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THE TURAEV-VIRO INVARIANT FOR 3-MANIFOLDS IS A SUM OF THREE INVARIANTS

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ABSTRACT. We show that every Turaev-Viro invariant for 3-manifolds is a sum of three new invariants and discuss their properties. We also find a solution of a conjecture of L. H. Kauffman and S. Lins. Tables of the invariants for closed orientable 3-manifolds of complexity ≤ 3 are presented at the end of the paper.

0. **Introduction.** In 1990 Turaev and Viro obtained an infinite set of 3-manifold numerical invariants $\mathrm{TV}_{r,p}$ [7]. The invariants are parameterized by pairs (r,p) of non-negative coprime integers with $r > p > 0$ and $r > 2$.

We show that every Turaev-Viro invariant is a sum of three new invariants and discuss their properties. The paper also answers (negatively) a conjecture due to L. H. Kauffman and S. Lins [2] and studies the case $r = 4$ (for the case $r = 3$ see [7]). At the end of the paper we present tables of the invariants with $r < 8$ for closed orientable 3-manifolds of complexity ≤ 3. (For complexity theory see [3].)

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1. **Definition and preliminaries.** A 2-dimensional connected polyhedron X is called *special* if it satisfies the following conditions: 1) the link of any point of X is homeomorphic to a circle, a circle with two radii, or a circle with three radii; 2) every connected component of the set of 2-manifold points of X is a 2-cell. A special polyhedron X is called a *special spine* of a compact 3-manifold M^3 with $\partial M^3 \neq \emptyset$ if there exists an imbedding $i: X \longrightarrow M^3$ such that $M^3 \searrow i(X)$, *i.e.*, M^3 collapses onto $i(X)$. A special polyhedron X is called a *special spine* of a closed 3-manifold M^3 if X is a special spine of M^3 with an open ball removed. It is known that every compact connected 3-manifold has a special spine ([1], [4]).

We shall describe elementary moves on special polyhedra. The move $\mathcal{M}$ changes a small neighborhood of some edge in a fashion indicated in Fig. 1. The move $\mathcal{M}^{-1}$ is the inverse of $\mathcal{M}$ (see [4] for details). We call two special polyhedra *$\mathcal{M}$-equivalent* if one can be obtained from the other by a finite sequence of moves $\mathcal{M}^{\pm 1}$. Let X_1 be a special spine of 3-manifold M^3 and let X_2 be a special polyhedron. Let both X_1 and X_2 have more

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than one vertex. Then in order for X_2 also to be a special spine of M^3, it is necessary and sufficient that X_1 and X_2 be $\mathcal{M}$-equivalent (See [4], [5]).

Let X be a special spine of a 3-manifold M^3. We fix a pair (r, p) of coprime integers with $r > p > 0$ and $r > 2$. Let $\Gamma_1, \ldots, \Gamma_b$ be the 2-components of X. By a *coloring* of X we mean an arbitrary mapping $\varphi: \{\Gamma_1, \ldots, \Gamma_b\} \longrightarrow \mathbf{Z}_{r-1} = \{0, 1, \ldots, r-2\}$. A coloring φ is called *admissible* if for all triples of 2-components Γ_i, Γ_j, Γ_k meeting on the same edge we have:

$$2r-4 \geq \varphi(\Gamma_i) + \varphi(\Gamma_j) + \varphi(\Gamma_k) \equiv 0 \pmod 2,$$
$$|\varphi(\Gamma_i) - \varphi(\Gamma_j)| \leq \varphi(\Gamma_k) \leq \varphi(\Gamma_i) + \varphi(\Gamma_j).$$

Let us denote the set of admissible colorings of X by $\mathrm{Adm}\,X$.

Let Γ_i, Γ_j, Γ_k be 2-components incident to an edge E of X and let $\varphi \in \mathrm{Adm}\,X$. We shall say that a triple $\big(\varphi(\Gamma_i), \varphi(\Gamma_j), \varphi(\Gamma_k)\big)$ is a *color of edge E*. There are six distinct 2-components incident to any vertex of a special spine. Suppose for a vertex v they receive under φ the values $i, j, k, l, m, n \in \mathbf{Z}_{r-1}$. A 6-tuple $\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$ is called a *color of vertex* v if (i, j, k) is a color of some edge incident to v and (i, l), (j, m), (k, n) are the pairs of colors of opposite 2-components incident to v.

