NONLINEAR ANELASTIC MODAL THEORY FOR SOLAR CONVECTION*

(Invited Review)

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Abstract. Preliminary solar envelope models have been computed using the single-mode anelastic equations as a description of turbulent convection. This approach provides estimates for the variation with depth of the largest convective cellular flows, akin to giant cells, with horizontal sizes comparable to the total depth of the convection zone. These modal nonlinear treatments are capable of describing compressible motions occurring over many density scale heights. Single-mode anelastic solutions have been constructed for a solar envelope whose mean stratification is nearly adiabatic over most of its vertical extent because of the enthalpy (or convective) flux explicitly carried by the big cell; a sub-grid scale representation of turbulent heat transport is incorporated into the treatment near the surface. The single-mode equations admit two solutions for the same horizontal wavelength, and these are distinguished by the sense of the vertical velocity at the center of the three-dimensional cell. It is striking that the upward directed flows experience large pressure effects when they penetrate into regions where the vertical scale height has become small compared to their horizontal scale. The fluctuating pressure can modify the density fluctuations so that the sense of the buoyancy force is changed, with buoyancy braking actually achieved near the top of the convection zone. The pressure and buoyancy work in the shallow but unstable H^+ and He^+ ionization regions can serve to decelerate the vertical motions and deflect them laterally, leading to strong horizontal shearing motions. It appears that such dynamical processes may explain why the amplitudes of flows related to the largest scales of convection are so feeble in the solar atmosphere.

1. Introduction

The structure of the solar atmosphere is determined largely by the convection just below the surface and the waves that it can generate. The coupling of these turbulent motions with magnetic fields must cause most of what is observed on the Sun. However, theoretical understanding of the dynamics of the solar convection zone and associated motions in the atmosphere is still very incomplete. For instance, no detailed theoretical

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explanations are available for the observed discrete scales of motion seen as granulation, mesogranulation and supergranulation. Nor are there reliable predictions for differential rotation with both latitude and depth, nor for the magnetic dynamo action as evidenced by global field reversals. Nor is it clear why giant cells, or convective flows with horizontal scales comparable to the total depth of the convection zone, are essentially undetectable at the surface of the Sun. What are required are far more explicit dynamical theories to try to resolve these issues, though all such analyses of turbulent flows are quite formidable tasks. Certainly strides have been made recently through nonlinear studies of incompressible convection in rotating spherical shells (e.g. Gilman, 1978, 1980) and of dynamo action therein (Gilman and Miller, 1981), through linear instability and nonlinear modal analysis of compressible fluids in similar configurations (e.g. Glatzmaier and Gilman, 1981; Marcus, 1980), and by nonlinear simulations of fully compressible convection in planar geometries (Hurlburt *et al.*, 1982). Further, recent dynamical models of granulation have included fairly realistic equations of state and effects of radiative transfer in the optically thin atmosphere (Nelson and Musman, 1977, 1978; Dravins *et al.*, 1981; Nordlund, 1980, 1982). Still, although progress has been made in describing the dynamics of convection in a more detailed (and thus presumably more reliable) manner than that of the mixing-length approach, many of the basic consequences of convection in the Sun have yet to be explained.

We have been developing theoretical descriptions for solar and stellar convection that use anelastic modal equations. These result after making two approximations to the full dynamical equations. First, an anelastic approximation is used to filter out the high frequency acoustic waves that might be present in the compressible motions but which would not normally contribute much to the convective transport. Second, we treat the convection as if it had only a discrete spectrum of horizontal scales. The horizontal structure is expanded in a finite number of horizontal planforms or modes. Such a truncation is used to make the problem tractable, choosing to emphasize accurate representation of the vertical and temporal structure at the expense of the horizontal. This truncated modal approach is motivated by the seemingly cellular character of turbulent convection, and similar methods have worked reasonably well in theoretical descriptions of laboratory convection (Toomre *et al.*, 1977, 1982). The resulting nonlinear anelastic modal equations are capable of describing compressible convection over multiple scale heights and with all the complexities arising from realistic equations of state in a star. The equations have a spatial differential order, $(6n + 3)$, with respect to the vertical coordinate z which depends upon the number of horizontal modes, n, retained in the analysis (Latour *et al.*, 1976, hereafter called as Paper I). Although the differential order is high and their solution difficult, these equations have the distinct advantage that they can be generalized to describe the coupling of convection with rotation, magnetic fields and pulsation in the star.

