Can. J. Math. Vol. 47 (3), 1995 pp. 474–499

GENERALIZED STIRLING NUMBERS, CONVOLUTION FORMULAE AND p, q-ANALOGUES

ANNE DE MÉDICIS AND PIERRE LEROUX

ABSTRACT. In this paper, we study two generalizations of the Stirling numbers of the first and second kinds, inspired from their combinatorial interpretation in terms of 0–1 tableaux. They are the $\mathfrak{A}$-*Stirling numbers* and the *partial Stirling numbers*. In particular, we give a q and a p, q-analogue of convolution formulae for Stirling numbers of the second kind, due to Chen and Verde-Star, and we extend these formulae to Stirling numbers of the first kind. Included in this study are the *a, d-progressive Stirling numbers*, corresponding to 0–1 tableaux with column lengths from an arithmetic progression $\{a + id\}_{i \geq 0}$.

RÉSUMÉ. Dans cet article, nous étudions deux généralisations des nombres de Stirling de première et deuxième espèces, inspirées par leur interprétation combinatoire en termes de tableaux 0–1. Il s'agit des $\mathfrak{A}$-*nombres de Stirling* et des *nombres de Stirling partiels*. Nous donnons en particulier des q et p, q-analogues de formules de convolution des nombres de Stirling de deuxième espèce, dues à Chen et Verde-Star, et nous étendons ces formules aux nombres de Stirling de première espèce. Les *nombres de Stirling a, d-progressifs*, correspondant aux tableaux 0–1 dont les longueurs des colonnes font partie d'une progression arithmétique $\{a+id\}_{i \geq 0}$, sont également inclus dans cette étude.

1. **Introduction.** A *0–1 tableau* is a pair $\varphi = (\lambda, f)$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of an integer $m = |\lambda|$, and $f = (f_{ij})_{1 \leq j \leq \lambda_i}$ is a "filling" of the cells of the corresponding Ferrers diagram of shape λ with 0's and 1's, such that there is exactly one 1 in each column.

For example, Figure 1 represents the 0–1 tableau $\varphi = (\lambda, f)$, where $\lambda = (8, 7, 6, 2)$ and $f_{15} = f_{17} = f_{18} = f_{21} = f_{23} = f_{32} = f_{34} = f_{36} = 1, f_{ij} = 0$ elsewhere $(1 \leq j \leq \lambda_i)$.

0	0	0	0	1	0	1	1
1	0	1	0	0	0	0	
0	1	0	1	0	1		
0	0						

FIGURE 1: 0–1 TABLEAU φ

The first author was supported by NSERC (Canada) and FCAR (Québec) Scholarships.
The second author was supported by NSERC (Canada) and FCAR (Québec).
Received by the editors June 14, 1993.
AMS subject classification: Primary: 05A15; secondary: 11B65, 11B73.

We define two statistics on these objects: first, the *inversion number* of φ, denoted by $\mathrm{inv}(\varphi)$, which is equal to the number of 0's below a 1 in φ; and the *non-inversion number* of φ, denoted by $\mathrm{nin}(\varphi)$, which is equal to the number of 0's above a 1 in φ. For instance, for the 0–1 tableau in Figure 1, we compute $\mathrm{inv}(\varphi) = 7$ and $\mathrm{nin}(\varphi) = 8$.

We then define the *p,q-Stirling numbers of first and second kinds* to be the polynomials

$$c_{p,q}[n,k] := \sum_{\varphi \in Td(n-1,n-k)} p^{\mathrm{nin}(\varphi)} q^{\mathrm{inv}(\varphi)}, \tag{1.1}$$

and

$$S_{p,q}[n,k] := \sum_{\varphi \in T(k,n-k)} p^{\mathrm{nin}(\varphi)} q^{\mathrm{inv}(\varphi)}, \tag{1.2}$$

respectively, where $T(h,r)$ denotes the set of all 0–1 tableaux (λ,f) such that the number of non-zero parts of λ is at most equal to h, and the first part of λ is exactly equal to r, for $h \geq 0$, $r \geq 0$, and $Td(h,r)$ denotes the subset of $T(h,r)$ containing all 0–1 tableaux for which the conjugate partition λ' of λ has distinct parts. In other words, a 0–1 tableau φ in $T(h,r)$ has exactly r colums of (non-zero) lengths less than or equal to h. Furthermore, if the lengths of the columns are all distinct, then $\varphi \in Td(h,r)$.

The classical Stirling numbers, $c(n,k)$ and $S(n,k)$, and their q-analogues, $c_q[n,k]$ and $S_q[n,k]$, are obtained by setting

$$c(n,k) = c_{1,1}[n,k], \quad c_q[n,k] = c_{q,1}[n,k], \tag{1.3}$$

$$S(n,k) = S_{1,1}[n,k], \quad S_q[n,k] = S_{q,1}[n,k]. \tag{1.4}$$

The combinatorial study of 0–1 tableaux has led to many interesting developments in the theory of q-analogues of classical Stirling numbers. In particular, these tableaux were used to establish a conjecture of L. Butler on q-logarithmic concavity of q-Stirling numbers (see Butler [Bu] and Leroux [Le]). Moreover, we presented in a previous paper (de Médicis and Leroux [dMLe]) a systematic study of q and p,q-analogues of identities involving Stirling numbers of both kinds, using algebraic and combinatorial methods, based on the combinatorics of 0–1 tableaux. We obtained a number of remarkable identities.

The object of the present paper is to answer two questions that were asked following that work. The first one is to give a q and p,q-analogue of W. Y. Chen's convolution formula, recently published (see [Ch])

$$S(m+n,k) = \sum_{i+j \geq k} \binom{m}{j} i^{m-j} S(n,i) S(j,k-i), \tag{1.5}$$

and to also find a similar formula for Stirling numbers of the first kind. The second question that was raised is to determine which properties remain valid when some constraints are imposed on the lengths of the columns in 0–1 tableaux, such as requiring lengths to be part of an arithmetic or a geometric progression.

In view of these problems, we consider two natural generalizations of Stirling numbers in the context of 0–1 tableaux, the $\mathfrak{A}$*-Stirling numbers* and the *partial Stirling numbers*.

The $\mathfrak{A}$-Stirling numbers are obtained by restricting the possible choices of lengths of columns and also by replacing the p,q-counting of fillings of columns by some weight on the columns. More precisely, let $\mathfrak{A} = (\mathcal{A}, w)$, where $\mathcal{A} = (a_i)_{i\geq 0}$ denotes a strictly increasing sequence of non-negative integers (the column lengths allowed), and $w\colon \mathbb{N} \longrightarrow K$ denotes a function from $\mathbb{N}$ to a ring K (column weights according to length). We define an $\mathcal{A}$-tableau to be a list ϕ of columns c of a Ferrers diagram of a partition λ (by decreasing order of length) such that the lengths $|c|$ are part of the sequence $\mathcal{A}$, and we set

$$w_{\mathfrak{A}}(\phi) = \prod_{c\in\phi} w(|c|). \tag{1.6}$$

Note that ϕ might contain a finite number of columns of length zero if 0 is part of the sequence $\mathcal{A}$ and if $w(0) \neq 0$. We then define the $\mathfrak{A}$*-Stirling numbers of the first kind (without sign) and of the second kind* to be respectively

$$c^{\mathfrak{A}}(n,k) := \sum_{\phi\in Td^{\mathcal{A}}(n-1,n-k)} w_{\mathfrak{A}}(\phi), \tag{1.7}$$

and

$$S^{\mathfrak{A}}(n,k) := \sum_{\phi\in T^{\mathcal{A}}(k,n-k)} w_{\mathfrak{A}}(\phi), \tag{1.8}$$

where $T^{\mathcal{A}}(h,r)$ denotes the set of $\mathcal{A}$-tableaux with exactly r columns whose lengths are in the set $\{a_0, a_1, \ldots, a_h\}$, and $Td^{\mathcal{A}}(h,r)$ denotes the subset of $T^{\mathcal{A}}(h,r)$ containing all $\mathcal{A}$-tableaux with columns of distinct lengths.

As for the p,q-Stirling numbers, by considering the maximum possible length of columns of $\mathcal{A}$-tableaux and their format, we get the following recurrences, where we set $w_i = w(a_i)$:

$$c^{\mathfrak{A}}(n,k) = c^{\mathfrak{A}}(n-1,k-1) + w_{n-1}c^{\mathfrak{A}}(n-1,k), \tag{1.9}$$

and

$$S^{\mathfrak{A}}(n,k) = S^{\mathfrak{A}}(n-1,k-1) + w_k S^{\mathfrak{A}}(n-1,k), \tag{1.10}$$

with initial boundary conditions $c^{\mathfrak{A}}(0,k) = \delta_{0,k} = S^{\mathfrak{A}}(0,k)$, $c^{\mathfrak{A}}(n,0) = w_0 w_1 \cdots w_{n-1}$ and $S^{\mathfrak{A}}(n,0) = w_0^n$.

Similar generalizations of Stirling numbers can be found in the literature. We are thus extending, from a combinatorial point of view, previous works such as those of L. Comtet [Co], M. Koutras [Ko], B. Voigt [Vo] and L. Verde-Star [VS].

Many classical families of numbers or polynomials arise as particular cases of $\mathfrak{A}$-Stirling numbers. Table 1 gives an outlook of some of the families that occur in combinatorics. In Section 2, we display the main properties shared by all of these families, as well as some connections that might occur between two related families, such as when a weight function w^* is obtained by multiplying another weight function w by a constant, $w^*(x) = a \cdot w(x)$, or is obtained by adding a constant, $w^*(x) = a + w(x)$, or still yet when the length sequence is obtained by translation $\delta^m \mathcal{A}$ of a given sequence $\mathcal{A}$. It is in this context that the convolution formulae extending Chen's formula (1.5) will be stated in the greatest generality (see Theorem 2.6, and also [VS], Proposition 5.4 and 5.5).

	w_i	$S^{\mathfrak{A}}(n,k)$	References	$c^{\mathfrak{A}}(n,k)$
1	1	$\binom{n}{k}$	binomial coefficients	$\binom{n}{k}$
2	i	$S(n,k)$	Stirling numbers	$c(n,k)$
3	i^2	$T(2n,2k)$	[Ri] (Riordan) pp. 212–249	$(-1)^{n-k}t(2n,2k)$
4	$(2i)^2$	$S^{\mathrm{imp}}(2n,2k)$	partitions with odd blocks [Co]	
5	$(2i+1)^2$	$S^{\mathrm{imp}}(2n+1,2k+1)$	partitions with odd blocks [Co]	
6	q^i	$\begin{bmatrix} n \\ k \end{bmatrix}_q$	q-binomial coefficients	$q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$
7	q^{i+1}	$q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q$	affine subspaces	$q^{\binom{n-k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$
8	$[i]_{p,q}$	$S_{p,q}[n,k]$	p,q-Stirling numbers	$c_{p,q}[n,k]$
9	$q^i - 1$	$(q-1)^{n-k} S_q[n,k]$	q-tableaux with no null columns, Section 2	$(q-1)^{n-k} c_q[n,k]$
10	$a+i$	$\sum_m \binom{n}{m} a^{n-m} S(m,k)$	Section 4	$\sum_m \binom{m}{k} a^{m-k} c(n,m)$
11	$-a+i$	$S_{(a)}(n,k)$	non-central Stirling numbers [Ko], Section 2	$c_{(a)}(n,k)$
12	$1+i\alpha$	$W_G(n,k)$	Whitney numbers of Dowling lattices [Do]	$(-1)^{n-k} w_G(n,k)$
13	$[a+id]_{p,q}$	$S^{a,d}_{p,q}[n,k]$	a,d-progressive Stirling numbers, Section 4	$c^{a,d}_{p,q}[n,k]$