Let us recall how to compute the Turaev-Viro invariant for compact connected 3-manifold M^3 with special spine X. For an integer $n > 0$ set

$$[n] = \sin(np\pi/r)/\sin(p\pi/r),$$
$$[n]! = [n][n-1]\cdots[2][1].$$

Set also $[0] = [0]! = 1$. For a color (i, j, k) of an edge set

$$\Delta(i,j,k) = \left(\frac{[\underline{i}+\underline{j}-\underline{k}]!\,[\underline{i}+\underline{k}-\underline{j}]!\,[\underline{j}+\underline{k}-\underline{i}]!}{[\underline{i}+\underline{j}+\underline{k}+1]!}\right)^{1/2}$$

where $\underline{i} = i/2$. Note that the expression in the round brackets presents a real number. By the square root $x^{1/2}$ of a real number x we mean the positive root of $|x|$ multiplied by $\sqrt{-1}$ if $x < 0$.

Let $\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$ be a color of some vertex v. A *symbol* of v is computed by the following formula

$$|T_v^{\varphi}| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$
$$= (\sqrt{-1})^{-(i+j+k+l+m+n)}\Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m)\begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix},$$

where

$$\begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix} = \sum_z (-1)^z[z+1]!\{[z-\underline{i}-\underline{j}-\underline{k}]!\,[z-\underline{i}-\underline{m}-\underline{n}]!\,[z-\underline{j}-\underline{l}-\underline{n}]!\,[z-\underline{k}-\underline{l}-\underline{m}]!$$
$$\times[\underline{i}+\underline{j}+\underline{l}+\underline{m}-z]!\,[\underline{i}+\underline{k}+\underline{l}+\underline{n}-z]!\,[\underline{j}+\underline{k}+\underline{m}+\underline{n}-z]!\}^{-1}.$$

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Here z runs through the non-negative integers such a way that all expressions in the square brackets are non-negative. For $i \in \mathbf{Z}_{r-1}$ put

$$w_i(r,p) = (\sqrt{-1})^i[i+1]^{1/2}.$$

For $\varphi \in \operatorname{Adm} X$ put

$$|X|_\varphi(r,p) = \prod_{i=1}^{b} w^2_{\varphi(\Gamma_i)} \prod_{j=1}^{d} |T^{\varphi}_{v_j}|,$$

where $v_1, \ldots, v_d$ are the vertices of X. The number

$$\mathrm{TV}_{r,p}(M^3) = w^{-2\chi(X)} \sum_{\varphi \in \operatorname{Adm} X} |X|_\varphi(r,p)$$

is the *Turaev-Viro invariant* for 3-manifold M^3, where $w = \pm\sqrt{2r}/\bigl(2\sin(p\pi/r)\bigr)$ and $\chi(X)$ is Euler characteristic of X. (See [7] for details.)

The set of 2-components that receive odd colors under an admissible coloring φ forms a closed surface embedded in X. We shall denote this surface by $S(\varphi)$. Present the set $\operatorname{Adm} X$ as a disjoint union of subsets $\operatorname{Adm}_0 X$, $\operatorname{Adm}_1 X$, and $\operatorname{Adm}_2 X$, where

0) $\varphi \in \operatorname{Adm}_0 X \Leftrightarrow (\varphi \in \operatorname{Adm} X)\ \&\ \bigl(S(\varphi) = \emptyset\bigr)$;

1) $\varphi \in \operatorname{Adm}_1 X \Leftrightarrow (\varphi \in \operatorname{Adm} X)\ \&\ \Bigl(\chi\bigl(S(\varphi)\bigr) \equiv 1 \pmod 2\Bigr)$;

2) $\varphi \in \operatorname{Adm}_2 X \Leftrightarrow (\varphi \in \operatorname{Adm} X)\ \&\ \bigl(S(\varphi) \neq \emptyset\bigr)\ \&\ \Bigl(\chi\bigl(S(\varphi)\bigr) \equiv 0 \pmod 2\Bigr)$.

Here $\chi\bigl(S(\varphi)\bigr)$ is Euler characteristic of surface $S(\varphi)$. If $\varphi \in \operatorname{Adm}_N X$, where $N = 0$, 1, or 2, then we say that N is the *index of coloring* φ.

