



# Epsilon constants and orthogonal representations

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## ABSTRACT

In this paper we suppose  $G$  is a finite group acting tamely on a regular projective curve  $\mathcal{X}$  over  $\mathbb{Z}$  and  $V$  is an orthogonal representation of  $G$  of dimension 0 and trivial determinant. Our main result determines the sign of the  $\epsilon$ -constant  $\epsilon(\mathcal{X}/G, V)$  in terms of data associated to the archimedean place and to the crossing points of irreducible components of finite fibers of  $\mathcal{X}$ , subject to certain standard hypotheses about these fibers.

## 1. Introduction

This section will state the main questions and results of the paper, specify notation, and give some background. Let  $\mathcal{X}$  be an arithmetic scheme of dimension  $d + 1$  which is flat, regular, and projective over  $\mathbb{Z}$ . We suppose that  $f : \mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$  is the structure morphism and that its fibers are all of dimension  $d$ . Let  $G$  be a finite group which acts tamely on  $\mathcal{X}$  in the sense that, for each closed point  $x \in \mathcal{X}$ , the order of the inertia group of  $x$  is relatively prime to the residue characteristic of  $x$ . Define  $\mathcal{Y}$  to be the quotient scheme  $\mathcal{X}/G$ . We assume that  $\mathcal{Y}$  is regular, and that for all finite places  $v$  the fiber  $\mathcal{Y}_v = (\mathcal{X}_v)/G = \mathcal{Y} \otimes_{\mathbb{Z}} (\mathbb{Z}/p(v))$  has normal crossings and smooth irreducible components with multiplicities relatively prime to the residue characteristic of  $v$ . Finally, let  $V$  be a representation of  $G$  over  $\overline{\mathbb{Q}}$ .

Associated to these data, there are well-known  $\zeta$ -functions and  $L$ -functions, both functions of a complex variable  $s$ . The  $L$ -function is conjectured to have a functional equation of the form

$$L(s, \mathcal{Y}, V) = \epsilon(\mathcal{Y}, V) A(\mathcal{Y}, V)^{-s} L(d + 1 - s, \mathcal{Y}, V^*)$$

in which  $A(\mathcal{Y}, V)$  is a positive integer called the conductor,  $V^*$  is the dual representation of  $V$  and the  $\epsilon$ -constant  $\epsilon(\mathcal{Y}, V)$  is a non-zero algebraic number. In recent years, many authors have studied the problem of determining these  $\epsilon$ -constants, which may be defined unconditionally after we choose an auxiliary prime  $\ell$ .

This paper concerns the case where  $V$  is an orthogonal representation, meaning that there is a non-degenerate symmetric  $G$ -invariant bilinear form  $V \times V \rightarrow \overline{\mathbb{Q}} \subseteq \mathbb{C}$  (where we fix an embedding of  $\overline{\mathbb{Q}}$  into  $\mathbb{C}$ ). In order to get the strongest results, we will furthermore make the technical hypotheses that  $V$  is a virtual representation of trivial determinant and dimension 0. In other words,  $V$  will be a linear combination of orthogonal representations such that the weighted sum of their dimensions is zero and the product of their determinants is trivial.

We can now state in general terms the main result of this paper.

**THEOREM 1.1.** *If  $d = 1$  and  $V$  is an orthogonal virtual representation of degree 0 and trivial determinant, then the sign of the constant  $\epsilon(\mathcal{Y}, V) \in \mathbb{R}^*$  can be determined from the  $\epsilon$ -constant  $\epsilon_\infty(\mathcal{Y}, V)$  and from the restriction of the  $G$ -cover  $\mathcal{X} \rightarrow \mathcal{Y}$  over the finite set of closed points  $z$  of  $\mathcal{Y}$  where two distinct irreducible components of a fiber of  $\mathcal{Y}$  over  $\text{Spec}(\mathbb{Z})$  intersect.*

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The constant  $\epsilon_\infty(\mathcal{Y}, V)$  which comes up in this formula is the archimedean  $\epsilon$ -constant defined by Deligne [Del73, § 8] using the action of the group  $G$  and of complex conjugation on the Hodge cohomology groups  $H^{p,q}(\mathcal{X}, \mathbb{C})$ . Section 2 of this paper recalls this and other definitions of  $\epsilon$ -constants, as well as work done by Deligne, Fröhlich, Queyrut, Chinburg, Erez, Pappas, and Taylor in computing  $\epsilon$ -constants associated to situations similar to those in Theorem 1.1. In § 3, we make a more precise statement of the main theorem and prove it. The proof uses formulae of Saito, class-field theory, and several of the results discussed in § 2.