TABLE 1

The second generalization, *partial Stirling numbers*, is addressed in Section 3. It is obtained by considering *weak 0–1 tableaux, i.e.* 0–1 tableaux that are weak in the sense that some columns may contain only 0's, contributing to the non-inversion statistic. They correspond, in the traditional combinatorial interpretations, to partial partitions and to permutations with some marked cycles. The partial p,q-Stirling numbers $\mathcal{P}c_{p,q}[n,m,k]$ and $\mathcal{P}S_{p,q}[n,m,k]$ are defined by p,q-counting weak 0–1 tableaux with a fixed number of columns filled with 0's. When the number of columns filled with 0's is not fixed, the corresponding polynomial families, $\mathcal{P}c_{p,q}[n,k] = \sum_m \mathcal{P}c_{p,q}[n,m,k]$ and $\mathcal{P}S_{p,q}[n,k] =$

https://doi.org/10.4153/CJM-1995-027-x Published online by Cambridge University Press

$\sum_m \mathcal{P}S_{p,q}[n,m,k]$ form a particular case of $\mathfrak{A}$-Stirling numbers, with $a_i = i$ and $w(i) = p^i + [i]_{p,q}$.

In Section 3, we extend to weak 0–1 tableaux the natural bijections between 0–1 tableaux and partitions or permutations (see de Médicis and Leroux [dMLe]). We then present a study of identities involving partial p,q-Stirling numbers. The main point in this section is the resolution of the first problem mentioned above, finding p,q-analogues and extensions of the convolution formula (1.5) (see Proposition 3.10). The specialization $p = 1$ leads to the following convolution formulae for q-Stirling numbers:

$$(1.11)\qquad c_q[m+n,k] = \sum_{i+j\geq k}\left\{\binom{m+k-i-j}{k-i} q^{n(i+j-k)}[n]_q^{m-j} c_q[n,i]c_q[m,m+k-i-j]\right\},$$

$$(1.12)\qquad S_q[m+n,k] = \sum_{i+j\geq k}\binom{m}{j} q^{i(i+j-k)}[i]_q^{m-j} S_q[n,i]S_q[j,k-i];$$

$$(1.13)\qquad c_q[n+1,k+l+1] = \sum_{i=0}^{n}\sum_{j=0}^{n-l-i}\left\{\binom{l+j}{j} q^{(i+1)(n-l-i-j)}[i+1]_q^j c_q[i,k]c_q[n-i,l+j]\right\},$$

and

$$(1.14)\qquad S_q[n+1,k+l+1] = \sum_{i=0}^{n}\sum_{j=0}^{n-l-i}\left\{\binom{n-i}{j} q^{(k+1)(n-l-i-j)}[k+1]_q^j S_q[i,k]S_q[n-i-j,l]\right\}.$$

The model of weak 0–1 tableaux also appears in the analysis of *a,d-progressive p,q-Stirling numbers* $c_{p,q}^{a,d}[n,k]$ and $S_{p,q}^{a,d}[n,k]$, introduced in the last section. They are particular cases of $\mathfrak{A}$-Stirling numbers where the a_i form an arithmetic progression $a_i = a + id$, $a,d \in \mathbb{N}$, and where $w(x) = [x]_{p,q}$.

Many interesting cases (*e.g.* items 1, 2, 8, 10 to 13 of Table 1) are covered by these polynomials. Here we deduce, from the combinatorial models, properties that are valid in general, in particular the p,q-analogues and extensions of convolution formulae due to L. Verde-Star (see Proposition 4.5). For $p = q = 1$, the a,d-progressive Stirling numbers were also studied by J. B. Remmel and M. Wachs [ReWa] and by A. Ruciński and B. Voigt [RuVo], who showed that if $a + d > 0$, the numbers $S_{1,1}^{a,d}(n,k)$ satisfy a local limit theorem.

2. **General study of $\mathfrak{A}$-stirling numbers.** In this section, we present the basic properties of $\mathfrak{A}$-Stirling numbers and some results linking two related families $\mathfrak{A}$ and $\mathfrak{A}^*$. The first proposition contains the most elementary results generalizing properties of p,q-Stirling numbers (see de Médicis and Leroux [dMLe]). These identities can be proven similarly to those in [dMLe], using either the combinatorics of $\mathcal{A}$-tableaux (bijectively or by applying an involution), or algebraically. Details are left to the reader.

PROPOSITION 2.1. *Let $\left(c^{\mathfrak{A}}(n,k)\right)_{n,k}$ and $\left(S^{\mathfrak{A}}(n,k)\right)_{n,k}$ be the families of polynomials defined by the recurrence relations (1.9) and (1.10). Then the following identities hold:*

a) (symmetric functions)

$$(2.1)\qquad c^{\mathfrak{A}}(n,k) = e_{n-k}(w_0, w_1, w_2, \ldots, w_{n-1}) = \sum_{0 \le i_1 < \cdots < i_{n-k} \le n-1} w_{i_1} \cdots w_{i_{n-k}}, \quad [\text{Co}, (12)]$$

$$(2.2)\qquad S^{\mathfrak{A}}(n,k) = h_{n-k}(w_0, w_1, w_2, \ldots, w_k) = \sum_{0 \le i_1 \le \cdots \le i_{n-k} \le k} w_{i_1} \cdots w_{i_{n-k}}, \quad [\text{Co}, (10)]$$

where $e_n(x_1, x_2, \ldots, x_k)$ and $h_n(x_1, x_2, \ldots, x_k)$ denote respectively the elementary and complete symmetric functions of degree n in k variables;

b) (generating functions)

For $n \geq 0$,

$$(2.3)\qquad \sum_{r \ge 0} c^{\mathfrak{A}}(n,r) y^{n-r} x^r = (x + w_0 y)(x + w_1 y) \cdots (x + w_{n-1} y), \quad [\text{Co}, (1)]$$

for $k \geq 0$,

$$(2.4)\qquad \sum_{r \ge 0} S^{\mathfrak{A}}(k+r,k) x^r = \frac{1}{(1-w_0 x)} \frac{1}{(1-w_1 x)} \cdots \frac{1}{(1-w_k x)}, \quad [\text{Co}, (8)]$$

$$(2.5)\qquad x^n = \sum_{k=0}^{n} S^{\mathfrak{A}}(n,k)(x-w_0)(x-w_1)\cdots(x-w_{k-1}); \quad [\text{Co}, (2)]$$

c) (recurrence relations)

$$(2.6)\qquad c^{\mathfrak{A}}(n+1,k+1) = \sum_{j=k}^{n} w_n w_{n-1} \cdots w_{j+1} c^{\mathfrak{A}}(j,k), \quad [\text{Co}, (7)]$$

$$(2.7)\qquad S^{\mathfrak{A}}(n+1,k+1) = \sum_{j=k}^{n} w_{k+1}^{n-j} S^{\mathfrak{A}}(j,k), \quad [\text{Co}, (6)]$$

$$(2.8)\qquad c^{\mathfrak{A}}(n,k) = \sum_{j=k}^{n} (-1)^{j-k} w_n^{j-k} c^{\mathfrak{A}}(n+1,j+1), \quad [\text{Co}, (5)]$$

$$(2.9)\qquad S^{\mathfrak{A}}(n,k) = \sum_{j=0}^{n-k} (-1)^j w_{k+1} w_{k+2} \cdots w_{k+j} S^{\mathfrak{A}}(n+1,k+j+1);$$

d) (orthogonality relations)

$$(2.10)\qquad \sum_{k=m}^{n} s^{\mathfrak{A}}(n,k) S^{\mathfrak{A}}(k,m) = \delta_{n,m},$$

$$(2.11)\qquad \sum_{k=m}^{n} S^{\mathfrak{A}}(n,k) s^{\mathfrak{A}}(k,m) = \delta_{n,m},$$

where $s^{\mathfrak{A}}(n,k) = (-1)^{n-k} c^{\mathfrak{A}}(n,k)$ are the $\mathfrak{A}$-Stirling numbers of the first kind with signs;

e) (other identities)

Let $\xi_j\mathcal{A}$ be the sequence obtained from $\mathcal{A} = (a_i)_{i\geq 0}$ by erasing the term a_j, and let $\xi_j\mathfrak{A} = (\xi_j\mathcal{A}, w)$. Then

$$(n-k)c^{\mathfrak{A}}(n,k) = \sum_{j=0}^{n-1} w_j c^{\xi_j\mathfrak{A}}(n-1,k), \tag{2.12}$$

$$(n-k)S^{\mathfrak{A}}(n,k) = \sum_{j=0}^{n-k-1} S^{\mathfrak{A}}(n-1-j,k)(w_0^{j+1} + w_1^{j+1} + \cdots + w_k^{j+1}), \tag{2.13}$$

$$\sum_{k=2}^{n}(1-w_1)(1-w_2)\cdots(1-w_{k-1})S^{\mathfrak{A}}(n,k) = 1 + w_0 + \cdots + w_0^{n-2}. \tag{2.14}$$

∎

Other explicit formulae, in terms of divided differences, Taylor expansions or systems of linear equations, can be found in Comtet [Co], Voigt [Vo] and Verde-Star [VS]. For example,

$$(-1)^{n-k}c^{\mathfrak{A}}(n,k) = \frac{1}{k!}\left\{\frac{d^k}{dx^k}(x;\mathfrak{A})_n\right|_{x=0}, \quad [\text{Co}, (13)]$$

where $(x;\mathfrak{A})_n = \prod_{j=0}^{n-1}(x-w_i)$, and as usual we write $w_i = w(a_i)$. If all the w_i's are distinct, then we have

$$S^{\mathfrak{A}}(n,k) = [w_0, w_1, \ldots, w_k]x^n, \quad [\text{Co}, (14)]$$

the k-th divided difference of x^n.