2. **Main theorem.**

THEOREM 2.1. *Let M^3 be a compact 3-manifold and X_1 be its special spine. Then the numbers*

$$|M^3|_0(r,p) = \sum_{\varphi \in \operatorname{Adm}_0 X_1} |X_1|_\varphi(r,p),$$
$$|M^3|_1(r,p) = \sum_{\varphi \in \operatorname{Adm}_1 X_1} |X_1|_\varphi(r,p),$$
$$|M^3|_2(r,p) = \sum_{\varphi \in \operatorname{Adm}_2 X_1} |X_1|_\varphi(r,p)$$

are invariants of M^3 and

$$\mathrm{TV}_{r,p}(M^3) = w^{-2\chi(X)}\bigl(|M^3|_0(r,p) + |M^3|_1(r,p) + |M^3|_2(r,p)\bigr).$$

PROOF. Let X_2 denote the spine, which is obtained from X_1 by a single elementary move $\mathcal{M}$. It suffices to prove that

$$\sum_{\varphi \in \operatorname{Adm}_N X_1} |X_1|_\varphi(r,p) = \sum_{\varphi \in \operatorname{Adm}_N X_2} |X_2|_\varphi(r,p)$$

for every $N \in \{0, 1, 2\}$.

By definition of the $\mathcal{M}$-move, there exist imbeddings $i_1: P_1 \longrightarrow X_1$ and $i_2: P_2 \longrightarrow X_2$ such that $X_1 - i_1(P_1) = X_2 - i_2(P_2)$, where P_1 and P_2 are the polyhedra in Fig. 1. It is evident that for any $\varphi_1 \in \mathrm{Adm}\, X_1$ and $\varphi_2 \in \mathrm{Adm}\, X_2$ such that $\varphi_1|_{X_1 - i_1(P_1)} = \varphi_2|_{X_2 - i_2(P_2)}$ the indices of colorings φ_1 and φ_2 are the same, *i.e.*, there exists a unique number $N \in \{0, 1, 2\}$ such that $\varphi_1 \in \mathrm{Adm}_N X_1$ and $\varphi_2 \in \mathrm{Adm}_N X_2$. The remaining part of the Proof coincides with the Proof of 4.4.B from [7]. ∎

The following Lemma shows that the invariants $|M^3|_N(r, p)$ are not independent.

LEMMA 2.2. *Let M^3 be a compact orientable 3-manifold. Then*
1) *For all parameters (r, p) we have $|M^3|_N(r, p) = (-1)^N |M^3|_N(r, r-p)$, where $N \in \{0, 1, 2\}$.*
2) $|M^3|_0(3, 1) = |M^3|_0(3, 2) = 1$.

PROOF. 1) It is easily proved that for any orientable 3-manifold M^3 with special spine X and for any $\varphi \in \mathrm{Adm}\, X$ we have

$$|X|_\varphi(r,p) = (-1)^{\chi(S(\varphi))} |X|_\varphi(r, r-p) \tag{1}$$

(for proof see [6]). It remains to check that $\chi\big(S(\varphi)\big) \equiv N \pmod 2$.

2) If $r = 3$ then $\mathrm{Adm}_0 X = \{\varphi_0\}$, where φ_0 is the zero–coloring. It is known that $|X|_{\varphi_0}(3, 1) = 1$. ∎

3. **The Kauffman-Lins conjecture. The case r=4.** Throughout this section, $\mathrm{TV}^*_{r.p}$ will be the Turaev-Viro invariant without the factor $w^{-2\chi(X)}$, *i.e.*,

$$\mathrm{TV}^*_{r.p}(M^3) = \sum_{\varphi \in \mathrm{Adm}\, X} |X|_\varphi(r,p).$$

(Without this factor the invariant depends on removing open balls from M^3. But this fact is irrelevant here.)

In [2] L. H. Kauffman and S. Lins put forward the following conjecture.

