semilinear mapping $\beta:M \to V'$ which induces π.

Let $\beta_0(u_i) = u_i'$ and $\beta_0(v_i) = v_i'$, $1 \leqq i \leqq n$, and $V = M_0k \perp W$ where W is a subspace of V with Witt index $i(\xi|W) = 0$. The polar space $S(\xi')$ may be associated with a (σ', ϵ')-hermitian form f', or with a (σ', ϵ')-quadratic form q' with hermitian form f'. In either case, f' can be normalized by changing to a proportional form so that

$$f'(u_i', v_j') = \delta_{ij}, \quad 1 \leqq i, j \leqq n,$$

$$\varphi(\epsilon) = \epsilon' \quad \text{and}$$

$$\varphi(a^\sigma) = \varphi(a)^{\sigma'} \quad \text{for all } a \in A.$$

Let

$$V_0' = \perp_{i=1}^{n}(u_i'k' + v_i'k') \quad \text{and} \quad V' = V_0' \perp W'$$

where W' is a subspace of V' with Witt index $i(\xi'|W') = 0$. Then

$$\beta_0(M_0)k' = V_0'.$$

Let M_W be the set of all $w \in W$ for which there exists $c \in A$ with

$$f(w, w) = c + c^\sigma\epsilon \quad \text{and} \quad q(w) = c + k_{\sigma,\epsilon}$$

when $S(\xi)$ is associated with a pseudo-quadratic form q. We will prove that M_W is an A-module in W. Clearly $wa \in M_W$ for all $w \in M_W$ and $a \in A$, so it suffices to prove additivity. Let $w \in M_W$ so there exists $c \in A$ with

$$f(w, w) = c + c^\sigma\epsilon$$

(and $q(w) = c + k_{\sigma,\epsilon}$). Put

$$x = u_1 - v_1c + w.$$

Then $X = xk$ is a point in $S(\xi)$ and by the properties of π, its image

$$\pi X = (u_1'd' - v_1'c' + w')k' \quad \text{for some } c', d' \in k' \text{ and } w' \in W'.$$

Let

$$r = u_1 + v_1c^{\sigma}\epsilon + u_2 - v_2c^{\sigma}\epsilon.$$

Then

$$f(x, r) = f(r, r) = 0$$

so that $X + rk$ is a line in $S(\xi)$. Hence

$$f'(\beta_0(r), \pi X) = 0$$

and consequently $c' = \varphi(c)d'$. If $d' = 0$, then $c' = 0$ and $w' = 0$ since

$$i(\xi'|W') = 0.$$

As this is impossible, $d' \neq 0$ and we may assume after changing w' by a scalar, that $d' = 1$ and $c' = \varphi(c)$. Now let z be a second element in M_W. Then

$$f(z, z) = a + a^{\sigma}\epsilon$$

(and $q(z) = a + k_{\sigma,\epsilon}$) for some $a \in A$. Let

$$b = f(w, z) \in k.$$

If we show $b \in A$ then it follows that $w + z \in M_W$ since

$$f(w + z, w + z) = e + e^{\sigma}\epsilon \quad \text{with } e = a + b + c \in A$$

(and $q(w + z) = e + k_{\sigma,\epsilon}$). Let Y be the point $(v_1b + u_2 - v_2a - z)k$ in $S(\xi)$. Then $X + Y$ is a line in $S(\xi)$. Analogously to the argument with X above,

$$\pi Y = (v_1'b' + u_2' - v_2'\varphi(a) - z')k'$$

for some $b' \in k'$ and $z' \in W'$; it is impossible for $\pi Y = v_1'k'$ since

$$f'(\pi X, \pi Y) = 0.$$

Thus

$$b' = f'(w', z').$$

Also $Y + (u_1 - v_2b^{\sigma}\epsilon)k$ is a line in $S(\xi)$. If $b \notin A$, then

$$\pi(u_1 - v_2b^{\sigma}\epsilon)k = v_2'k'$$

forces

$$f'(\pi Y, v_2') = 0,$$

a contradiction. Hence $b \in A$ and then $b' = \varphi(b)$. Thus M_W is an A-module. We have also shown that

$$\varphi(f(w, z)) = f'(w', z'),$$

and when $S(\xi)$ and $S(\xi')$ are both associated with pseudo-quadratic forms, that

$$q'(w') = \varphi(q(w)).$$

The above argument also shows that if $w \in M_W$, so there exists $c \in A$ with $f(w, w) = c + c^\sigma\epsilon$, then the image of the point $X = xk$ where $x = u_1 - v_1c + w$ is

$$\pi X = (u_1' - v_1'\varphi(c) + w')k' \quad \text{for some } w' \in W'.$$

Define $\beta_W: M_W \to W'$ by $\beta_W(w) = w'$. We will prove that β_W is a φ-semilinear mapping. If $c \in \mathfrak{m}$, the unique two-sided ideal of A, then

$$f'(w', w') = 0$$

(and $q'(w') = 0$). Since $i(\xi'|W') = 0$, it then follows that $w' = 0$. In particular,

$$\beta_W(wa) = 0 = \beta_W(w)\varphi(a) \quad \text{for all } w \in M_W \text{ and } a \in \mathfrak{m}.$$

For $w \in M_W$, let $t = v + w$ where v is a primitive element of M_0 (so $v \notin M_0\mathfrak{m}$) with $tk \in S(\xi)$. Then, by an argument analogous to one used before,

$$\pi tk = (\beta_0(v) + w'')k' \quad \text{for some } w'' \in W'.$$

(In fact, after changing the basis of M_0, we may assume $v = \bar{u}_1 - \bar{v}_1 a$ with $a \in A$ in some new basis $\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_n, \bar{v}_n$ associated with a polar frame, and the result follows from the φ-semilinearity of β_0 on M_0.) We next show $w'' = \beta_W(w)$. For those $v \in M_0$ for which $f(x, t) = 0$ this is clear since $(x - t)k$ lies in $S(\xi) \cap \mathbf{P}(M_0k)$, so that $\pi(x - t)k$ must be collinear with πX and πtk. In the remaining case, it follows after constructing a new point of this same type orthogonal to both X and tk. This construction is possible since $n \geqq 3$. Thus

$$\pi(v + w)k = (\beta_0(v) + \beta_W(w))k'.$$

It is now easily seen that β_W is an additive homomorphism and

$$\beta_W(wa) = \beta_W(w)\varphi(a) \quad \text{for all units } a \in A.$$

Thus β_W is a φ-semilinear mapping.