The anelastic modal procedure has been used to study convection both in A-type stars and in the Sun. Although the stellar applications have been carried out so far mainly with the single-mode equations, the results obtained for convection in the outer envelope of an A star have turned out to be very instructive. The A stars have proved to be a useful framework of developing and refining the modal anelastic procedure, for here, unlike in the Sun, the mean structure is not overly sensitive to the details of the convection. The nonlinear modal studies have revealed that the two convection zones in A stars are dynamically coupled by the convective motions penetrating through the intervening stable material (Toomre *et al.*, 1976, Paper II; Toomre, 1980; Zahn, 1980; Nelson, 1980; Latour *et al.*, 1981, Paper III). The convection associated with cells of large horizontal scale is found to be driven principally by buoyancy forces in the deeper He^{++} unstable zone. It is striking that these motions of supergranular scale are able to penetrate upward all the way to the surface of the star, contrary to mixing-length predictions. Thus the convection is not simply confined to the unstable regions, and for A stars this means that diffusive gravitational separation of elements cannot be occurring in what previously was supposed to be a quiescent region between the H^+ and He^{++} convection zones. Another noteworthy result concerns the nature of the supergranular scale flow in the shallow H^+ zone. Analysis of the buoyancy and pressure work terms reveals that pressure effects dominate in the upper zone. The predominately vertical motions in the convection zone deeper down are turned into strong horizontal shear flows in the upper zone, largely as a result of the strong braking of vertical momentum in this region. This serves to diminish the vertical velocity amplitudes that are actually visible in the atmosphere. The convection of small horizontal scales, like granulation, can experience buoyancy driving in the H^+ zone, while supergranular scales are strongly braked. In the latter, significant pressure fluctuations can serve to change the sense of the density fluctuations, so that a rising fluid element near the top of this unstable zone is heavier than its surroundings and experiences buoyancy braking. Net work is extracted by both scales of convection from this highly unstable zone, though the effects are very different.

The work with A stars led to the suggestion that supergranulation in the Sun may well possess strong horizontal flows in the H^+ zone just below the surface (e.g. Toomre, 1980). Thus a shallow but highly unstable region may be able to effectively prevent large-scale cellular motions from getting through into the atmosphere with any significant portion of their original momentum. The deflection of large-scale flows appears to be a consequence of the rapidly decreasing scale height just below the surface of the star; similar behavior is also seen in modal analysis of convection in polytropes when the effects of stratification are significant (Massaguer and Zahn, 1980; Latour *et al.*, 1982). If such predictions continue to be borne out by modal solutions for solar convection, then this may explain why the observed vertical velocity amplitudes in supergranular flows in the atmosphere are so small. This may also explain why the giant cells of global scale, suggested by the magnetic field patterns, are below the present level of detection for velocities in the atmosphere. These giant cell flows may well be deflected by the shrinking scale height at the depth of the He^+ ionization in the Sun.

The basic problem in dealing with the solar convection zone is that the overall mean structure of the outer envelope is very sensitive to the detailed treatment of the convective motions just below the surface. Since the convection is responsible for almost all the transport, the mean structure is close to adiabatic throughout most of the convection zone. However, which particular adiabat is chosen depends sensitively upon the very rapid transition from radiative to adiabatic conditions just below the surface. Thus the details of the convection at the very top of the zone strongly influence the choice of adiabat, which in turn controls the depth of the convection zone in such envelope models. This means that a single-mode anelastic description of the convection must be supplemented by some representation of the heat flux being carried by other scales of motions. Certainly the longer term goal will be to explicitly include a sufficient number of horizontal modes so that the dominant scales of convection are being computed in a self-consistent manner. However, that poses a formidable task mathematically, for one may expect that at the very least about four modes will be required in the analysis. These four horizontal modes probably need to be of disparate scale, much like granulation, mesogranulation, supergranulation and giant cells.

We sought to minimize our computational difficulties at first by simply using a mean structure for the Sun constructed from mixing-length models to test the gross behavior of the modal convection as the horizontal scale is varied. The use of these highly simplified models, where the feedback between the convective flux and the mean structure is largely severed, has suggested that convection cells with the large horizontal scales of supergranulation are driven mainly by He^{++} and display strong horizontal shear layers just below the surface. Modes of intermediate scale like that of mesogranulation primarily feel the effects of He^{+}, while only cells with the small scale of granulation get much buoyancy driving in the H^{+} zone. We have now advanced to considerably more realistic descriptions in which the nonlinear feedback of the single anelastic mode upon the mean structure is fully implemented. We have accomplished this by introducing the effects of unresolved small scales of convection and turbulence as a diffusion of mean entropy, while treating the dynamics of the large-scale convection cell explicitly by solving the full anelastic equations. Such a procedure has worked out quite well in building preliminary models of supergranulation and giant cells: with the diffusive scale of such an eddy process restricted to be less than 1 Mm, we find that a single large-scale mode can transport nearly the full solar flux over most of the convection zone without developing any noticeable pathologies in the mean stratification. Further, we note that the cellular motions in our solar model extend over multiple density scale heights, much as we anticipated from our work with A stars.

2. Formulating the Problem

In this preliminary study of the hydrodynamics of the solar convection zone, we shall use the anelastic modal equations in their simplest form by retaining only a single mode. The class of solutions investigated has an upwelling flow at the center of the three-dimensional hexagonal convection cell. The computations are relevant only to the largest convective cellular flows in the Sun, with their horizontal scales being comparable to the overall depth of the convection zone. Our notation and formulation of the equations will be identical to that used in Paper II, though we will introduce additional terms to represent the transport of heat and the diffusion of momentum by turbulent motions with scales much smaller than the explicit cellular mode.