CONJECTURE. Consider an arbitrary closed 3-manifold M^3, and let X be a special spine for M^3. Let n_e be the number of closed surfaces contained in X that have even Euler characteristic and n_o the number of closed surfaces in X that have odd Euler characteristic. Then (i) either $n_e = n_o$, or $n_o = 0$, and this alternative is invariant for every special spine of M^3; (ii) $n_o = 0 \Leftrightarrow \mathrm{TV}^*_{4.1}(M^3) = \mathrm{TV}^*_{4.3}(M^3) \in \mathbf{Z}$.

Note that if M^3 is an orientable manifold, then part (i) of the conjecture follows from 8.3–8.4 in [7]. If M^3 is a nonorientable manifold, then there exists a counter example offered by S. V. Matveev. It is the manifold $\mathbf{R}P^2 \times S^1$; a neighborhood of the 1-skeleton of a special spine for $\mathbf{R}P^2 \times S^1$ is shown on Fig. 2a. For this spine we have: $n_o = 1$ and $n_e = 3$. This disproves the part (i) of the conjecture. (Let us give some explanation of Fig. 2a. The manifold $\mathbf{R}P^2 \times S^1$ with open ball B^3 removed collapses onto $M \times S^1 \cup_{\partial M} D^2$, where M is Möbius band and D^2 is a disk attached along ∂M. The manifold $M \times S^1$ collapses onto torus $S^1 \times S^1$, so $\mathbf{R}P^2 \times S^1 - B^3$ collapses onto $S^1 \times S^1$ with disk D^2,

	$L_{3.1}$	S^3	$L_{5.2}$	$L_{4.1}$	$\mathbf{R}P^3$	$L_{5.1}$	$L_{7.2}$	$L_{8.3}$	S^3/Q_8	$L_{6.1}$	$L_{9.2}$	$L_{10.3}$	$L_{11.4}$	$L_{12.5}$	$L_{13.5}$	S^3/Q_{12}
(3,1)	0.500	0.500	0.500	1.000	0.000	0.500	0.500	1.000	2.000	0.000	0.500	0.000	0.500	1.000	0.500	1.000
(3,2)	0.500	0.500	0.500	1.000	1.000	0.500	0.500	1.000	2.000	1.000	0.500	1.000	0.500	1.000	0.500	1.000
(4,1)	0.250	0.250	0.250	0.500	0.146	0.250	0.250	1.000	2.500	0.853	0.250	0.853	0.250	0.500	0.250	0.500
(4,3)	0.250	0.250	0.250	0.500	0.853	0.250	0.250	1.000	2.500	0.146	0.250	0.146	0.250	0.500	0.250	0.500
(5,1)	0.362	0.138	0.000	0.276	0.000	0.500	0.362	0.724	2.553	0.000	0.138	0.000	0.138	0.724	0.362	1.276
(5,4)	0.362	0.138	0.000	0.276	0.724	0.500	0.362	0.724	2.553	0.276	0.138	0.000	0.138	0.724	0.362	1.276
(5,2)	1.382	0.362	0.000	0.724	0.276	0.500	0.138	0.276	3.447	0.724	0.362	0.000	0.362	0.276	0.138	1.724
(5,3)	1.382	0.362	0.000	0.724	0.000	0.500	0.138	0.276	3.447	0.000	0.362	0.000	0.362	0.276	0.138	1.724
(6,1)	0.250	0.083	0.083	0.333	0.045	0.083	0.083	0.333	2.333	0.500	0.250	0.622	0.083	1.000	0.083	1.333
(6,5)	0.250	0.083	0.083	0.333	0.622	0.083	0.083	0.333	2.333	0.500	0.250	0.045	0.083	1.000	0.083	1.333
(7,1)	0.175	0.054	0.272	0.349	0.000	0.272	0.000	0.108	2.043	0.000	0.272	0.000	0.175	0.543	0.054	0.758
(7,6)	0.175	0.054	0.272	0.349	0.543	0.272	0.000	0.108	2.043	0.108	0.272	0.349	0.175	0.543	0.054	0.758
(7,2)	0.272	0.175	0.054	0.543	0.108	0.054	0.000	0.349	4.268	0.349	0.054	0.543	0.272	0.108	0.175	0.806
(7,5)	0.272	0.175	0.054	0.543	0.000	0.054	0.000	0.349	4.268	0.000	0.054	0.000	0.272	0.108	0.175	0.806
(7,3)	0.054	0.272	0.175	0.108	0.000	0.175	0.000	0.543	3.689	0.000	0.175	0.000	0.054	0.349	0.272	1.436
(7,4)	0.054	0.272	0.175	0.108	0.349	0.175	0.000	0.543	3.689	0.543	0.175	0.108	0.054	0.349	0.272	1.436