$$\sum_{n\geq k} S^{\mathfrak{A}}(n,k)\frac{t^n}{n!} = \sum_{j=0}^{k}\frac{e^{tw_j}}{(w_j)_k}, \quad [\text{Co}, (9)]$$

where, for $j \leq k$, $(w_j)_k = \prod_{l=0, l\neq j}^{k}(w_j - w_l)$. And if all w_i's are distinct, then

$$\begin{pmatrix} s^{\mathfrak{A}}(n,0) \\ s^{\mathfrak{A}}(n,1) \\ \vdots \\ s^{\mathfrak{A}}(n,n-1)\end{pmatrix} = -\begin{pmatrix} 1 & w_0 & w_0^2 & \cdots & w_0^{n-1} \\ 1 & w_1 & w_1^2 & \cdots & w_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_{n-1} & w_{n-1}^2 & \cdots & w_{n-1}^{n-1}\end{pmatrix}^{-1}\begin{pmatrix} w_0^n \\ w_1^n \\ \vdots \\ w_{n-1}^n\end{pmatrix}. \quad [\text{Co}, (11)]$$

The q-Stirling numbers of the second kind were originally introduced by L. Carlitz [Ca1, Ca2] as the polynomial family $S_q[n,k]$ satisfying the following identity:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=k}^{n}\binom{n}{j}(q-1)^{j-k}S_q[j,k]. \tag{2.15}$$

THEOREM 2.2 (GENERALIZATIONS OF CARLITZ FORMULAE). *Let $\mathcal{A}$ be an increasing sequence of non-negative integers, and $w\colon \mathbb{N} \longrightarrow K$ and $w^*\colon \mathbb{N} \longrightarrow K$ two functions such that $w(i) = w_0 + w^*(i)$, $\forall i \in \mathbb{N}$. Let $\mathfrak{A} = (\mathcal{A}, w)$, $\mathfrak{A}^* = (\mathcal{A}, w^*)$ and $\mathfrak{A}_0 = (\mathcal{A}, w_0)$, where w_0 denotes the constant function equal to w_0, then*

$$S^{\mathfrak{A}}(n,k) = \sum_{j=k}^{n}\binom{n}{j}w_0^{n-j}S^{\mathfrak{A}^*}(j,k). \tag{2.16}$$

Moreover, the following identities, obtained by applying various inversions, are equivalent to (2.16).

$$c^{\mathfrak{A}}(n,s)=\sum_{j=s}^{n}\binom{j}{s}w_0^{j-s}c^{\mathfrak{A}^*}(n,j); \tag{2.17}$$

$$S^{\mathfrak{A}^*}(n,s)=\sum_{j=s}^{n}(-w_0)^{n-j}\binom{n}{j}S^{\mathfrak{A}}(j,s); \tag{2.18}$$

$$c^{\mathfrak{A}^*}(n,s)=\sum_{j=s}^{n}(-w_0)^{j-s}\binom{j}{s}c^{\mathfrak{A}}(n,j); \tag{2.19}$$

$$S^{\mathfrak{A}_0}(n,s)=\sum_{j=s}^{n}(-1)^{j-s}c^{\mathfrak{A}^*}(j,s)S^{\mathfrak{A}}(n,j); \tag{2.20}$$

$$c^{\mathfrak{A}_0}(n,s)=\sum_{j=s}^{n}(-1)^{n-j}c^{\mathfrak{A}}(j,s)S^{\mathfrak{A}^*}(n,j). \tag{2.21}$$

PROOF. The lifting of certain identities on q-Stirling numbers to the level of $\mathfrak{A}$-Stirling numbers sometimes depends on the weight function w. When the weight function takes the form of a canonical finite sum of elements of a ring K, we can distribute the different terms of the sums on the $\mathcal{A}$-tableaux ϕ, assigning to each column, as weight, a single term of the sum corresponding to its length rather than the entire sum. In that case, we will say that ϕ is a *w-distributed $\mathcal{A}$-tableau*. Thus, for the choice $a_i = i$ and $w(i) = [i]_{p,q}$, we recover the combinatorial interpretation of the p,q-Stirling numbers of both kinds in terms of 0–1 tableaux. Indeed, instead of attributing a global weight of the form $[i]_{p,q} = p^{i-1}+p^{i-2}q+\cdots+q^{i-1}$ to a column of length i, we associate to each monomial p^jq^{i-j-1} the choice of the $(j+1)$-th position for the 1, giving j non-inversions and $(i-j-1)$ inversions.

Coming back to Theorem 2.2, since w (in the pair $\mathfrak{A}=(\mathcal{A},w)$) is a canonical sum of two terms, we can talk about w-distributed $\mathcal{A}$-tableaux. For these $\mathcal{A}$-tableaux, each column of length a_i will have weight either w_0 or $w^*(a_i)$. We give here the general idea of the combinatorial proofs, leaving out the details. See [dMLe] (Theorem 4.1 and Proposition 4.2) for similar proofs.

To prove identities (2.16) and (2.17), essentially choose the columns of the w-distributed $\mathcal{A}$-tableaux with weights different from w_0.

For (2.18) and (2.19), the right-hand side counts w-distributed $\mathcal{A}$-tableaux with some columns of weight w_0 distinguished, each contributing a factor (-1) to the global weight. We can construct a weight-preserving sign-reversing involution on these tableaux, such that the fixed points are all w-distributed $\mathcal{A}$-tableaux with no column of weight w_0, hence giving the left-hand side of the identities. The involution just "changes the status" of the leftmost column of weight w_0 in the w-distributed $\mathcal{A}$-tableaux.

For identities (2.20) and (2.21), weight-preserving sign-reversing involutions resembling those used to show orthogonality can be designed. ∎

Theorem 2.2 deals essentially with interactions between $\mathfrak{A}$-Stirling numbers where the weight functions differ by an additive constant. The next proposition examines what happens when the weight functions differ by a multiplicative constant.

PROPOSITION 2.3. *Let $\mathfrak{A} = (\mathcal{A}, w)$ where $w\colon \mathbb{N} \longrightarrow K$, and let $a \in K$. Denote by aw the function from $\mathbb{N}$ to the ring K obtained by multiplication of w by the constant a. Then,*

$$c^{(\mathcal{A},aw)}(n,k) = a^{n-k}c^{(\mathcal{A},w)}(n,k), \tag{2.22}$$

$$S^{(\mathcal{A},aw)}(n,k) = a^{n-k}S^{(\mathcal{A},w)}(n,k). \tag{2.23}$$

PROOF. It suffices to note that for every $\mathcal{A}$-tableau ϕ having k columns, the weight of ϕ according to weight function aw, $(aw)_{(\mathcal{A},aw)}(\phi)$, is related to the weight $w_{\mathfrak{A}}(\phi) = w_{(\mathcal{A},w)}(\phi)$ in the following way:

$$(aw)_{(\mathcal{A},aw)}(\phi) = a^k \cdot w_{(\mathcal{A},w)}(\phi). \qquad \blacksquare$$

REMARK 2.4. Observe that the exact algebraic expressions for cases (7) and (9) of Table 1 can be obtained from cases (6) and (8) (with $p = 1$) respectively, via Proposition 2.3.

The particular case $a_i = i$, $w_0 = 1$, and $w^*(i) = q^i - 1$ in Theorem 2.2 gives Carlitz's identity and its close relatives (*cf.* [dMLe], (4.5) to (4.10)), corresponding to cases (6) and (9) of Table 1. The original combinatorial proofs used a unified combinatorial model, *q-tableaux*, that is fillings of Ferrers diagram with elements of the set $\{0, 1, 2, \ldots, q-1\}, q \in \mathbb{N}$.

Another application of Theorem 2.2 is the following: M. Koutras [Ko] defined the *non-central Stirling numbers of the first and second kinds* $c_{(a)}(n,k)$ and $S_{(a)}(n,k)$ (case (11) of Table 1) to be the polynomials satisfying

$$t(t-1)\cdots(t-n+1) = \sum_{k=0}^{n}(-1)^{n-k}c_{(a)}(n,k)(t-a)^k \tag{2.24}$$

and

$$(t-a)^n = \sum_{k=0}^{n} S_{(a)}(n,k)t(t-1)\cdots(t-k+1), \tag{2.25}$$

Setting $x = (t-a)$ in the previous identities and comparing with (2.3) and (2.5), we deduce that $c_{(a)}(n,k) = c^{\mathfrak{A}}(n,k)$ and $S_{(a)}(n,k) = S^{\mathfrak{A}}(n,k)$, for $\mathfrak{A} = (\mathcal{A}, w)$, where $\mathcal{A} = (i)_{i\geq 0}$, and $w(i) = -a + i$. Formulae (2.16) and (2.17) of Theorem 2.2 then let us express the non-central Stirling numbers of Koutras in terms of the usual "central" Stirling numbers (case (2) of Table 1):

$$c_{(a)}(n,s) = \sum_{j=s}^{n}\binom{j}{s}(-a)^{j-s}c(n,j), \tag{2.26}$$

(2.27) $$S_{(a)}(n,k)=\sum_{j=k}^{n}\binom{n}{j}(-a)^{n-j}S(j,k).$$

COROLLARY 2.5. *Let $\mathfrak{A}=(\mathcal{A},w)$. If the following equality holds for all $i\geq 0$*

$$w(a_i)=w_0+w_0z+w_0z^2+\cdots+w_0z^{i-1}=w_0[i]_z\in\mathbb{N}[w_0,z],$$

then

(2.28) $$c^{\mathfrak{A}}(n+1,k+1)=\sum_{j=k}^{n}\binom{j}{k}w_0^{j-k}z^{n-j}c^{\mathfrak{A}}(n,j),$$

(2.29) $$S^{\mathfrak{A}}(n+1,k+1)=\sum_{m=k}^{n}\binom{n}{m}w_0^{n-m}z^{m-k}S^{\mathfrak{A}}(m,k).$$

PROOF. Identity (2.28) (respectively (2.29)) follows from (2.17) and (2.22) (respectively (2.16) and (2.23)). The combinatorial proofs are similar to the ones of (2.11) and (2.12) of [dMLe], using w-distributed $\mathcal{A}$-tableaux. ∎

THEOREM 2.6 (CONVOLUTION FORMULAE). *Let $\mathfrak{A}=(\mathcal{A},w)$, where $\mathcal{A}=(a_0,a_1,a_2,\ldots)$. Denote by $\delta\mathcal{A}$ the sequence $(a_1,a_2,\ldots)$, and, more generally, by $\delta^n\mathcal{A}=\delta(\delta^{n-1}\mathcal{A})$ the sequence $(a_n,a_{n+1},a_{n+2},\ldots)$. Moreover, denote $\delta^n\mathfrak{A}=(\delta^n\mathcal{A},w)$. Then*

(2.30) $$c^{\mathfrak{A}}(m+j,n)=\sum_{k=0}^{n}c^{\mathfrak{A}}(m,k)c^{\delta^m\mathfrak{A}}(j,n-k),$$

(2.31) $$S^{\mathfrak{A}}(m+j,n)=\sum_{k=0}^{n}S^{\mathfrak{A}}(m,k)S^{\delta^k\mathfrak{A}}(j,n-k).\quad\text{[VS, (5.18)]}$$

Likewise,

(2.32) $$c^{\mathfrak{A}}(n+1,m+j+1)=\sum_{k=0}^{n}c^{\mathfrak{A}}(k,m)c^{\delta^{k+1}\mathfrak{A}}(n-k,j),$$

(2.33) $$S^{\mathfrak{A}}(n+1,m+j+1)=\sum_{k=0}^{n}S^{\mathfrak{A}}(k,m)S^{\delta^{m+1}\mathfrak{A}}(n-k,j).\quad\text{[VS, (5.19)]}$$

PROOF. To prove these formulae combinatorially, one must strategically section the $\mathcal{A}$-tableaux into two parts.