A. SINGLE-MODE ANELASTIC APPROACH

We shall here briefly recall some of our notation for the single-mode anelastic representation of the convection, but for the sake of brevity ask the reader to refer to Paper III for a detailed exposition of the equations. Each thermodynamic variable is separated into its horizontal mean, which depends only on the vertical coordinate z and time t, and its fluctuations relative to that mean value. The coordinate z is taken as the depth below the surface of the Sun and thus increases downwards. The fluctuations will be factorized so as to separate their amplitude functions from their specified horizontal planform function $f(x, y)$. The temperature field will thus be represented as

$$T(x, y, z, t) = \overline{T}(z, t)[1 + f(x, y)\Theta(z, t)], \tag{2.1}$$

where $\overline{T}$ is the mean temperature and θ the amplitude function of its relative fluctuations. The density field is similarly described by its mean value $\bar{\rho}$ and the amplitude function Λ of its relative fluctuation, and so too pressure in terms of $\overline{P}$ and Π. Likewise enthalpy, entropy and thermal conductivity, with $\overline{H}$, $\overline{S}$, $\overline{K}$ being their mean values and H, S, K their absolute fluctuation functions. The momentum vector $m_i = \rho v_i$ is divergence free within the anelastic approximation if we ignore the explicit time dependence of $\bar{\rho}$ associated with possible radial pulsations. Thus the momentum vector can be represented as

$$m_i = \left\{a^{-2}\frac{\partial f}{\partial x}DW, \quad a^{-2}\frac{\partial f}{\partial y}DW, \quad fW\right\}, \tag{2.2}$$

assuming that the vertical component of vorticity is negligible and denoting $D \equiv \partial/\partial z$. The horizontal planform fluctuations satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f + a^2 f = 0, \qquad \overline{(f)^2} = 1, \tag{2.3}$$

where the overbar denotes horizontal averaging. The planform functions within a single-mode representation are characterized by their horizontal wavenumber a and their self-interaction constant $C = \frac{1}{2}\overline{(f)^3}$, which is zero for two-dimensional rolls and $6^{-1/2}$ for the three-dimensional cells with a hexagonal planform considered here.

The use of such mean and fluctuating variables yields a system of nonlinear partial differential equations with z and t as independent variables. The single-mode anelastic equations are stated as Equations (2.7) to (2.12) in Paper III, and involve a fluctuating horizontal vorticity equation and a fluctuating heat equation for the particular mode under consideration, plus a mean-momentum equation and a mean-heat equation which are coupled by various nonlinear terms to the fluctuations. The single-mode system of equations is of 3rd order in time derivatives, and of 9th order in spatial derivatives. The nonlinear differential system will be solved by finite differences, the details of which are also provided in Paper III.

B. MEAN MODEL AND BOUNDARY CONDITIONS

Our computational domain in the vertical extends over much of the depth of the solar convection zone, with the lower boundary placed at a depth of about 256 Mm below the solar surface so that the effects of motions in the stable stratification below the convection zone can be resolved. The placement of the upper boundary of the computational domain presents greater difficulties. It would be most appropriate to extend the domain well into the solar chromosphere in order to study the possible upward penetration of these largest scales of convection into the stable atmosphere. However, severe computational difficulties arise even when dealing with the full complexities of the hydrogen ionization zone just below the surface: the pressure scale height varies by a factor of about 20 in going from a depth of 7 Mm (where H is fully ionized) to the photosphere, and the conductivity $\overline{K}$ varies by several orders of magnitude as a consequence of hydrogen ionization. Indeed, the pressure scale height ranges from $H_p \simeq 0.1$ Mm in the photosphere to $H_p \simeq 50$ Mm near the bottom of the convection zone (at a depth of about 200 Mm), thereby changing by a factor of about 500. The numerical difficulties in describing compressible flows over such a range of properties are most formidable, and we preferred in these preliminary calculations to avoid some of them by placing our upper boundary at a depth where hydrogen is already largely (80%) ionized. Thus our computational domain starts at a depth of 5.7 Mm and extends downwards a further 250 Mm, with the scale height H_p varying by a factor of about 25 between the top and bottom of the domain.

We have adopted a fairly standard chemical composition of $X = 0.73$, $Y = 0.25$, and $Z = 0.02$ for our solar model, used the Cox and Stewart (1965) opacity tables, and have imposed the mean temperature and density at our upper boundary at $z = 5.7$ Mm to have the values $\overline{T}_0 = 3.774 \times 10^4$ K and $\overline{\rho}_0 = 1.880 \times 10^{-4}$ g cm^{-3} in keeping with standard calibrated mixing-length envelope models (cf. Gough and Weiss, 1976). At the lower boundary we assert that the energy flux is purely radiative and thus specify the mean temperature gradient there. We require the vertical momentum W and the temperature fluctuation θ to vanish at the upper and lower boundaries, which are otherwise stress-free. Such an imposition of conditions at the upper artificial boundary is certainly arbitrary and we have tried others, but the choice made here has the modest advantage of being the simplest.