TABLE 1

attached to it such that ∂D^2 winds twice around $S^1 \times \{*\}$. There is an isotopy of ∂D^2 in the torus $S^1 \times S^1$ transforming our polyhedron to special one with neighborhood of the 1-skeleton pictured on Fig. 2a. Union of 2-components marked by I, II, III is the torus $S^1 \times S^1$, unmarked curve is the boundary of the disk D^2.)

Let us consider the manifold S^3/Q_{16} (A neighborhood of the 1-skelton of a special spine for S^3/Q_{16} is shown on Fig. 2b, see [3]). We can see that $\mathrm{TV}^*_{4.1}(S^3/Q_{16}) = \mathrm{TV}^*_{4.3}(S^3/Q_{16}) = 6$, but $n_o \neq 0$ for this manifold. So the implication $\Leftarrow$ of part (ii) in the conjecture is not true. But if $n_o = 0$ then $\mathrm{TV}^*_{4.1}(M^3) = \mathrm{TV}^*_{4.3}(M^3) \in \mathbf{Z}$. It follows from the next Lemma. (Let us remark that $\mathrm{TV}^*_{r.p}(M^3) = \mathrm{TV}^*_{r.r-p}(M^3)$ in the case $n_o = 0$ for any (r, p) and any compact orientable 3-manifold M^3. It follows from the equation (1)).

Let us denote by $S_i(\varphi)$ the union of 2-components $\Gamma_j \subset X$ such that $\varphi(\Gamma_j) = i$.

LEMMA 3.1. *Let X be a special spine of a compact 3-manifold M^3, let $r = 4$, and $\varphi \in \mathrm{Adm}\, X$. Then*

$$|X|_\varphi(4,1) = (-\sqrt{2})^{\chi(S_1(\varphi))}(-1)^n, \tag{2}$$

$$|X|_\varphi(4,3) = (\sqrt{2})^{\chi(S_1(\varphi))}(-1)^n, \tag{2'}$$

where n is the number of vertices in the graph $G = \partial S_2(\varphi)$.

PROOF. Values of symbols for vertices and weights for 2-cells of the spine X in the case $r = 4$ are given in [2]. Their products give necessary equalities. ■