For instance, one gets identity (2.30) (respectively (2.33)) by separating the columns of the $\mathcal{A}$-tableaux in the set $Td^{\mathcal{A}}(m+j-1,m+j-n)$ (respectively $T^{\mathcal{A}}(m+j+1,n-m-j)$) to form an $\mathcal{A}$-tableau with column lengths in $\{a_0,a_1,\ldots,a_{m-1}\}$ (respectively $\{a_0,a_1,\ldots,a_m\}$), and another $\mathcal{A}$-tableau with column lengths in $\{a_m,a_{m+1},\ldots,a_{m+j-1}\}$ (respectively $\{a_{m+1},a_{m+2},\ldots,a_{m+j+1}\}$ for (2.33)).

For identities (2.31) and (2.32), if $\phi\in T^{\mathcal{A}}(n,m+j-n)$ (respectively $\phi'\in Td^{\mathcal{A}}(n,n-m-j)$), we must first determine the unique integer k_0, $0\leq k_0\leq n$ (respectively k_1, $m\leq k_1\leq n$), such that

(2.34) $$(k_0-1)+|\phi|_0+|\phi|_1+\cdots+|\phi|_{k_0-1}<m\leq k_0+|\phi|_0+|\phi|_1+\cdots+|\phi|_{k_0}$$

(respectively

$$(2.35)\quad k_1 - |\phi'|_0 - |\phi'|_1 - \cdots - |\phi'|_{k_1-1} = m < (k_1+1) - |\phi'|_0 - |\phi'|_1 - \cdots - |\phi'|_{k_1}),$$

where $|\phi|_i$ denotes the number of columns of length a_i in ϕ. Note that since ϕ' has columns of distinct lengths, (2.35) can only be satisfed if $|\phi'|_{k_1} = 0$. The $\mathcal{A}$-tableau ϕ (respectively ϕ') is then divided into an $\mathcal{A}$-tableau containing the $(m - k_0)$ last columns of ϕ (respectively $(k_1 - m)$ last columns of ϕ'), with column lengths in the set $\{a_0, a_1, \ldots, a_{k_0}\}$ (respectively $\{a_0, a_1, \ldots, a_{k_1-1}\}$), and the remaining $\mathcal{A}$-tableau, with column lengths in the set $\{a_{k_0}, a_{k_0+1}, \ldots, a_n\}$ (respectively $\{a_{k_1+1}, a_{k_1+2}, \ldots, a_n\}$). ■

These identities are generalizations of the convolution formulae (1.11) to (1.14) stated in the introduction, where $a_i = i$ and $w(i) = [i]_q$.

As observed in Proposition 2.1, the $\mathfrak{A}$-Stirling numbers can be expressed in terms of the elementary and complete symmetric functions respectively, as $c^{\mathfrak{A}}(n,k) = e_{n-k}(w_0, w_1, \ldots, w_{n-1})$ and $S^{\mathfrak{A}}(n,k) = h_{n-k}(w_0, w_1, \ldots, w_k)$ respectively. Some interesting instances of these families of numbers are given by Whitney numbers of supersolvable lattices (*cf.* Stanley [St1, St2]). Other combinatorial properties of generalized Stirling numbers will be inherited from properties of symmetric functions. An example is logarithmic concavity (see Comtet [Co], Habsieger [Ha], Leroux [Le], and Sagan [Sa]), an aspect which is not treated in the present paper.

3. **Partial stirling numbers and weak 0–1 tableaux.**

DEFINITIONS 3.1. A *weak 0–1 tableau* ψ is a triple $\psi = (\lambda, l, f)$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of an integer m, l is an integer greater or equal to λ_1, and $f = (f_{ij})_{1 \leq j \leq \lambda_i}$ is a filling of the cells of the corresponding Ferrers diagram of shape λ with 0's and 1's, such that there is *at most* one 1 in each column. We will say that ψ contains l columns, $(l - \lambda_1)$ of which being of length zero and considered filled with 0's.

For example, the weak 0–1 tableau $\psi = (\lambda, l, f)$ where $\lambda = (5,4,2,2)$, $l = 6$ and $f_{13} = f_{24} = f_{32} = 1, f_{ij} = 0$ elsewhere $(1 \leq j \leq \lambda_i)$ is illustrated in Figure 2.

0	0	1	0	0
0	0	0	1	
0	1			
0	0			

FIGURE 2: WEAK 0–1 TABLEAU ψ

As for the usual 0–1 tableaux, the *inversion number* of a weak 0–1 tableau ψ, denoted by $\mathrm{inv}(\psi)$, is equal to the number of 0's below a 1 in ψ. However, the *non-inversions number* of ψ, denoted by $\mathrm{nin}(\psi)$, is equal to the number of 0's above a 1 in ψ, plus the number of 0's in the columns filled with 0's. For instance, for ψ in Figure 2, we compute $\mathrm{inv}(\psi) = 2$ and $\mathrm{nin}(\psi) = 8$.

We will denote by $T_w(h,r)$ (respectively $T_w(h,s,r)$), the set of all weak 0–1 tableaux $\psi=(\lambda,r,f)$ containing r columns of length less or equal to h (and having exactly $(r-s)$ 1's in the filling f respectively), and by $Td_w(h,r)$ (respectively $Td_w(h,s,r)$), the subset of $T_w(h,r)$ (respectively $T_w(h,r,s)$) consisting of all weak 0–1 tableaux with distinct column lengths.

DEFINITIONS 3.2. The *partial* p,q*-Stirling numbers* are given by:

$$\mathcal{P}c_{p,q}[n,m,k]:=\sum_{\psi\in Td_w(n-1,n-m,n-k)} p^{\mathrm{nin}(\psi)}q^{\mathrm{inv}(\psi)}; \tag{3.1}$$

$$\mathcal{P}S_{p,q}[n,m,k]:=\sum_{\psi\in T_w(k,n-m,n-k)} p^{\mathrm{nin}(\psi)}q^{\mathrm{inv}(\psi)}; \tag{3.2}$$

$$\mathcal{P}c_{p,q}[n,k]:=\sum_{\psi\in Td_w(n-1,n-k)} p^{\mathrm{nin}(\psi)}q^{\mathrm{inv}(\psi)}; \tag{3.3}$$

$$\mathcal{P}S_{p,q}[n,k]:=\sum_{\psi\in T_w(k,n-k)} p^{\mathrm{nin}(\psi)}q^{\mathrm{inv}(\psi)}. \tag{3.4}$$

Note that these polynomials are not symmetric in p and q, and that

$$\mathcal{P}c_{p,q}[n,k]=\sum_{m=k}^{n}\mathcal{P}c_{p,q}[n,m,k], \tag{3.5}$$

$$\mathcal{P}S_{p,q}[n,k]=\sum_{m=k}^{n}\mathcal{P}S_{p,q}[n,m,k]. \tag{3.6}$$

Like classical Stirling numbers, the partial Stirling numbers have a combinatorial interpretation in terms of permutations and set partitions, or more precisely, in terms of *permutations with marked cycles* and *partial set partitions*. To show this, we shall exhibit bijections between weak 0–1 tableaux and these objects. The corresponding statistics, obtained by carrying the inversion number and the non-inversion number along the bijections will then provide an alternate interpretation for the partial p,q-Stirling numbers. We need some definitions and notations.

DEFINITIONS 3.3. Let $\sigma\in\mathfrak{S}(n,l,k)$ be a permutation of $\{1,2,\ldots,n\}=[\![n]\!]$, with k cycles such that l of them are marked. We will say that σ is written as *standard product of cycles* if σ is written as a product of disjoint cycles, each cycle starting with its minimum, and the cycles are ordered by increasing minima. We will denote by $w(\sigma)$ the word obtained by supressing the parenthesis in the permutation σ written as a standard product of cycles. The minimum of each cycle is called the *cycle leader*. To indicate a marked cycle, we will underline its cycle leader. For instance, $\sigma=(1,9,3,5)(\underline{2},7)(\underline{4})(6,10,8)\in\mathfrak{S}(10,2,4)$ is written as a standard product of cycles, both cycles $(\underline{2},7)$ and $(\underline{4})$ are marked, the cycle leaders are respectively 1, 2, 4 and 6, and $w(\sigma)=1\ 9\ 3\ 5\ 2\ 7\ 4\ 6\ 10\ 8$.

There is a natural bijection Φ_1 between the sets $Td_w(n-1,l,n-k+l)$ and $\mathfrak{S}(n,l,k)$, that is inspired from the recurrence relations (3.20) or (3.26). Let $\psi\in Td_w(n-1,l,n-k+l)$. We construct by induction a sequence of permutations with marked cycles $\sigma_0,\sigma_1,\ldots,\sigma_n$ such that σ_i has support $[\![i]\!]$. The image of ψ under Φ_1 will then be $\Phi_1(\psi)=\sigma_n$. We start with

$\sigma_0 = \emptyset$, the empty permutation. Suppose we have already determined $\sigma_0, \sigma_1, \ldots, \sigma_{i-1}$, $1 \le i \le n$. Then σ_i is obtained from σ_{i-1} in the following way:

i) if ψ does not contain a column of length $(i-1)$, σ_i coincides with σ_{i-1} on the set $[\![i-1]\!]$, and i forms a new unmarked cycle of length one in σ_i;

ii) if ψ contains a column of length $(i-1)$ filled with 0's, σ_i coincides with σ_{i-1} on the set $[\![i-1]\!]$, and i forms a new marked cycle of length one in σ_i;

iii) if ψ contains a column of length $(i-1)$ with a 1 in the j-th cell (from top to bottom), $1 \le j \le (i-1)$, then σ_i is obtained from σ_{i-1} by inserting i as the image of the j-th letter in $w(\sigma_{i-1})$. The image of i is then set equal to the image of that letter in σ_{i-1}.