C. REPRESENTATION OF SMALL-SCALE TURBULENCE

The single mode of large horizontal scale is likely to be able to transport a significant fraction of the solar flux over only the deeper portions of the convection zone, with smaller cellular scales having roles at shallower depths. Further, a cascade process must be present that transfers energy from the largest cellular scales being described by the modal equations to the smaller scales at which viscous dissipation occurs. We introduced in Paper III a representation for the energy cascade to smaller scales by means of a turbulent viscosity $\bar{\nu}_T$, with the latter a function of the shear being experienced by the modal flow. The turbulent viscosity serves to parameterize sub-grid shearing instabilities and consequent diffusion of vorticity by a succession of small-scale eddies.

We must here also introduce a representation for the turbulent heat transport associated with the much smaller scales of motion. Our imposition of an artificial upper boundary means that transport by the large mode becomes ineffective in a narrow region below this boundary. If radiation alone were required to carry all the flux there, then the temperature gradient that would result would be so steep that the whole structure of the convection zone would be drastically changed, making it much shallower than acceptable. Such a thermal boundary layer can be avoided by allowing the flux to be carried by the much smaller turbulent scales that must be present there, for the mean stratification is still highly unstable to convection and the pressure scale height is small. We therefore incorporate a turbulent transport term into the mean heat equation that serves to diffuse the mean entropy field should its gradient attempt to become very steep. Such turbulent heat transport is analogous to what would result from a local mixing-length approximation, though by limiting the length scale of mixing to be less than about 1 Mm, we can confine these processes to the vicinity of the upper boundary.

The turbulent heat transport by small scales can be accommodated by introducing the additional term

$$-D(\bar{\rho}\bar{T}\chi_T D\bar{S}) \tag{2.4}$$

to the left-hand side of the mean-heat equation (2.12) in Paper III, with this term representing the divergence of the turbulent heat flux. Here χ_T is a turbulent diffusivity $u_T l$, with l the mixing length linked to the local pressure scale height but possessing an imposed upper bound. The effective turbulent velocity u_T is estimated, as in the mixing-length approach, from the kinetic energy acquired by a small adiabatic parcel of fluid traveling a distance l under the acceleration of buoyancy, thus obtaining

$$\chi_T = l^2\left[-g\left(\frac{\partial \bar{S}}{\partial \ln \bar{\rho}}\right)^{-1} D\bar{S}\right]^{1/2} . \tag{2.5}$$

The modified mean-heat equations (2.12) from Paper III may then be restated in conservative form to clearly identify the sources and sinks, with the equation becoming

$$\bar{\rho}\bar{T}\frac{\partial \bar{S}}{\partial t} + D\left[WH - W\frac{P}{\bar{\rho}} - \bar{K}D\bar{T} - \bar{\rho}\bar{T}\chi_T D\bar{S}\right] = gW\Lambda - WPD\left(\frac{1}{\bar{\rho}}\right) + \\ + \frac{(\bar{\nu}+\bar{\nu}_T)}{\bar{\rho}}\left[a^{-2}(\phi + 2a^2 W)^2 + 4(DW)^2\right], \tag{2.6}$$

with all the notation the same as that introduced in Paper III.

Similarly, we modify the fluctuating heat equation (2.9) in Paper III by introducing a term to its right-hand side which represents turbulent diffusion acting on the fluctuating entropy, with the expression being

$$+\bar{\rho}\bar{T}\chi_T(D^2 - a^2)S\,. \tag{2.7}$$

This term is just the fluctuating analog of expression (2.4) to leading order.

We will find that the turbulent heat transport (2.4) is of importance only near our artificially imposed upper boundary and negligible over most of the computational domain. By adjusting the value of l in that term we can adjust the depth of the unstable zone so that it has a reasonable value as judged by properly evolved full solar models (e.g., Gough and Weiss, 1976). Thus the term is of key importance in selecting the adiabat for the convection zone, but does not otherwise control the dynamics of the large-scale mode which will serve to keep the stratification in the rest of the zone adiabatic.

3. Results and Discussions

We have obtained steady solutions for a number of single-mode convective flows with a hexagonal planform, and in all these there has been a full feedback of the nonlinear convection upon the mean stratification. We will concentrate here on discussing the properties of one of these solutions of large horizontal scale, for it will serve to show what is characteristic of all these anelastic modal solutions for solar convection. The steady states are attained by time evolving the solutions from various different initial conditions, sometimes from an initial mean structure based on the usual mixing-length description and with the modal perturbation fields of very small amplitude, but more often from other evolved modal solutions nearby in parameter space.