COROLLARY 3.2. *If M^3 is a compact orientable 3-manifold, then*

$$|M^3|_0(4,1) = |M^3|_0(4,3) = \mathrm{TV}^*_{3.2}(M^3).$$

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	$L_{3.1}$	S^3	$L_{5.2}$	$L_{4.1}$	$\mathbf{R}P^3$	$L_{5.1}$	$L_{7.2}$	$L_{8.3}$	S^3/Q_8	$L_{6.1}$	$L_{9.2}$	$L_{10.3}$	$L_{11.4}$	$L_{12.5}$	$L_{13.5}$	S^3/Q_{12}
(3,1)0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(3,1)1	0.000	0.000	0.000	0.000	−1.000	0.000	0.000	0.000	0.000	−1.000	0.000	−1.000	0.000	0.000	0.000	0.000
(3,1)2	0.000	0.000	0.000	1.000	0.000	0.000	0.000	1.000	3.000	0.000	0.000	0.000	0.000	1.000	0.000	1.000
(4,1)0	1.000	1.000	1.000	2.000	2.000	1.000	1.000	2.000	4.000	2.000	1.000	2.000	1.000	2.000	1.000	2.000
(4,1)1	0.000	0.000	0.000	0.000	−1.414	0.000	0.000	0.000	0.000	1.414	0.000	1.414	0.000	0.000	0.000	0.000
(4,1)2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.000	6.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
(5,1)0	2.618	1.000	0.000	1.000	2.618	3.618	2.618	2.618	4.618	1.000	1.000	0.000	1.000	2.618	2.618	4.618
(5,1)1	0.000	0.000	0.000	0.000	−2.618	0.000	0.000	0.000	0.000	−1.000	0.000	0.000	0.000	0.000	0.000	0.000
(5,1)2	0.000	0.000	0.000	1.000	0.000	0.000	0.000	2.618	13.854	0.000	0.000	0.000	0.000	2.618	0.000	4.618
(5,2)0	0.382	1.000	0.000	1.000	0.382	1.382	0.382	0.382	2.382	1.000	1.000	0.000	1.000	0.382	0.382	2.382
(5,2)1	0.000	0.000	0.000	0.000	0.382	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000
(5,2)2	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.382	7.146	0.000	0.000	0.000	0.000	0.382	0.000	2.382
(6,1)0	3.000	1.000	1.000	4.000	4.000	1.000	1.000	4.000	10.000	6.000	3.000	4.000	1.000	6.000	1.000	10.000
(6,1)1	0.000	0.000	0.000	0.000	−3.464	0.000	0.000	0.000	0.000	0.000	0.000	3.464	0.000	0.000	0.000	0.000
(6,1)2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	18.000	0.000	0.000	0.000	0.000	6.000	0.000	6.000
(7,1)0	3.247	1.000	5.049	3.247	5.049	5.049	0.000	1.000	9.494	1.000	5.049	3.247	3.247	5.049	1.000	7.049
(7,1)1	0.000	0.000	0.000	0.000	−5.049	0.000	0.000	0.000	0.000	−1.000	0.000	−3.247	0.000	0.000	0.000	0.000
(7,1)2	0.000	0.000	0.000	3.247	0.000	0.000	0.000	1.000	28.482	0.000	0.000	0.000	0.000	5.049	0.000	7.049
(7,2)0	1.555	1.000	0.308	1.555	0.308	0.308	0.000	1.000	6.110	1.000	0.308	1.555	1.555	0.308	1.000	2.308
(7,2)1	0.000	0.000	0.000	0.000	0.308	0.000	0.000	0.000	0.000	1.000	0.000	1.555	0.000	0.000	0.000	0.000
(7,2)2	0.000	0.000	0.000	1.555	0.000	0.000	0.000	1.000	18.330	0.000	0.000	0.000	0.000	0.308	0.000	2.308
(7,3)0	1.198	1.000	0.643	0.198	0.643	0.643	0.000	1.000	3.396	1.000	0.643	0.198	0.198	0.643	1.000	2.643
(7,3)1	0.000	0.000	0.000	0.000	−0.643	0.000	0.000	0.000	0.000	−1.000	0.000	−0.198	0.000	0.000	0.000	0.000
(7,3)2	0.000	0.000	0.000	0.198	0.000	0.000	0.000	1.000	10.188	0.000	0.000	0.000	0.000	0.643	0.000	2.643

TABLE 2

PROOF. Let X be a special spine of M^3. If $\varphi \in \mathrm{Adm}_0 X$, then $\chi\big(S_1(\varphi)\big)$ and n in (2) and (2′) are equal to 0, so we have $|M^3|_0(4,1) = |M^3|_0(4,3) = |\,\mathrm{Adm}_0 X|$. Consider the coloring $\underline{\varphi}\colon \{\Gamma_1,\ldots,\Gamma_b\} \longrightarrow \mathbf{Z}_2$ such that $\underline{\varphi}(\Gamma_i) = \varphi(\Gamma_i)/2$ for every $1 \le i \le b$. The mapping $\varphi \longmapsto \underline{\varphi}$ gives a bijection between $\mathrm{Adm}_0 X$ in the case $r = 4$ and $\mathrm{Adm}\, X$ in the case $r = 3$. The equation $\mathrm{TV}^*_{3,2}(M^3) = |\,\mathrm{Adm}\, X|$ follows from 8.3 in [7]. ∎

4. **The tables.** We computed the values for invariants $\mathrm{TV}_{r,p}(M^3)$ and $|M^3|_N(r,p)$ with $r \le 7$ for all prime closed orientable 3-manifolds of complexity ≤ 6. The computed values allowed us to conjecture that the sums $\sum_{p<r/2} |M^3|_0(r,p)$ and $\sum_{p<r/2} |M^3|_2(r,p)$ for any closed orientable 3-manifold M^3 are integers. Analysis of computation shows that the invariants with $r \le 7$ identify 45 manifolds from 61 of complexity < 6. There are two pairs and four triples of 3-manifolds with the same computed invariants.