For example, for $\psi \in Td_w(5,2,5)$ given in Figure 3, $\sigma_0 = \emptyset$, $\sigma_1 = (\underline{1})$, $\sigma_2 = (\underline{1},2)$, $\sigma_3 = (\underline{1},2)(\underline{3})$, $\sigma_4 = (\underline{1},2)(\underline{3})(4)$, $\sigma_5 = (\underline{1},2)(\underline{3},5)(4)$, and $\Phi_1(\psi) = \sigma_6 = (\underline{1},2)(\underline{3},5,6)(4)$. It is easy to see that Φ_1 is a bijection. Details are left to the reader.

$$\begin{array}{|c|c|c|c|}
\hline
0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 \\
\cline{1-3}
0 & 1 \\
\cline{1-2}
1 & 0 \\
\cline{1-2}
0 \\
\cline{1-1}
\end{array}$$

FIGURE 3: WEAK 0–1 TABLEAU $\psi \in Td_f(5,5,2)$

The bijection Φ_1 lets us define inversion and non-inversion statistics on permutations with marked cycles:

$$\text{(3.7)} \qquad \operatorname{inv}(\sigma) = \operatorname{inv}\bigl(\Phi_1^{-1}(\sigma)\bigr), \quad \text{and} \quad \operatorname{nin}(\sigma) = \operatorname{nin}\bigl(\Phi_1^{-1}(\sigma)\bigr).$$

In fact, it is not hard to see that the *inversion number* of $\sigma \in \mathfrak{S}(n,l,k)$ is equal to

$$\text{(3.8)} \qquad \operatorname{inv}(\sigma) = \operatorname{inv}\bigl(w(\sigma)\bigr),$$

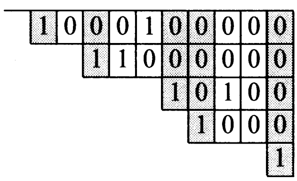
where $\operatorname{inv}\bigl(w(\sigma)\bigr)$ is the number of inversions in the word $w(\sigma)$, that is the number of pairs of letters (i,j) such that j lies to the right of i in $w(\sigma)$, but $j < i$, and the *non-inversion number* is equal to

$$\text{(3.9)} \qquad \operatorname{nin}(\sigma) = \sum_i \bigl(i - 2 - \operatorname{inv}_\sigma(i)\bigr) + \sum_j (j-1),$$

where the first sum ranges over all integer $i \in [\![n]\!]$ not cycle leaders, $\operatorname{inv}_\sigma(i)$ is just the number of inversions (i,j) in $w(\sigma)$ with i as first coordinate, and the second sum ranges over the marked cycles leaders.

DEFINITIONS 3.4. A *partial partition* $(E;\pi) \in \mathcal{PP}(n,l,k)$ of $[\![n]\!]$ is a pair $(E;\pi)$ such that E is a subset of $[\![n]\!] = \{1,2,\ldots,n\}$ of cardinality l, and $\pi = \pi_1, \pi_2, \ldots, \pi_k$ is a partition with exactly k blocks of the set $[\![n]\!] - E$. We will say that $(E;\pi)$ is written in *standard form* if the elements of E appear in increasing order, each block of π is written by increasing order of its elements, and the blocks are ordered by increasing minima. The blocks minima, $a_1, a_2, \ldots, a_k$ respectively, are called *block leaders*. For example, $(E;\pi) \in \mathcal{PP}(10,2,3)$, with $E = \{5,9\}$ and $\pi = \pi_1, \pi_2, \pi_3 = \{1,3,7\}, \{2,6,10\}, \{4,8\}$, is written in standard form and the block leaders of π are equal to $a_1 = 1$, $a_2 = 2$ and $a_3 = 4$.

We now construct a bijection Φ_2 between the sets $T_w(k,l,n-k+l)$ and $\mathcal{PP}(n,l,k)$, inspired from the recurrence relations (3.21) or (3.27). Let $\psi \in T_w(k,l,n-k+l)$. Transport ψ in the third quadrant by a vertical reflection (so that the columns are now by increasing order of length). For $1 \le i \le k$, add a column of length i, with a 1 in the bottom cell, to the left of the columns of length greater or equal to i (in ψ reflected). Then add cells filled with 0's to the bottom of each column so that they reach length k. We obtain a $k \times n$ matrix with at most one 1 in each column (these manipulation are shown in Figure 4 for the weak 0–1 tableau of Figure 2). It is the row-to-column representation of the *restricted-growth function* associated with the partial partition $\Phi_2(\psi) = (E;\pi)$ (*cf.* Milne [Mi]). More precisely, if the j-th column of the matrix contains only 0's, then $j \in E$, and if the j-th column contains a 1 in the m-th row, then j lies in the m-th block of the partition π, written in standard form. For example, for $\psi \in T_w(5,3,6)$ illustrated in Figure 2, we obtain the matrix of Figure 4, and $\Phi_2(\psi) = (E;\pi) = (\{1,3,10\}; \{2,6\}, \{4,5\}, \{7,9\}, \{8\}, \{11\}) \in \mathcal{PP}(11,3,5)$. Φ_2 is clearly a bijection.



$$\Updownarrow$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

FIGURE 4: CONSTRUCTION OF A MATRIX FROM A WEAK 0–1 TABLEAU

Once again, carrying over Φ_2 the inversion and non-inversion statistics, we obtain the *inversion number* of a partial partition $(E;\pi)$,

$$\mathrm{inv}(E;\pi) = \sum_{i \in [\![n]\!] - E} \mathrm{inv}_\pi(i), \tag{3.10}$$

https://doi.org/10.4153/CJM-1995-027-x Published online by Cambridge University Press

where, if $a_1 < a_2 < \cdots < a_k$ are the block leaders of π and i lies in the m-th block π_j, $\mathrm{inv}_\pi(i)$ is equal to the number of a_j, $m < j \leq k$, such that $a_j < i$.

Moreover, if we let

$$\mathrm{nin}_\pi(i) = \begin{cases} |\{a_j \mid a_j < i\}| & \text{if } i \in E, \\ |\{a_j \mid 1 \leq j < m \text{ and } a_j < i\}| & \text{if } i \text{ lies in the } m\text{-th block of } \pi \\ \quad (= m-1) & \text{and } i \text{ is not a block leader,} \\ 0 & \text{if } i \text{ is a block leader,} \end{cases} \tag{3.11}$$

then the *non-inversion number* of a partial partition $(E;\pi)$ equals

$$\begin{aligned} \mathrm{nin}(E;\pi) &= \sum_{i\in[\![n]\!]} \mathrm{nin}_\pi(i), \\ &= \sum_{i=0}^{k-1} i(|\pi_{i+1}|-1) + \sum_{j\in E} \mathrm{nin}_\pi(j). \end{aligned} \tag{3.12}$$

From these two bijections Φ_1 and Φ_2, we can deduce

PROPOSITION 3.5 (ALTERNATIVE COMBINATORIAL MODELS). *With the preceding notations and definitions,*

$$\mathcal{P}c_{p,q}[n,m,k] = \sum_{\sigma\in\mathfrak{S}(n,n-m,n-m+k)} p^{\mathrm{nin}(\sigma)} q^{\mathrm{inv}(\sigma)}, \tag{3.13}$$

$$\mathcal{P}S_{p,q}[n,m,k] = \sum_{(E;\pi)\in\mathcal{PP}(n,n-m,k)} p^{\mathrm{nin}(E;\pi)} q^{\mathrm{inv}(E;\pi)}. \tag{3.14}$$

Using these alternate interpretations, the generating series for the $p = q = 1$ case are easy to find:

PROPOSITION 3.6 (GENERATING SERIES). *We have*

$$\sum_{n,m,k\geq 0} \mathcal{P}c_{1,1}[n,n-m,k] a^m u^k \frac{x^n}{n!} = (1-x)^{-(a+u)}, \tag{3.15}$$

$$\sum_{n,m,k\geq 0} \mathcal{P}S_{1,1}[n,n-m,k] a^m u^k \frac{x^n}{n!} = e^{ax} e^{u(e^x-1)}, \tag{3.16}$$

$$\sum_{n,k\geq 0} \mathcal{P}c_{1,1}[n,k] u^k \frac{x^n}{n!} = (1-x)^{-(1+u)}, \tag{3.17}$$

$$\sum_{n,k\geq 0} \mathcal{P}S_{1,1}[n,k] u^k \frac{x^n}{n!} = e^x e^{u(e^x-1)}. \tag{3.18}$$

PROOF. The generating series of permutations according to the number of cycles (weighted by u) is well-known, it is

$$\sum_{n\geq 0} \sum_{k\geq 0} c(n,k) u^k \frac{x^n}{n!} = \left(\frac{1}{1-x}\right)^u. \tag{3.19}$$

Identity (3.15) can be deduced from the fact that a permutation with marked cycles decomposes naturally into two permutations, one containing the marked cycles (weighted by a), and the other the unmarked cycles (weighted by u).

Identity (3.16) expresses the fact that a partial partition $(E;\pi)$ consists of a set E (each element weighted by a) and a partition π, which is a set of non-empty blocks (with each block weighted by u).

Identities (3.17) and (3.18) follow from (3.15) and (3.16) and the connections (3.5) and (3.6) between the partial p,q-Stirling numbers with three parameters and the partial p,q-Stirling numbers with two parameters. ∎

REMARK 3.7. The basic recurrence relations for the partial p,q-Stirling numbers with two parameters are given by

$$\mathcal{P}c_{p,q}[n+1,k] = \mathcal{P}c_{p,q}[n,k-1] + (p^n + [n]_{p,q})\mathcal{P}c_{p,q}[n,k], \tag{3.20}$$

$$\mathcal{P}S_{p,q}[n+1,k] = \mathcal{P}S_{p,q}[n,k-1] + (p^k + [k]_{p,q})\mathcal{P}S_{p,q}[n,k], \tag{3.21}$$

with initial boundary conditions $\mathcal{P}c_{p,q}[n,k] = 0 = \mathcal{P}S_{p,q}[n,k]$ if $k > n$, $\mathcal{P}c_{p,q}[n,0] = (p+[1]_{p,q})(p^2+[2]_{p,q})\cdots(p^{n-1}+[n-1]_{p,q})$, $\mathcal{P}S_{p,q}[n,0] = 1$ and $\mathcal{P}c_{p,q}[0,k] = \delta_{0,k} = \mathcal{P}S_{p,q}[0,k]$.

Separating the different possibilities for the leftmost column of weak 0–1 tableaux (length not equal to the maximal allowed value, or length equal to the maximal value with the filling containing a 1 or no 1) gives a combinatorial proof of these recurrence relations.

Comparing (3.20) and (3.21) to (1.9) and (1.10) yields the conclusion that these families of polynomials are special cases of $\mathfrak{A}$-Stirling numbers (for $\mathfrak{A} = (\mathcal{A}, w)$, where $\mathcal{A} = (i)_{i\geq 0}$ and $w(i) = p^i + [i]_{p,q}$), and thus every identity in the previous section applies to them. However, it is not the case for the sequences $(\mathcal{P}c_{p,q}[n,m,k])$ and $(\mathcal{P}S_{p,q}[n,m,k])$, for which a more detailed analysis is necessary.