A. Flow momenta and thermodynamic fluctuations

Figure 1a presents the variation of the vertical and horizontal momentum amplitude functions, W and U, with depth z over our computational domain, with the solution displayed in equal increments of $\log\bar{P}$ so that the structures near the top, where the scale height gets small, are readily visible. The actual numerical solutions were constructed on a highly stretched grid so adjusted that all boundary-layer features are adequately resolved, with typically 300 mesh points being used for each variable in the vertical. The horizontal wavenumber of this particular solution is $a = 15$ when based on the 250 Mm depth of our computational domain, thus yielding a horizontal wavelength for this hexagonal cell of 140 Mm. The vertical momentum W in this solution peaks at $z \approx 60$ Mm and decreases sharply as the bottom of the convection zone is reached at $z \approx 200$ Mm. The vertical momentum also decreases steadily as the upper boundary is approached. Although this is partly the result of our boundary condition there, the decrease is primarily attributable to buoyancy and pressure forces, triggered by the shrinking scale height, that deflect the upward directed momentum at cell center into a strong horizontal shearing flow. The plot of horizontal momentum U in Figure 1a

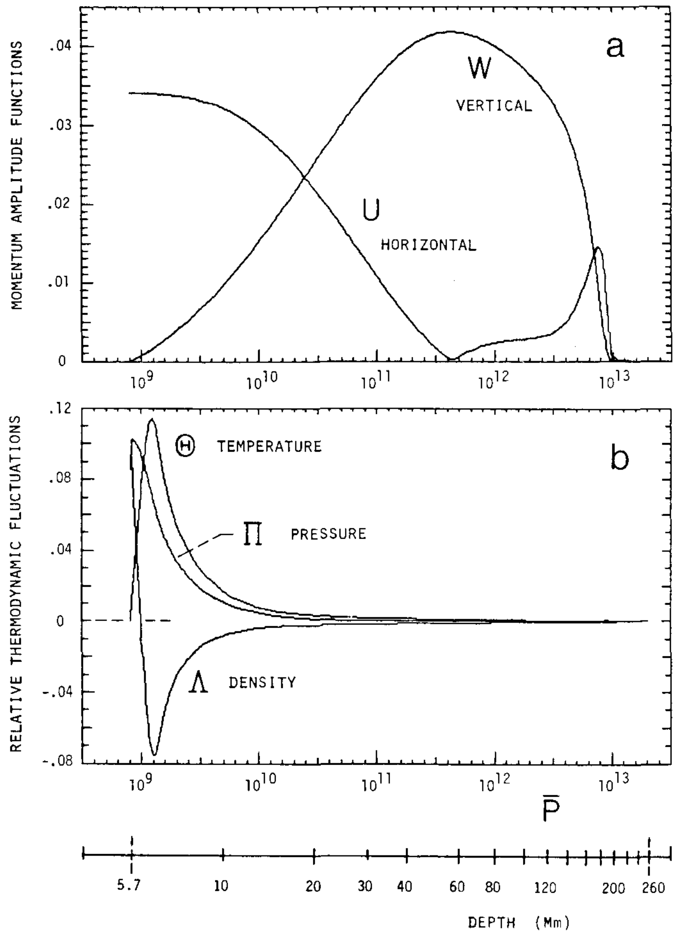

Fig. 1. Variation with depth (or $\log \bar{P}$) in the solar envelope of a typical single-mode nonlinear convection solution. The horizontal wavenumber for this hexagonal cell of large horizontal scale is $a = 14.0$, corresponding to a size of 140 Mm, and the flow is predominantly upward at cell center. Shown in (a) are the amplitude functions of the vertical momentum W and horizontal momentum U in dimensional units of ($\mathrm{g\,cm^{-2}\,s^{-1}}$), and in (b) the amplitude functions of the relative thermodynamic fluctuations for temperature, θ, density, Λ, and pressure, Π.

shows that this component is especially prominent from the upper boundary to a depth of about 30 Mm, with another shearing flow also present near the bottom of the convection zone. Thus this large-scale cellular flow occupies the full extent of the unstable domain here, unlike what might have been expected from mixing-length predictions; there is little to confine the motions to a mere scale height or so. Certainly some of the variations seen in the solution near the top of the computational domain will be artifacts arising from the placement of our arbitrary upper boundary, though the striking tendency for the cellular convection to extend over many vertical scale heights appears to be a robust result.

The amplitude functions for the relative thermodynamic fluctuations of temperature, density and pressure are shown as θ, Λ, and Π in Figure 1b. They are of the same order throughout the zone and peak near the middle of the He^+ ionization region at a depth of about 7 Mm (see also Figure 2b). The amplitudes of these relative thermodynamic fluctuations are small: about 0.12 for the maximum of θ, but only of order 0.01 or less over most of the convection zone. The linearized relation $\Lambda = a\Pi - b\theta$ that links the relative thermodynamic fluctuations within the anelastic approximation is therefore justified. (The positive coefficients a, b depend upon $\overline{T}$ and $\overline{\rho}$ and the equation of state, and would just be unity if the gas were perfect.) When the pressure fluctuation Π is negligible, as within the Boussinesq approximation, then the density fluctuation Λ is just the mirror image of the temperature fluctuation θ. However when the stratification effects are significant, as in the upper portions of the solar convection zone, Π may be of the same order as θ and then Λ responds to both fluctuations. This may lead to a rising fluid element actually experiencing buoyancy braking near the top of an unstable zone, largely because Π can even change the sign of Λ. Although there is a hint of this near the upper boundary in Figure 1a, we do not place much credence in this because of our arbitrary boundary conditions there. However, such buoyancy braking arising from the effect of Π on Λ has a major role in the nonlinear modal solutions for convection studied in polytropes by Massaguer and Zahn (1980) and in the A-star convective envelopes of Paper III. Thus buoyancy braking caused by effects of compressibility where the scale height is small may also turn out to be an important process in the Sun, though we would have to extend the computational zone into the atmosphere to be sure of this.

