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A SIMPLE PROOF OF A THEOREM ON REDUCED RINGS

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We give a simple proof of a theorem by Andrunakievič and Rjabuhin which states that a reduced ring is a subdirect product of entire rings. Our proof makes no use of m-systems and is in some sense similar to the proof of the corresponding theorem in the commutative case due to Krull.

A reduced ring is a ring without non-zero nilpotent elements. It is well-known that if a reduced ring is commutative, then it is a subdirect product of integral domains [2]. This result has been generalized to arbitrary reduced rings [1]. The proof in the general case is somewhat complicated. We present a simple proposition which leads to a simple proof of the general case.

If Q is an ideal of a ring R and R/Q is reduced, we say that Q is a reduced ideal. Observe that if Q is a reduced ideal of R and $a, b \in R$ satisfy $ab \in Q$, then $ba \in Q$. Indeed, $ab \in Q$ implies $(ba)^2 \in Q$, so $ba \in Q$ since Q is reduced.

PROPOSITION. *Let Q be a reduced ideal of a ring R. If A is the left* (*or right*) *annihilator* mod Q *of any subset $S \subseteq R$, then A is a reduced ideal.*

Proof. We may assume that S contains only one element s, since the intersection of reduced ideals is reduced. So let $A = \{r \in R \mid rs \in Q\}$ and we prove that A is a reduced ideal.

A is clearly a left ideal, so to prove that A is an ideal let $r \in A$, $x \in R$ and we show that $rx \in A$. Q is reduced and $rs \in Q$, so $sr \in Q$ and $srx \in Q$. This implies $rxs \in Q$, hence $rx \in A$.

To prove that A is reduced, it suffices to show that if $r^2 \in A$ then $r \in A$. So let $r^2 s \in Q$. It follows that $rsr \in Q$ and $(rs)^2 \in Q$. This implies $rs \in Q$, hence $r \in A$.

THEOREM (Andrunakievič and Rjabuhin). *If R is a reduced ring, then R is a subdirect product of entire rings* (= *rings without non-zero zero divisors*).

Proof. It suffices to prove that given $0 \neq x \in R$, there exists and ideal Q excluding x, such that R/Q is entire. Since the zero ideal is reduced, we can apply Zorn's lemma on the set of reduced ideals excluding x, and we obtain a maximal reduced ideal Q excluding x. We claim that R/Q is entire. Assume on the contrary that $ab \in Q$ and $a \notin Q$, $b \notin Q$. Let A be the left annihilator mod Q of b and let B be the right annihilator mod Q of A. By the proposition, A and B are reduced ideals. It is clear that $A \supseteq Q$, $B \supseteq Q$ and $AB \subseteq Q$. Moreover

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$A \neq Q$ since $a \in A$, and $B \neq Q$ since $b \in B$. It follows that $x \in A$ and $x \in B$, so $x^2 \in AB \subseteq Q$. Hence $x \in Q$ since Q is reduced, a contradiction.

References

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