PROPOSITION 3.8. *The following identities hold for the partial p,q-Stirling numbers with three parameters $\mathcal{P}c_{p,q}[n,m,k]$ and $\mathcal{P}S_{p,q}[n,m,k]$:*

a) (connections and special cases)

$$\mathcal{P}c_{p,q}[n,n,k] = c_{p,q}[n,k] \quad \text{and} \quad \mathcal{P}c_{p,q}[n,k,k] = p^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_p, \tag{3.22}$$

$$\mathcal{P}S_{p,q}[n,n,k] = S_{p,q}[n,k] \quad \text{and} \quad \mathcal{P}S_{p,q}[n,k,k] = \begin{bmatrix} n \\ k \end{bmatrix}_p, \tag{3.23}$$

$$\mathcal{P}c_{1,q}[n,m,k] = \binom{n-m+k}{k} c_q[n, n-m+k], \tag{3.24}$$

$$\mathcal{P}S_{1,q}[n,m,k] = \binom{n}{m} S_q[m,k]; \tag{3.25}$$

b) (recurrence relations)

$$\mathcal{P}c_{p,q}[n+1,m+1,k] = \mathcal{P}c_{p,q}[n,m,k-1] + p^n \mathcal{P}c_{p,q}[n,m+1,k] + [n]_{p,q}\mathcal{P}c_{p,q}[n,m,k], \tag{3.26}$$

$$\mathcal{P}S_{p,q}[n+1,m+1,k] = \mathcal{P}S_{p,q}[n,m,k-1] + p^k \mathcal{P}S_{p,q}[n,m+1,k] + [k]_{p,q}\mathcal{P}S_{p,q}[n,m,k], \tag{3.27}$$

with initial boundary conditions $\mathcal{P}c_{p,q}[n,m,k] = 0 = \mathcal{P}S_{p,q}[n,m,k]$ *if* $m > n$ *or if* $k > m$, $\mathcal{P}c_{p,q}[0,m,k] = \delta_{0,k}\delta_{0,m}$, $\mathcal{P}S_{p,q}[0,m,k] = \delta_{0,k}\delta_{0,m}$, $\mathcal{P}c_{p,q}[n,0,k] = p^{\binom{n}{2}}\delta_{k,0}$, $\mathcal{P}S_{p,q}[n,0,k] = \delta_{0,k}$, *and* $\mathcal{P}c_{p,q}[n,m,0] = p^{\binom{n}{2}-m}c_{1,q/p}[n,n-m]$, $\mathcal{P}S_{p,q}[n,m,0] = \delta_{m,0}$;

c) (generating series) For $n > 0$,

$$\sum_{r\geq 0}\sum_{m\geq 0} \mathcal{P}c_{p,q}[n,m,r]x^r y^{m-r} z^{n-m} = (x+z)(x+pz+[1]_{p,q}y)\cdots(x+p^{n-1}z+[n-1]_{p,q}y), \tag{3.28}$$

for $k \geq 0$,

$$\sum_{r\geq 0}\sum_{l\geq 0} \mathcal{P}S_{p,q}[k+r,k+l,k]x^l y^{r-l} = \frac{1}{(1-y)}\frac{1}{(1-py-[1]_{p,q}x)}\frac{1}{(1-p^2y-[2]_{p,q}x)}\cdots\frac{1}{(1-p^ky-[k]_{p,q}x)}. \tag{3.29}$$

PROOF. a) When $n = m$, this forces all the columns of the weak 0–1 tableaux to contain a 1 so $\mathcal{P}c_{p,q}[n,n,k]$ and $\mathcal{P}S_{p,q}[n,n,k]$ are counting usual 0–1 tableaux according to inversion and non-inversion numbers.

When $m = k$, the corresponding weak 0–1 tableaux must be filled with 0's only, each 0 contributing to a factor p, thus yielding the combinatorial interpretation of the q-binomial coefficients in terms of partitions fitting in a box.

Identities (3.24) and (3.25) become obvious with the alternate combinatorial interpretations of Proposition 3.5. Indeed, $\mathcal{P}c_{1,q}[n,m,k]$ is q-counting (according to inversion number) permutations of $[\![n]\!]$ with $(n-m+k)$ cycles (factor $c_q[n,n-m+k]$ on the right-hand side of (3.24)), such that $(n-m)$ of these cycles are marked (there is $\binom{n-m+k}{n-m}$ ways of choosing them among the $(n-m+k)$ cycles of the permutations). Likewise, $\mathcal{P}S_{1,q}[n,m,k]$ is q-counting (according to inversion number) partial partitions $(E;\pi)$ of the set $[\![n]\!]$ such that $|E| = (n-m)$ (the factor $\binom{n}{m}$ on the right-hand side of (3.25) corresponds to the choice of E) and π is a partition with exactly k blocks (corresponding to the factor $S_q[m,k]$).

b) The proofs of (3.28) and (3.29) are similar to those of (3.20) and (3.21). Note that the weak 0–1 tableaux $\psi \in Td_w(n-1,n-m,n)$ all have a "staircase" shape (the lengths of the columns are respectively $(n-1),(n-2),\ldots,1$ and 0). To compute their non-inversion number, we need only substract the inversion number and the number m of 1's in the filling to the total number of cells of ψ. This gives the initial boundary condition $\mathcal{P}c_{p,q}[n,m,0] = p^{\binom{n}{2}-m}c_{1,q/p}[n,n-m]$.

c) (3.28) and (3.29) easily follow from the combinatorial definition. ■

In addition to identities (3.24) and (3.25), partial Stirling numbers can also be expressed in terms of usual q-Stirling numbers in the following way (for $p = 1$):

PROPOSITION 3.9. *We have:*

$$\mathcal{P}c_{1,q}[n+1,k+1] = \sum_{j=k}^{n}\left[\binom{j+1}{k+1} + \binom{j}{k}\right]2^{j-k-1}q^{n-j}c_q[n,j], \tag{3.30}$$

$$\mathcal{P}S_{1,q}[n+1,k+1] = \sum_{j=0}^{n}\left[\sum_{i=0}^{n-j}\binom{n-i}{j}2^{n-j-i}\right]q^{j-k}S_q[j,k], \tag{3.31}$$

$$\begin{aligned}&\mathcal{P}c_{1,q}[n+1,m+1,k+1]\\ &\quad= \sum_{j=k}^{n}\left[\binom{j}{k}\binom{j-k}{n-m} + \binom{j}{k+1}\binom{j-k-1}{n-m-1}\right]q^{n-j}c_q[n,j],\end{aligned} \tag{3.32}$$

$$\mathcal{P}S_{1,q}[n+1,m+1,k+1] = \sum_{j=0}^{n}\left[\sum_{i=0}^{n-j}\binom{n-i}{j}\binom{n-j-i}{n-m-i}\right]q^{j-k}S_q[j,k]. \tag{3.33}$$

PROOF. These identities are proven similarly to (2.11) and (2.12) in [dMLe]. We essentially remove the columns causing no inversions. Thus all the remaining columns must contain a 0 in the bottom cell and a 1 somewhere above it. By deleting these bottom cells, we obtain a general 0–1 tableau. The difference with the identities (2.11) and (2.12) of [dMLe] is that there are two kinds of columns that do not cause any inversion: the columns filled exclusively with 0's and the columns which contain a 1 positioned in the bottom cell. Details are left to the reader. ■

PROPOSITION 3.10 (CONVOLUTION FORMULAE). *We have:*

$$c_{p,q}[m+n,k] = \sum_{i+j\geq k} p^{n(i+j-k)}[n]_{p,q}^{m-j}c_{p,q}[n,i]\mathcal{P}c_{q,p}[m,j,k-i], \tag{3.34}$$

$$S_{p,q}[m+n,k] = \sum_{i+j\geq k} p^{i(i+j-k)}[i]_{p,q}^{m-j}S_{p,q}[n,i]\mathcal{P}S_{q,p}[m,j,k-i]. \tag{3.35}$$

$$c_{p,q}[n+1,m+j+1] = \sum_{k=0}^{n}\sum_{i=j}^{n-k} p^{(k+1)(i-j)}[k+1]_{p,q}^{n-k-i}c_{p,q}[k,m]\mathcal{P}c_{q,p}[n-k,i,j], \tag{3.36}$$

$$S_{p,q}[n+1,m+j+1] = \sum_{k=0}^{n}\sum_{i=j}^{n-k} p^{(m+1)(i-j)}[m+1]_{p,q}^{n-k-i}S_{p,q}[k,m]\mathcal{P}S_{q,p}[n-k,i,j]. \tag{3.37}$$

PROOF. These identities are particular cases of Theorem 2.6, with $\mathcal{A} = (i)_{i\geq 0}$ and $w(i) = [i]_{p,q}$. For these parameters, $c^{\mathfrak{A}}(n,k) = c_{p,q}[n,k]$, $S^{\mathfrak{A}}(n,k) = S_{p,q}[n,k]$,

$$c^{\delta^m\mathfrak{A}}(n,k) = \sum_{\varphi\in Td^{\geq m}(n-1,n-k)} p^{\mathrm{nin}(\varphi)}q^{\mathrm{inv}(\varphi)},$$

and

$$S^{\delta^m \mathfrak{A}}(n,k) = \sum_{\varphi \in T^{\geq m}(k,n-k)} p^{\mathrm{nin}(\varphi)} q^{\mathrm{inv}(\varphi)},$$

where $T^{\geq m}(h,r)$ (respectively $Td^{\geq m}(h,r)$) denotes the subset of $T(h+m,r)$ (respectively $Td(h+m,r)$) containing all the 0–1 tableaux φ such that the length of each column is at least equal to m. The notations $T(h,r)$ and $Td(h,r)$ were introduced in Section 1.

The 0–1 tableaux $\varphi \in Td^{\geq m}(n-1,n-k)$ (respectively $\varphi \in T^{\geq m}(k,n-k)$) can be decomposed into a rectangular weak 0–1 tableau ψ_1 of format $m \times (n-k)$, and a general weak 0–1 tableau $\psi_2 \in Td_w(n-1,n-k)$ (or $\psi_2 \in T_w(k,n-k)$ respectively), such that the columns of ψ_1 containing a 1 correspond exactly to the columns of ψ_2 filled with 0's, and vice-versa. The enumeration of these pairs of weak 0–1 tableaux according to inversion and non-inversion numbers leads to

$$c^{\delta^m \mathfrak{A}}(n,k) = \sum_{i=k}^{n} p^{m(i-k)} [m]_{p,q}^{n-i} \mathcal{P}c_{q,p}[n,i,k], \tag{3.38}$$

$$S^{\delta^m \mathfrak{A}}(n,k) = \sum_{i=k}^{n} p^{m(i-k)} [m]_{p,q}^{n-i} \mathcal{P}S_{q,p}[n,i,k]. \tag{3.39}$$

Details are left to the reader. ■

Note that (3.34) to (3.37) are p,q-analogues of identities (1.11) to (1.14), thus providing a complete answer to the first question that was addressed to us.