B. What drives the motions

In order to understand the dynamics of such large-scale convection, it is useful to examine the rates of working by buoyancy and pressure forces, for these will largely control the power integrals for these flows. The expression $E_B = gW\Lambda$ represents work done by the mean pressure, or thus by buoyancy, while that by fluctuating pressure is $E_P = W\Pi\overline{P}D(1/\overline{\rho})$. In addition, a term E_V would represent the work done by viscous forces, but will not be shown explicitly. We should note that E_P represents the work done to modify the volume of a fluid parcel as it moves vertically, with this term vanishing in the abscence of density stratification, either in Boussinesq convection or in the comparable assumptions made in mixing-length treatments. The notation for E_P and E_B is such that positive terms increase the local kinetic energy in the flow. The net work produced by fluctuating pressure, or thus the integral in the vertical of E_P, plus that done by buoyancy, is converted into heat by viscous dissipation. These processes are however usually not in local balance, and thus flux terms serve to redistribute the energy vertically across the layer.

Figure 2a shows the variation with depth of E_B and E_P in the representative solution. We note that rates of working by buoyancy and pressure possess a relatively narrow peak at a depth of about 7 Mm in the middle of the He^+ ionization region (see Figure 2b). The buoyancy term E_B is consistently of greater amplitude than that of the pressure term E_P, and near the upper boundary a small region of buoyancy braking is

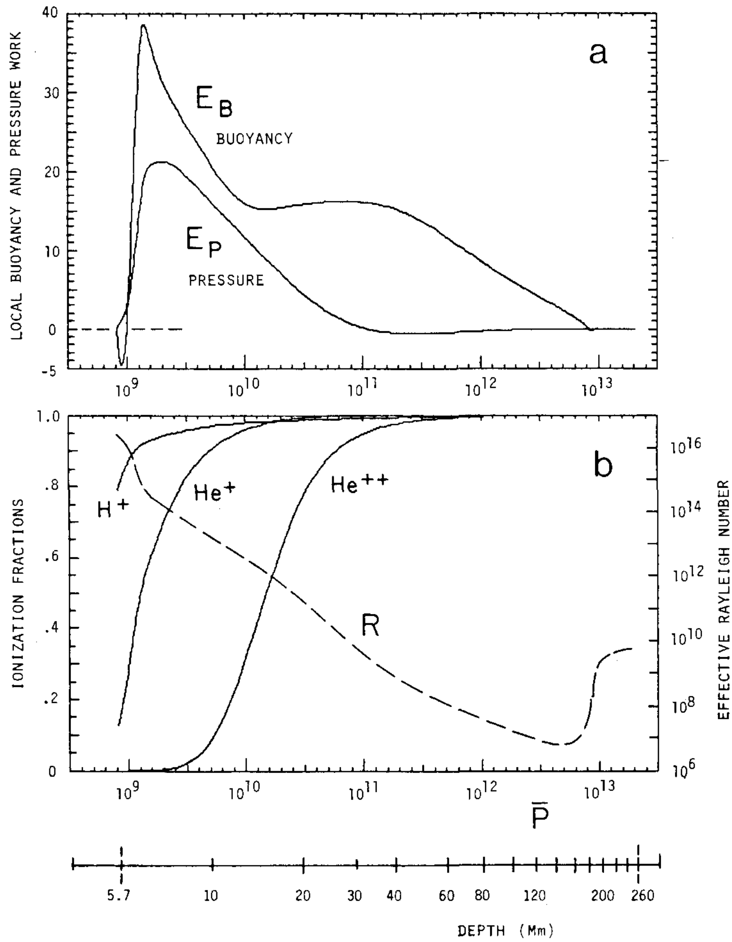

Fig. 2. Further properties of the nonlinear convection in the large-scale hexagon solution shown in Figure 1. (a) Variation with depth (or $\log\bar{P}$) of the local work done by the buoyancy (E_B) and by the fluctuating pressure (E_P). (b) Ionization fractions for H^+, He^+, and He^{++} with depth in this solar envelope, and the variation of the local effective Rayleigh number R. E_B and E_P are shown in the same arbitrary units.