## 2. Background

In this section, we look at some of the work that others have done in order to compute  $\epsilon$ -constants in various situations.

Fröhlich and Queyrut look at computing  $\epsilon$ -constants in the case where  $\mathcal{X}$  and  $\mathcal{Y}$  are of relative dimension 0 over  $\mathbb{Z}$  and  $V$  is an orthogonal representation. In [FQ73] they are able to prove the following result.

**THEOREM 2.1.** *If  $d = 0$  and  $V$  is an orthogonal representation of  $G$ , then  $\epsilon(\mathcal{Y}, V)$  is positive.*

We now recall some elements of Deligne’s theory of local constants, which are essential for our work.

**DEFINITION 2.2.** Let  $\mathcal{X}$  and  $\mathcal{Y} = \mathcal{X}/G$  be as above. Let  $V$  be any virtual complex representation of  $G$ .

- a) Let  $\epsilon_{v,0}(\mathcal{Y}, V)$  be the Deligne local constant defined in [CEPT97a] (see also [Del73]). In particular, the definition of  $\epsilon_{v,0}(\mathcal{Y}, V)$  requires that one chooses an auxiliary prime  $\ell \neq v$ , a non-trivial continuous complex character of  $\mathbb{Q}_v$  which we denote by  $\psi_v$ , and a Haar measure  $dx_v$  on  $\mathbb{Q}_v$ . In the case where  $V$  has trivial determinant and is of dimension 0, then  $\epsilon_{v,0}(\mathcal{Y}, V)$  is independent of these choices (see Proposition 2.4.1 of [CEPT97a]). This term is well defined for  $v = \infty$  as well as for finite places  $v$ .
- b) Let  $X$  be a variety of dimension  $d$  which is defined over a finite field of characteristic  $p$ . Let  $\ell$  be a prime different from  $p$  and let  $j_\ell : \overline{\mathbb{Q}_\ell} \rightarrow \mathbb{C}$  be an embedding. Finally, define  $V_\ell$  to be a virtual representation of  $G$  over  $\overline{\mathbb{Q}_\ell}$  such that  $j_\ell(\chi_{V_\ell}) = \chi_V$ . Define

$$\epsilon(X, V) = j_\ell(\det(-F|(H_{\text{ét}}^*(\overline{\mathbb{F}_p} \times_{\mathbb{F}_p} X, \mathbb{Q}_\ell) \otimes V_\ell^*)^G)),$$

where  $F$  is the geometric Frobenius automorphism. This number is independent of all choices.

- c) For finite places  $v$  of  $\mathbb{Q}$ , we let  $\epsilon_v(\mathcal{Y}, V) = \epsilon_{v,0}(\mathcal{Y}, V)\epsilon(\mathcal{Y}_v, V)$ , where  $\epsilon(\mathcal{Y}_v, V)$  is defined as an  $\epsilon$ -constant over a finite field. Furthermore, in the case where  $v = \infty$ , we let  $\epsilon(\mathcal{Y}_v, V) = 1$  so that in particular  $\epsilon_\infty(\mathcal{Y}, V) = \epsilon_{\infty,0}(\mathcal{Y}, V)$ .
- d) The global  $\epsilon$ -constant associated to  $V$  is defined by  $\epsilon(\mathcal{Y}, V) = \prod_v \epsilon_v(\mathcal{Y}, V)$  where the product is over all places  $v$  of  $\mathbb{Q}$ .