4. **Identities related to 0–1 tableaux whose column lengths are restricted to an arithmetic progression.** In this section, we study $\mathcal{A}$-tableaux for which $w(a_i) = [a+id]_{p,q}$, $a,d \in \mathbb{N}$, the p,q-analogue of an arithmetic progression. This particular case can be realized in two different ways.

The first possibility $\mathcal{A}_{ap} = (a+id)_{i\geq 0}$ and $w_{ap}\colon \mathbb{N} \longrightarrow \mathbb{N}[p,q]$ such that $w_{ap}(k) = [k]_{p,q}$, corresponds combinatorially to the p,q-counting of 0–1 tableaux whose column lengths are part of the arithmetic progression $\{a+id\}_{i\geq 0}$. More precisely, fix a and $d \in \mathbb{N}$. We will denote by $T^{a,d}(h,r)$ (respectively $Td^{a,d}(h,r)$) the subset of $T(a+hd,r)$ (respectively $Td(a+hd,r)$) containing all 0–1 tableaux with column lengths belonging to the set $\{a, a+d, a+2d, \ldots, a+hd\}$. The notations $T(h,r)$ and $Td(h,r)$ were introduced in Section 1. Note that if $a=0$, since $[0]_{p,q}=0$, the column lengths will be elements of the set $\{d, 2d, \ldots, hd\}$. If $d=0$, the 0–1 tableaux are degenerate and contain only columns of length a. However, this case can be included if we distinguish between the different lengths $(a+i\cdot 0)$, $i \geq 0$. A way to do this is to associate to the 0–1 tableaux φ of format $a \times m$ partitions $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0$, corresponding to these different i's.

We then have

$$c^{(\mathcal{A}_{ap},w_{ap})}(n,k) = \sum_{\varphi \in Td^{a,d}(n-1,n-k)} p^{\mathrm{nin}(\varphi)} q^{\mathrm{inv}(\varphi)}, \tag{4.1}$$

and

$$S^{(\mathcal{A}_{ap},w_{ap})}(n,k) = \sum_{\varphi \in T^{a,d}(k,n-k)} p^{\mathrm{nin}(\varphi)} q^{\mathrm{inv}(\varphi)}. \tag{4.2}$$

The second possibility is $\mathfrak{U}_{cw} = (\mathcal{A}_{cw}, w_{cw})$, where $\mathcal{A}_{cw} = (i)_{i\geq 0}$ and

$$w_{cw}(k) = [a+kd]_{p,q} = p^{kd}[a]_{p,q} + q^a[d]_{p,q}[k]_{p^d,q^d}. \tag{4.3}$$

This decomposition of the weight function suggests a combinatorial interpretation in terms of p, q-counting of coloured weak 0–1 tableaux.

DEFINITION 4.1. A *coloured weak 0–1 tableau* consists of a pair (ψ, γ) such that $\psi = (\lambda, l, f)$ is a weak 0–1 tableau, and $\gamma\colon C_\psi \longrightarrow \{0, 1, 2, \ldots, a+d-1\}$ is a "colouring" of the columns c of ψ with colors $\gamma(c)$, satisfying the following conditions:

i) if the column c is filled with 0's, then $0 \leq \gamma(c) \leq (a-1)$, and

ii) if the filling of c contains a 1, then $a \leq \gamma(c) \leq (a+d-1)$.

By convention, if $a = 0$, the weak 0–1 tableau ψ contains no columns filled with 0's, and if $d = 0$, ψ contains no columns with 1's.

We will denote by $T_{cw}(h,r)$ (respectively $Td_{cw}(h,r)$) the set of all coloured weak 0–1 tableaux (ψ,γ) such that $\psi \in T_w(h,r)$ (respectively $\psi \in Td_w(h,r)$). The notations $T_w(h,r)$ and $Td_w(h,r)$ were introduced in Section 3.

DEFINITION 4.2. Let (ψ,γ) be a coloured weak 0–1 tableau. The *p, q-weight $w_{p,q}(\gamma)$ of the colouring* γ is given by the following expression:

$$w_{p,q}(\gamma) = \prod_{c\in C_\psi} w_p(c)w_q(c), \tag{4.4}$$

where C_ψ denotes the set of columns (of length possibly null) of the weak 0–1 tableau $\psi = (\lambda, l, f)$,

$$w_p(c) = \begin{cases} p^{a-1-\gamma(c)} & \text{if } c \text{ is filled with 0's,} \\ p^{a+d-1-\gamma(c)} & \text{if } c \text{ contains a 1;} \end{cases} \tag{4.5}$$

and

$$w_q(c) = q^{\gamma(c)}. \tag{4.6}$$

For example, if $a = 2$, $d = 3$, and ψ is the weak 0–1 tableau illustrated in Figure 2, a possible colouring γ of ψ would be $\gamma(c_1) = 0$, $\gamma(c_2) = 4$, $\gamma(c_3) = 4$, $\gamma(c_4) = 2$, $\gamma(c_5) = 1$, and $\gamma(c_6) = 1$, where $c_1, c_2, \ldots, c_6$ denote the columns of ψ from left to right, and the p, q-weight of γ is equal to $w_{p,q}(\psi,\gamma) = p^{1+0+0+2+0+0}q^{0+4+4+2+1+1} = p^3q^{12}$.

PROPOSITION 4.3. *With the preceding notations,*

$$c^{(\mathcal{A}_{cw},w_{cw})}(n,k) = \sum_{(\psi,\gamma)\in Td_{cw}(n-1,n-k)} p^{d\times \mathrm{nin}(\psi)}q^{d\times \mathrm{inv}(\psi)}w_{p,q}(\gamma), \tag{4.7}$$

and

$$S^{(\mathcal{A}_{cw},w_{cw})}(n,k) = \sum_{(\psi,\gamma)\in T_{cw}(k,n-k)} p^{d\times \mathrm{nin}(\psi)}q^{d\times \mathrm{inv}(\psi)}w_{p,q}(\gamma). \tag{4.8}$$

PROOF. We need only see that in

$$w_{cw}(k) = [a + kd]_{p,q} = p^{kd}[a]_{p,q} + q^a[d]_{p,q}[k]_{p^d,q^d},$$

the factors p^{kd} and $[k]_{p^d,q^d}$ correspond to d times the inversion and non-inversion numbers of weak 0–1 tableaux, and $[a]_{p,q}$ and $q^a[d]_{p,q}$ to the p,q-weight of colourings. ∎

There is a natural bijection preserving p,q-counting between coloured weak 0–1 tableaux and 0–1 tableaux whose column lengths lie in an arithmetic progression. Let $(\psi,\gamma) = \big((\lambda,l,f),\gamma\big) \in T_{cw}(h,r)$ (respectively $Td_{cw}(h,r)$). We map (ψ,γ) into a 0–1 tableau $\varphi = (\tilde{\lambda},\tilde{f}) \in T^{a,d}(h,r)$ (respectively $Td^{a,d}(h,r)$) in the following way: each column c_j in ψ of length i is replaced by a column of length $(a + id)$.

i) if the column c_j was filled with 0's and coloured with color $\gamma(c_j) = m$, $0 \leq m \leq (a-1)$, we put a 1 in position $(id+a-m)$ (from top to bottom) of the new column ($i.e.\ \tilde{f}_{id+a-m,j} = 1$), and

ii) if c_j had a 1 in position h, $1 \leq h \leq i$, and was coloured with color $\gamma(c_j) = m$, $a \leq m \leq (a+d-1)$, we put a 1 in position $(hd + a - m)$ ($i.e.\ \tilde{f}_{hd+a-m,j} = 1$).

For example, if $a = 2$, $d = 3$, $h = 5$ and $r = 6$, Figure 5 illustrates the correspondence. The first row (shaded in the figure) of the tableau on the left gives the different colors of the columns of the coloured tableau (which is the weak 0–1 tableau of Figure 2). Note that in this example, we have, as expected, $w_{p,q}(\gamma)p^{d\times \mathrm{nin}(\psi)}q^{d\times \mathrm{inv}(\psi)} = p^3q^{12}p^{3\times 8}q^{3\times 2} = p^{27}q^{18} = p^{\mathrm{nin}(\varphi)}q^{\mathrm{inv}(\varphi)}$. Details are left to the reader.

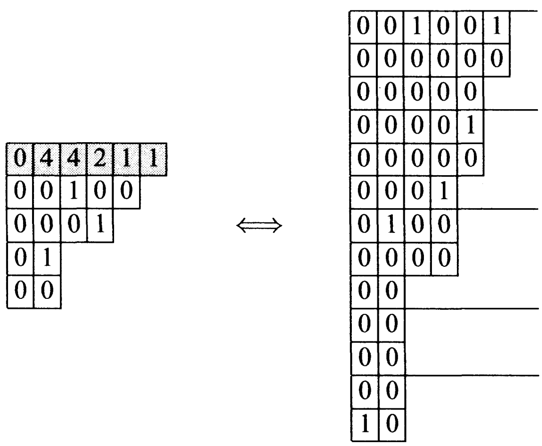


FIGURE 5: CORRESPONDENCE BETWEEN THE SETS $T_{cw}(h,r)$ AND $T^{a,d}(h,r)$

Let $c^{a,d}_{p,q}[n,k]$ and $S^{a,d}_{p,q}[n,k]$ denote

$$\begin{aligned} c^{a,d}_{p,q}[n,k] &= c^{(\mathcal{A}_{ap},w_{ap})}(n,k) \\ &= c^{(\mathcal{A}_{cw},w_{cw})}(n,k), \end{aligned} \tag{4.9}$$

and

$$S^{a,d}_{p,q}[n,k] = S^{(\mathcal{A}_{ap},w_{ap})}(n,k) \tag{4.10}$$
$$= S^{(\mathcal{A}_{cw},w_{cw})}(n,k).$$

We will call these polynomials *a, d-progressive p, q-Stirling numbers of first and second kinds*. As for the partial p, q-Stirling numbers, we can use the correspondence between weak 0–1 tableaux and permutations with marked cycles or partial set partitions described in Section 3 to obtain an alternate combinatorial interpretations in terms of these objects, enriched with some colouring.

In the remainder of this section, we present various identities on a, d-progressive p, q-Stirling numbers. We will use either combinatorial interpretations in terms of 0–1 tableaux to prove these identities, according to the context.