Next we prove $M_Wk = W$ so that $M = M_0 \perp M_W$ is an A-module with $Mk = V$. Let $w \in W$. Since f is trace-valued there exists $c \in k$ such that

$$f(w, w) = c + c^\sigma\epsilon$$

(and $q(w) = c + k_{\sigma,\epsilon}$ when $S(\xi)$ is associated with a pseudo-quadratic form q). If $c \in A$, then $w \in M_W$. Otherwise, $c^{-1} \in A$ and $wc^{-1} \in M_W$ since

$$f(wc^{-1}, wc^{-1}) = e + e^{\sigma}\epsilon$$

where

$$e = (c^{-1})^{\sigma} \in A.$$

Hence $M_W k = W$.

Define a φ-semilinear mapping $\beta:M \to V'$ by

$$\beta(v + w) = \beta_0(v) + \beta_W(w) \quad \text{for all } v \in M_0 \text{ and } w \in M_W.$$

Then β induces π. For let $X = xk \in S(\xi)$ where $x = v + w$ with $v \in M_0$ primitive and $w \in W'$. Then

$$f(w, w) = -f(v, v)$$

(and $q(w) = -q(v)$). Since $v \in M_0$ there now exists $c \in A$ such that

$$f(w, w) = c + c^{\sigma}\epsilon$$

(and $q(w) = c + k_{\sigma,\epsilon}$). Hence $w \in M_W$ and $x \in M$. Since $M \cap X = xA$ it follows that

$$\pi X = \beta(x)k' = \beta(M \cap X)k'.$$

This completes the proof of the main statement of the theorem. We have also shown that f and f' can be chosen such that

$$\varphi(f(x, y)) = f'(\beta(x), \beta(y)) \quad \text{for all } x, y \in M.$$

If $f(M_W, M_W) \subseteq \mathfrak{m}$, then (see note added in proof)

$$\beta(M_W) = 0 \quad \text{and} \quad \beta(M)k' = V_0'.$$

In this case take $M(1) = M_0$ and $M(2) = M_W$. Then

$$M = M(1) \perp M(2)$$

with $M(1)$ a free unimodular A-module of rank $2n$. It is possible to construct examples where $M(2)$ is not free.

Now assume there exist $w, z \in M_W$ with $f(w, z) = 1$. Let

$$\beta(w) = w' \quad \text{and} \quad \beta(z) = z'.$$

Then

$$f'(w', z') = \varphi(1) = 1$$

and hence $w' \neq 0$ and $z' \neq 0$. If $f(w, w)$ is a unit in A, then wA is a free, rank one, orthogonal direct summand of M_W. Likewise if $f(z, z)$ is a unit. If neither $f(w, w)$ nor $f(z, z)$ are units, then $wA + zA$ is a unimodular free, rank two, orthogonal direct summand of M_W. Since now

$$f'(w', w') = \varphi(f(w, w)) = 0,$$

the polar space $S(\xi')$ must be associated with a pseudo-quadratic form q' with $q'(w') \neq 0$ (and the characteristic of k' must be two). Proceeding in this fashion we obtain a splitting

$$M = M(1) \perp M(2)$$

where

$$M(1) = M_0 \perp B_1 \perp \ldots \perp B_m$$

with the B_i free unimodular A-modules of rank one or two, while $M(2)$ is an A-module with

$$f(M(2), M(2)) \subseteq \mathfrak{m}.$$

Thus $\beta(M(2)) = 0$. The A-module $M(1)$ is unimodular and free with rank equal to the dimension of $\beta(M)$ over $\varphi(A)$. This completes the proof of the theorem.

Remarks. In the splitting $M = M(1) \perp M(2)$ the components $M(1)$ and $M(2)$ are not uniquely determined. If the valuation ring A is discrete, then $M(2)$ will be free and $\mathfrak{m}$-modular and $M = M(1) \perp M(2)$ is just a splitting into Jordan components. In general, $M(2)$ will not be free.

Since there is no assumption that the collineation

$$\pi: S(\xi) \rightarrow S(\xi')$$

is surjective, in general

$$\varphi(A) \neq k' \quad \text{and} \quad \beta(M)k' \neq V'.$$

In fact, it is not necessarily true that $\beta(M)k'$ and V' have the same dimension over k'. However, if it is assumed that π is surjective, although not necessarily injective, it is easily seen that

$$\varphi(A) = k', \beta_0(M_0) = V'_0 \quad \text{and}$$

$$\beta(M) = \beta(M(1)) = V'.$$

Note added in proof. There is an exceptional situation for part of the theorem we had not noticed before. Assume the characteristic of k' is two, $S(\xi')$ is a polar space associated with a pseudo-quadratic form and π is not surjective. It is then possible for $\beta(M(2)) \neq 0$, so the statement about $M(1)$ and $M(2)$ in the theorem should be omitted for this situation. The main part of the theorem is not affected.

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