The  $\epsilon$ -constants associated to varieties defined over finite fields are studied by Chinburg, Erez, Pappas, and Taylor in [CEPT97b]. Other papers by these authors, such as [CEPT97a] and [CEPT98], prove results on computing  $\epsilon$ -constants associated to arithmetic schemes in the case where  $V$  is a symplectic representation. Recall that a symplectic representation is a representation  $V$  which is equipped with a non-degenerate alternating  $G$ -invariant bilinear form.

For many applications of  $\epsilon$ -constants it is not the actual  $\epsilon$ -constant we are interested in computing but merely the sign of this constant. We denote the sign of  $\epsilon(\mathcal{Y}, V)$  by  $W(\mathcal{Y}, V)$  and call this the root number of  $V$ .

### 3. Main results

#### 3.1 Reduction to fibral computations

Let  $\mathcal{X}$ ,  $G$ ,  $\mathcal{Y} = \mathcal{X}/G$  be as in § 1. Let  $S$  be the set of all finite places  $v$  of  $\mathbb{Q}$  where either the fiber  $\mathcal{Y}_v = \mathcal{Y} \otimes_{\mathbb{Z}} (\mathbb{Z}/p(v))$  is not smooth or the map  $\pi : \mathcal{X} \rightarrow \mathcal{Y}$  is ramified. Let  $D'$  be a horizontal divisor on  $\mathcal{Y}$  such that  $D' + \mathcal{Y}_T = K_{\mathcal{Y}} + \mathcal{Y}_S^{\text{red}}$ , where  $K_{\mathcal{Y}}$  is a canonical divisor on  $\mathcal{Y}$ ,  $\mathcal{Y}_S^{\text{red}}$  is the sum of the reductions of the fibers of  $\mathcal{Y}$  at the places in  $S$ ,  $T$  is a finite set of finite places of  $\mathbb{Q}$  which is disjoint from  $S$ , and  $\mathcal{Y}_T$  is the sum of the (necessarily reduced) fibers of  $\mathcal{Y}$  over the places in  $T$ . Thus  $\mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T)$  is isomorphic to the twist  $\omega_{\mathcal{Y}/\mathbb{Z}}(\mathcal{Y}_S^{\text{red}})$  of the relative dualizing sheaf  $\omega_{\mathcal{Y}/\mathbb{Z}}$  by  $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}})$ . We further wish to choose  $D'$  so that it intersects the non-smooth fibers  $\mathcal{Y}_v$  of  $\mathcal{Y}$  transversally at smooth points on the reduction of  $\mathcal{Y}_v$ . We can choose such a  $D'$  after a suitable base change due to the moving lemma proven as Proposition 9.1.3 in [CEPT97a]. The choice of this canonical divisor is not unique, but our calculation will show that the results are independent of the choice of  $D'$ . Note that we can define  $D$  to be the horizontal divisor on  $\mathcal{X}$  such that  $D' = D/G$  and in doing so we can define  $\epsilon$ -constants associated to  $D'$  as in Definition 2.2.

As stated above, we can only choose a horizontal divisor  $D'$  with the desired properties after a suitable base change. Thus, we need to consider how base changes will affect the  $\epsilon$ -constants. To be precise about how we make the base change, we will choose an odd prime  $\ell$  which is not in the set of bad primes  $S$ , and we denote by  $N_{\infty}$  the cyclotomic  $\mathbb{Z}_{\ell}$  extension of  $\mathbb{Q}$ . Because we have chosen  $\ell \notin S$ , this base extension is étale over  $S$ , and the pullback of a canonical divisor remains canonical up to a multiple of the fiber of  $\mathcal{Y}$  over  $\ell$ . Proposition 9.1.3 of [CEPT97a] shows that a horizontal divisor  $D'$  with the required properties exists after a base extension to the ring of integers of a finite extension of  $\mathbb{Q}$  inside  $N_{\infty}$ . This base extension, which we now fix, is of degree a power of  $\ell$ . Since  $\ell$  is not in the set  $S$ , the Hasse–Davenport Theorem together with Lemma 9.4.1 of [CEPT97a] show that the epsilon constants we will consider for the base change are the  $\ell^a$ th powers of the corresponding constants before the base change. Because we are primarily interested in the sign of the  $\epsilon$ -constant, we are free to make a base change of the above kind. If we were interested in preserving more information about  $\epsilon$ , we would place a stricter congruence condition on the prime  $\ell$ .