PROPOSITION 4.4. *a) (special cases)*

$$c^{0,d}_{p,q}[n,k] = [d]^{n-k}_{p,q} c_{p^d,q^d}[n,k], \tag{4.11}$$

$$S^{0,d}_{p,q}[n,k] = [d]^{n-k}_{p,q} S_{p^d,q^d}[n,k]. \tag{4.12}$$

In particular,

$$c^{0,1}_{p,q}[n,k] = c_{p,q}[n,k] \quad \textit{and} \quad S^{0,1}_{p,q}[n,k] = S_{p,q}[n,k]. \tag{4.13}$$

$$c^{a,0}_{p,q}[n,k] = [a]^{n-k}_{p,q} \binom{n}{k}, \tag{4.14}$$

$$S^{a,0}_{p,q}[n,k] = [a]^{n-k}_{p,q} \binom{n}{k}. \tag{4.15}$$

In particular,

$$c^{1,0}_{p,q}[n,k] = \binom{n}{k} = S^{1,0}_{p,q}[n,k]. \tag{4.16}$$

$$c^{1,1}_{p,q} = \sum_{m=k}^{n} q^{m-k} \mathcal{P} c_{p,q}[n,m,k], \tag{4.17}$$

$$S^{1,1}_{p,q} = \sum_{m=k}^{n} q^{m-k} \mathcal{P} S_{p,q}[n,m,k]. \tag{4.18}$$

b) (q-analogues of identities of Voigt [Vo])

$$c^{a,d}_{1,q}[n,j] = \sum_{k=j}^{n} \binom{k}{j} [a]^{k-j}_q q^{a(n-k)} c^{0,d}_{1,q}[n,k], \tag{4.19}$$

$$S^{a,d}_{1,q}[n,j] = \sum_{k=0}^{n-j} \binom{n}{k} [a]^{k}_q q^{a(n-k-j)} S^{0,d}_{1,q}[n-k,j]. \tag{4.20}$$

c) (connections with q-Stirling numbers)

$$c^{a,d}_{1,q}[n,j] = \sum_{k=j}^{n} \binom{k}{j} [a]^{k-j}_q (q^a[d]_q)^{n-k} c_{q^d}[n,k], \tag{4.21}$$

and

$$S^{a,d}_{1,q}[n,j] = \sum_{k=0}^{n-j} \binom{n}{k} [a]_q^k (q^a[d]_q)^{n-k-j} S_{q^d}[n-k,j]. \tag{4.22}$$

PROOF. a) For $a = 0$ and $d = 1$, both combinatorial models reduce to the usual p, q-counting (according to inversion and non-inversion numbers) of 0–1 tableaux in $Td(n-1, n-k)$ for $c^{0,1}_{p,q}[n,k]$ and in $T(k, n-k)$ for $S^{0,1}_{p,q}[n,k]$.

Identities (4.11) to (4.16) have simple proofs using either interpretations in terms of 0–1 tableaux. We will use the coloured weak 0–1 tableau model.

The choice $a = 0$ suppresses all weak 0–1 tableaux containing columns filled with 0's (since there are no "colors" to colour them with). Thus only usual 0–1 tableaux remain, for which each column is assigned a color between 0 and $(d-1)$. For a fixed 0–1 tableau φ, the p, q-weight of all possible colourings is $[d]_{p,q}^{\mathrm{nc}(\varphi)}$, where $\mathrm{nc}(\varphi)$ denotes the number of columns in φ. Moreover, since every inversion and non-inversion is counted d times, we obtain (4.11) and (4.12).

Choosing $d = 0$ forces the corresponding weak 0–1 tableaux to be filled with 0's only, each column being coloured with some color i between 0 and $(a-1)$. Note that since $d = 0$, inversion and non-inversion numbers of the weak 0–1 tableaux are not taken in account, and we are counting partitions (with $(n-k)$ columns of length possibly zero) that fit in a box, with some color associated to each column. Possible colourings correspond to the factor $[a]_{p,q}^{n-k}$ on the right-hand side of (4.14) and (4.15), and partitions (with distinct column lengths or not) to the factor $\binom{n}{k}$.

In the case $a = 1$ and $d = 1$, every column has a single choice of color, that is color 0 if it is filled with 0's, or color 1 if it contains a 1. Let (ψ, γ) be a coloured weak 0–1 tableau with such a colouring. Then it is easy to see that $w_{p,q}(\gamma) = q^j$, where j is equal to the number of columns in ψ containing a 1. Identities (4.17) and (4.18) follow from the combinatorial interpretation of partial p, q-Stirling numbers (with three parameters) in terms of weak 0–1 tableaux.

b) We want to isolate the contribution of parameter a in $c^{a,d}_{1,q}[n,j]$ and $S^{a,d}_{1,q}[n,j]$. Using the interpretation in terms of coloured weak 0–1 tableaux, this means separating the columns filled with 0's from the rest of the tableau, remembering their lengths and their positions (same reasoning as (2.28) and (2.29) or (3.30) to (3.33)). Next, we isolate the q-weight from their colourings (factor $[a]_q$ for each column filled with 0's), as well as the contribution q^a to the q-weight of the colouring of each column containing a 1. This yields to (4.19) and (4.20).

c) (4.21) is deduced from (4.19) and (4.11), and (4.22) is deduced from (4.20) and (4.12). ∎

The a, d-progressive Stirling numbers being a special case of $\mathfrak{A}$-Stirling numbers, all the results of Section 2 apply to them. In particular, we obtain the following convolution formulae from Theorem 2.10:

PROPOSITION 4.5 (CONVOLUTION FORMULAE).

$$c_{p,q}^{a,d}[m+j,n] = \sum_{k=0}^{n} c_{p,q}^{a,d}[m,k]c_{p,q}^{a+md,d}[j,n-k], \tag{4.23}$$

$$S_{p,q}^{a,d}[m+j,n] = \sum_{k=0}^{n} S_{p,q}^{a,d}[m,k]S_{p,q}^{a+kd,d}[j,n-k], \tag{4.24}$$

$$c_{p,q}^{a,d}[n+1,m+j+1] = \sum_{k=0}^{n} c_{p,q}^{a,d}[k,m]c_{p,q}^{a+(k+1)d,d}[n-k,j], \tag{4.25}$$

$$S_{p,q}^{a,d}[n+1,m+j+1] = \sum_{k=0}^{n} S_{p,q}^{a,d}[k,m]S_{p,q}^{a+(m+1)d,d}[n-k,j]. \tag{4.26}$$

Note that (4.24) and (4.26) are p,q-analogues of (6.24) and (6.25) of [VS].

PROPOSITION 4.6 (ANALOGUE OF COROLLARY 2.5).

$$c_{q,1}^{a,d}[n+1,k+1] = \sum_{i=0}^{1}[a]_{q,1}^{i} \sum_{j=k}^{n-i} \binom{j+i}{j-k} [d]_{q,1}^{j-k} q^{d(n-i-j)} c_{q,1}^{a,d}[n,j+i], \tag{4.27}$$

$$S_{q,1}^{a,d}[n+1,k+1] = \sum_{i=0}^{n-k}[a]_{q,1}^{i} \sum_{m=k}^{n-i} \binom{n-i}{m} [d]_{q,1}^{n-m-i} q^{d(m-k)} S_{q,1}^{a,d}[m,k]. \tag{4.28}$$

PROOF. The proof uses the same idea as the one of Corollary 2.5. Using the model in terms of 0–1 tableaux whose column lengths are part of an arithmetic progression $\{a+id\}_{i\geq 0}$, simply delete every column of length a and every column of length $a+jd$, $j \geq 1$, such that the 1 appears in one of the d first rows (from top to bottom). Details are left to the reader. ■

PROPOSITION 4.7 (GENERATING SERIES). *a) (J. B. Remmel and M. Wachs [ReWa])*

$$\sum_{n,k\geq 0} c_{1,1}^{a,d}[n,k]u^k \frac{x^n}{n!} = (1-dx)^{\frac{-(a+u)}{d}}, \tag{4.29}$$

$$\sum_{n,k\geq 0} S_{1,1}^{a,d}[n,k]u^k \frac{x^n}{n!} = e^{ax}e^{u(\frac{e^{dx}-1}{d})}. \tag{4.30}$$

b) (generalization of case $a=0$, $d=1$ due to I. Gessel [Ge, (6.1)])

$$\sum_{n\geq 0}\sum_{k\geq 0} c_{1,q}^{a,d}[n,k]u^k \frac{x^n}{[n]_{q^d}!} = \frac{(q^a[d]_q x; q^d)_\infty}{\Big(\big(q^a[d]_q + (1-q^d)([a]_q+u)\big)x; q^d\Big)_\infty}, \tag{4.31}$$

where $(p;q)_\infty = \prod_{i=0}^{\infty}(1-pq^i)$.

PROOF. a) These generating series are trivial using the alternate combinatorial models in terms of coloured permutations with marked cycles and coloured partial set partitions. Note that when $p = q = 1$, a and d can be considered as weights instead of colourings. On one hand, the parameter a accounts for the number of marked cycles in the permutations and the number of elements of E, in the partial set partitions $(E;\pi)$. On the other hand, d accounts for the number of elements of the permutations that are not cycle leaders, and the number of elements in the partial set partitions that are not in E nor a block leader. Details are left to the reader.

b) Identity (4.31) is obtained by algebraic manipulations on q-series. Let

$$F(u,t)=\sum_{n\geq 0}\sum_{k\geq 0}c_{1,q}^{a,d}[n,k]u^k\frac{t^n}{(q^d;q^d)_n}, \tag{4.32}$$

where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$. On one hand,

$$G(u,t)=\sum_{n\geq 0}\sum_{k\geq 0}c_{1,q}^{a,d}[n,k]u^k\frac{t^{n-1}}{(q^d;q^d)_{n-1}} \tag{4.33}$$

$$=\frac{1}{t}\big(F(u,t)-F(u,q^dt)\big). \tag{4.34}$$

And on the other hand, by replacing $c_{1,q}^{a,d}[n,k]$ in (4.33) by the right-hand side of the basic recurrence relation

$$c_{1,q}^{a,d}[n,k]=c_{1,q}^{a,d}[n-1,k-1]+[a+(n-1)d]_{1,q}c_{1,q}^{a,d}[n-1,k],$$

we compute

$$G(u,t)=[a]_qF(u,t)+q^a\frac{[d]_q}{(1-q^d)}\big(F(u,t)-F(u,q^dt)\big)+uF(u,t). \tag{4.35}$$

Identities (4.34) and (4.35) lead to the functional equation

$$F(u,t)=\frac{\left(1-\frac{q^d[d]_qt}{(1-q^d)}\right)}{\left(1-[a]_qt-\frac{q^a[d]_qt}{(1-q^d)}-ut\right)}F(u,q^dt). \tag{4.36}$$

Finally, iterating (4.36) and replacing t by $(1-q^d)x$ gives the generating series (4.31). ∎

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School of Mathematics
University of Minnesota
Minneapolis, Minnesota
U.S.A.

LACIM
Université du Québec à Montréal,
Montréal, Québec