evident. Further, the E_B profile in Figure 2a displays a plateau with a faint secondary maximum in the region where the second helium ionization is more than half completed, as seen in the curve of the ionization fraction of He^{++} in Figure 2b. Such a profile for E_B suggests that the buoyancy work used to drive the convective motions is largely concentrated in the ionization regions, with both the He^+ and He^{++} processes having a role in such driving. We should emphasize that the rate of working by the pressure terms through E_P has a significant effect on the flow, though the overall integral of E_B dominates. Closer inspection of the individual components that go into E_P reveal that pressure forces at depths shallower than about 20 Mm serve to decelerate the upward directed flow at the center of the cell and drive the consequent strong horizontal motions. The overall contributions by E_P in these regions is positive, meaning that net work is extracted by pressure terms from the unstable stratification, and so too by the buoyancy terms, but the pressure field has the key role in the initial deflection of the flow.

Figure 2b presents the degree of ionization of H and He with depth, and thus serves to quantify the locations of the ionization regions. Shown also is the variation of local Rayleigh number, based on the local degree of superadiabaticity, pressure scale height and enhanced turbulent viscosity. This Rayleigh number R ranges from about 10^{16} at the top to about 10^7 near the bottom of the convection zone, indicating that we have increased the viscosity typically by a factor of about 10^{12} over that of the natural molecular viscosity of the solar plasma. Such an enhancement of the viscosity, primarily by the turbulent viscosity $\bar{\nu}_T$, has made the numerical calculations tractable, and we have not found the resulting solutions to be overly sensitive to the functional forms used for $\bar{\nu}_T$.

C. The energy transports

Figure 3 displays how the various fluxes of energy in such modal convection vary with depth. Shown are the enthalpy or convective heat flux F_C, the radiative flux F_R, the viscous shear flux E_V, and the turbulent eddy heat flux F_E. These fluxes have been defined in Paper III, but for F_E, the divergence of which is introduced in Equation (2.4); all the fluxes have been normalized by the total flux. The conservation of energy in a steady state requires that the sum of these fluxes, together with the kinetic energy flux F_K not shown here, must be constant throughout the envelope. An interesting result is that F_C is dominant over most of the convection zone and is close to unity. Thus the large-scale convection mode is very efficient in transporting most of the solar flux. Near

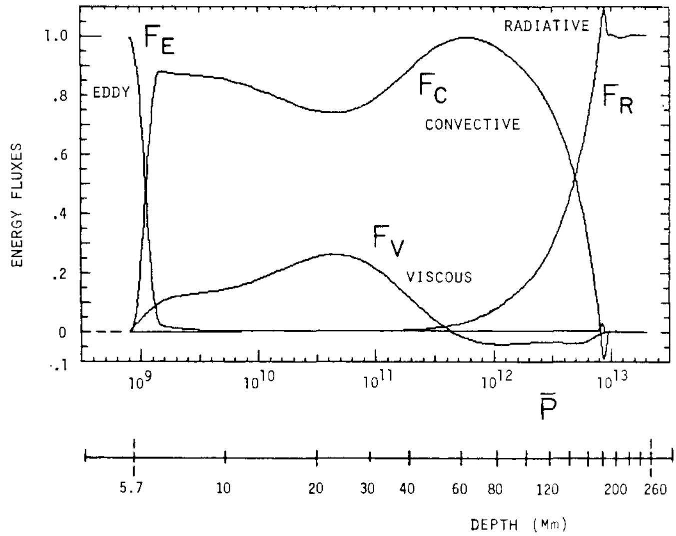

Fig. 3. Variation with depth (or $\log\bar{P}$) in the hexagon solution shown in Figure 1 of the enthalpy or convective heat flux F_C, the radiative flux F_R, the viscous shear flux F_V, and the small-scale turbulent eddy flux F_E. All fluxes have been scaled by the upward total flux.

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the bottom of the zone, F_C is naturally supplanted by the radiative flux F_R, and even becomes slightly negative in a narrow overshooting region, forcing F_R to be larger than unity. Near our artificial upper boundary, the heat transport F_E by small-scale eddies takes over, thereby preventing the formation of a thermal boundary layer there since we force F_C to vanish. The viscous shear flux F_V represents a net kinetic energy transport by viscous stresses arising from our enhanced turbulent viscosity. This flux is not negligible over a significant portion of the zone, with F_C and F_V together accounting for most of the total flux over a large depth range. Figure 3 thus serves to emphasize that even a single large-scale mode, akin to possibly a giant cell, can make the mean stratification nearly adiabatic over most of the vertical extent of the convection zone.