We present the tables of invariants for the prime closed orientable 3-manifolds of complexity < 4. The first table contains the original Turaev-Viro invariants $\mathrm{TV}_{r,p}(M^3)$.

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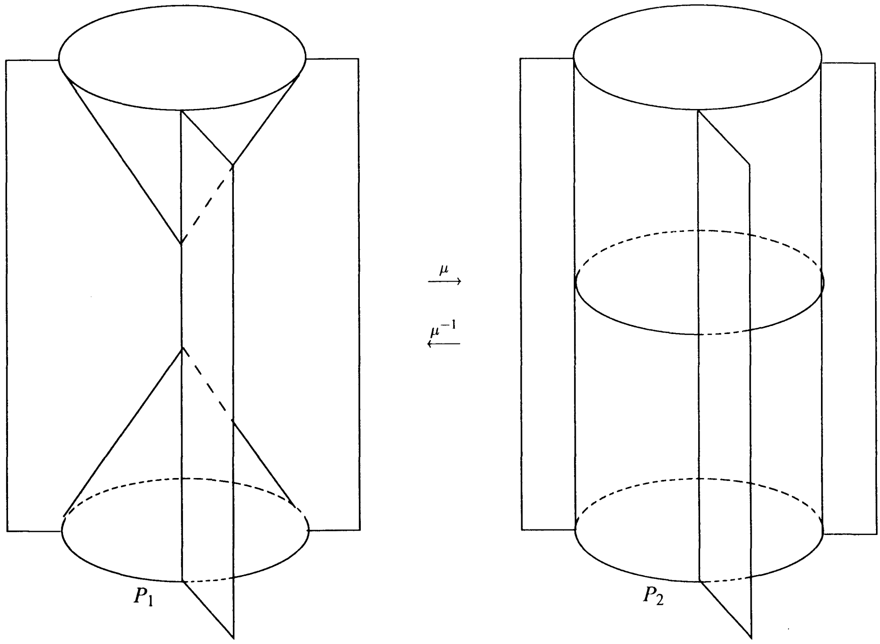
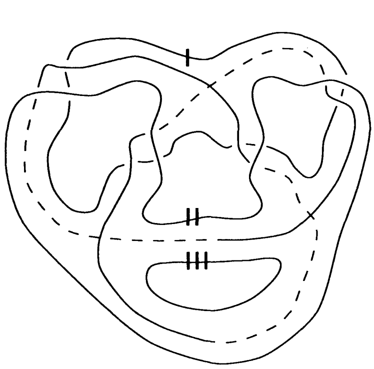
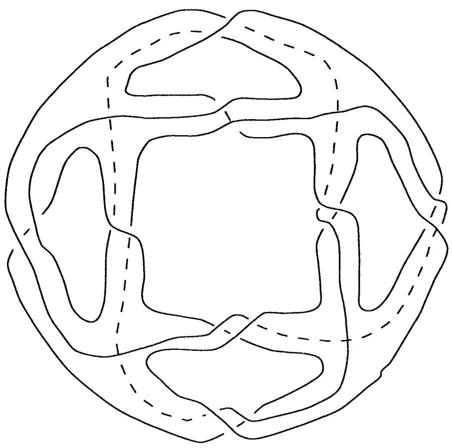




a) $\mathbb{R}P^2 \times S^1$



b) S^3/Q^{16}

The second table contains the invariants $|M^3|_N(r,p)$ (see section 2). The names we have chosen for the manifolds are taken from [3]. The numbers in the brackets in the first columns of tables are the parameters (r,p); the number 0, 1, or 2 after the brackets in the first column of the second table shows what kind of three invariants $|M^3|_0(r,p)$, $|M^3|_1(r,p)$, or $|M^3|_2(r,p)$ is presented in this line. The second table does not contain invariants with parameters $p \geq r/2$ because it has been proved in Lemma 2.2 that $|M^3|_N(r,p) = (-1)^N|M^3|_N(r,r-p)$.

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The invariants from the first table and the second one are related by the equality

$$\mathrm{TV}_{r,p}(M^3) = \left(2\sin^2(p\pi/r)/r\right) \times \left(|M^3|_0(r,p) + |M^3|_1(r,p) + |M^3|_2(r,p)\right).$$

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