4. Conclusions

The single-mode nonlinear solutions that we have studied all appear to suggest that the largest scales of convection in the Sun are prevented from getting through into the atmosphere with any significant fraction of their primary momentum. The shrinking scale height as the surface of the Sun is approached from below can produce major compressibility effects in the dynamics of large-scale convection that serve to deflect the cellular motions laterally. Thus fairly strong horizontal shearing flows may be present below the surface for scales of convection comparable to giant cells, and even some signature of them may extend into the atmosphere due to viscous stresses, whereas the vertical component of motion getting through into the photosphere is feeble at best. Certainly the preliminary anelastic modal solutions that we have considered so far only suggest such behavior, for we have imposed an artificial upper boundary in order to simplify the computational tasks, thereby ignoring much of what goes on explicitly in the hydrogen ionization zone. Also, our results are influenced by the enhanced turbulent viscosity that we have introduced to effectively reduce the local Rayleigh number and make the computations tractable. In particular, the vertical and horizontal momentum amplitude functions shown in Figure 1a translate into flow fields in which the maximum vertical velocity of about 10 m s^{-1} is attained at a depth of about 12 Mm and a peak horizontal velocity of about 100 m s^{-1} occurs at a depth of 7 Mm. These values can be changed by a factor of about 10 by changing the turbulent viscosity $\bar{\nu}_T$ by a factor of 10^5, thereby emphasizing that we are presently uncertain about the flow amplitudes that will be attained just below the surface. These modifications in $\bar{\nu}_T$ however appear to have little impact on the overall structure of the convective mode over the bulk of the zone where its convective heat transport is sufficient to account for most of the solar flux.

Future improvements to these preliminary anelastic models of solar convection will require a better representation of the small-scale turbulence and the inclusion of additional horizontal modes in the analysis. These modes will need to encompass horizontal scales of convection comparable to granulation, mesogranulation and supergranulation, in addition to those of giant cells, and it is not readily apparent just how sensitive the results will be to the number and choice of modes in such a representation of compressible turbulence. Clearly the analysis of the convective flows must also be extended into the atmosphere and radiative transfer effects there taken into account. We recognize that such steps are necessary before we can have any sense of comfort about the predictions of anelastic convection theory applied to the Sun, and we are thus engaged in implementing such improvements. What these preliminary single-mode solutions presently do provide is a sequence of nonlinear numerical experiments to help develop our intuition about highly compressible flows. Certainly many of the results are at variance with what is assumed in the effectively Boussinesq mixing-length approaches, for the convection here readily extends over multiple scale heights and is thereby decidedly nonlocal in character. The preliminary results to date suggest that prominent horizontal flows may be associated with the largest scales of convection at reasonably shallow depths below the surface. These may be capable of being sampled by the use of the five-minute oscillations as probes of such convective structures.

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References

Cox, A. N. and Stewart, J. N.: 1965, *Astrophys. J. Suppl.* **11**, 94.
Dravins, D., Lindegren, L., and Nordlund, A.: 1981, *Astron. Astrophys.* **96**, 345.
Gilman, P. A.: 1978, *Geophys. Astrophys. Fluid Dynamics* **11**, 157.
Gilman, P. A.: 1980, *Highlights Astron.* **5**, 91.
Gilman, P. A. and Miller, J.: 1981, *Astrophys. J. Suppl.* **46**, 211.
Glatzmaier, G. and Gilman, P. A.: 1981, *Astrophys. J. Suppl.* **45**, 351.
Gough, D. O. and Weiss, N. O.: 1976, *Monthly Notices Roy. Astron. Soc.* **176**, 589.
Hurlburt, N. E., Toomre, J., Massaguer, J. M., and Graham, E.: 1982, *Astron. Astrophys.*, submitted.
Latour, J., Spiegel, E. A., Toomre, J., and Zahn, J.-P.: 1976, *Astrophys. J.* **207**, 233 (Paper I).
Latour, J., Toomre, J., and Zahn, J.-P.: 1981, *Astrophys. J.* **248**, 1081 (Paper III).
Latour, J., Massaguer, J. M., Toomre, J., and Zahn, J.-P.: 1982, *Astron. Astrophys.*, submitted.
Marcus, P. S.: 1980, *Astrophys. J.* **240**, 203.
Massaguer, J. M. and Zahn, J.-P.: 1980, *Astron. Astrophys.* **87**, 315.
Nelson, G. D.: 1980, *Astrophys. J.* **238**, 659.
Nelson, G. D. and Musman, S.: 1977, *Astrophys. J.* **214**, 912.
Nelson, G. D. and Musman, S.: 1978, *Astrophys. J. Letters* **222**, L69.
Nordlund, A.: 1980, in D. F. Gray and J. L. Linsky (eds.), 'Stellar Turbulence', *IAU Colloq.* **51**, 213.
Nordlund, A.: 1982, *Astron. Astrophys.*, submitted.
Toomre, J.: 1980, *Highlights Astron.* **5**, 571.
Toomre, J., Zahn, J.-P., Latour, J., and Spiegel, E. A.: 1976, *Astrophys. J.* **207**, 545.
Toomre, J., Gough, D. O., and Spiegel, E. A.: 1977, *J. Fluid Mech.* **79**, 1.
Toomre, J., Gough, D. O., and Spiegel, E. A.: 1982, *J. Fluid Mech.*, in press.
Zahn, J.-P.: 1980, in D. F. Gray and J. L. Linsky (eds.), 'Stellar Turbulence', *IAU Colloq.* **51**, 1.