LEMMA 3.1. *For the infinite place,  $\epsilon_{\infty,0}(D', V) = 1$ .*

This lemma is an immediate corollary to Proposition 5.4.2 of [CEPT97a]. In particular, this proposition says that if  $d$  is odd then the archimedean epsilon constant associated to the canonical divisor and to any representation  $V$  of trivial determinant and dimension 0 is equal to one.  $D'$  differs from the canonical divisor only by vertical fibers, and thus the result applies.

LEMMA 3.2. *With  $\mathcal{Y}$ ,  $D'$ , and  $V$  chosen as above,  $\epsilon_{v,0}(\mathcal{Y}, V) = \epsilon_{v,0}(D', V)$  for all finite places  $v$  of  $\mathbb{Q}$ .*

*Proof.* For all places  $v \in S$ , this follows directly from [CEPT97a]. However, one can generalize their results in order to prove the lemma. In order to do this, we let  $\mathcal{C}_v$  be the set of irreducible components of  $\mathcal{Y}_v^{\text{red}}$ . For each  $C_i \in \mathcal{C}_v$  let  $\kappa_i$  be the Gauss sum associated to the restriction of the representation  $V$  to the inertia group of the generic point of  $C_i$  as defined in [CEPT97a]. They define  $c_i$  to be the  $\ell$ -adic Euler characteristic with compact support of the open subscheme of  $C_i$  consisting of points which are non-singular in  $\mathcal{Y}_v^{\text{red}}$ . The formulae developed by Saito as Theorems 1 and 2 of [Sai93] imply that  $\epsilon_{v,0}(\mathcal{X}, V) = \prod_{i \in \mathcal{C}_v} \kappa_i(V)^{c_i}$ .

For each  $C_i$  we compute as in § 9 of [CEPT97a] that  $\deg_{C_i}(\mathcal{O}_{\mathcal{Y}}(K_{\mathcal{Y}} + \mathcal{Y}_S^{\text{red}})) = -c_i f_i$ , where  $f_i$  is the index of the constant field extension  $[F_i : \mathbb{F}_p]$ . Changing views, we let  $\delta'$  be a point where  $\mathcal{Y}_v^{\text{red}}$  intersects the horizontal divisor  $D'$ . We define Gauss sums  $\kappa_{\delta'}$  in a similar way to the above defined  $\kappa_i$ , such that, in particular,  $\kappa_{\delta'} = \kappa_i^{[k(\delta'):F_i]}$ . Furthermore, the local epsilon constant

$\epsilon_{v,0}(D', V)$  is given by  $\prod_{\delta' \in D' \cap \mathcal{Y}_v^{\text{red}}} \kappa_{\delta'}$  (see [Sai93, p. 416]). The proof of the lemma in this case now reduces to counting intersection numbers and verifying that  $\kappa_i$  occurs as a factor the same number of times in both  $\epsilon_{v,0}(\mathcal{Y}, V)$  and  $\epsilon_{v,0}(D', V)$ .

For the finite places  $v$  which are not in  $S$  the argument is similar. It is only the intersection multiplicities of  $D'$  with certain vertical divisors that matter, and these numbers do not change in the event that we add new vertical fibers into the divisors. For this reason, the appearance of  $\mathcal{Y}_T$  in the equality  $D' + \mathcal{Y}_T = K_{\mathcal{Y}} + \mathcal{Y}_S^{\text{red}}$  makes no difference in the argument.  $\square$

With these lemmas in hand, we can make the following series of calculations:

$$\begin{aligned} \epsilon(\mathcal{Y}, V) &= \prod_v \epsilon_v(\mathcal{Y}, V) \\ &= \epsilon_{\infty}(\mathcal{Y}, V) \prod_{v \text{ finite}} \epsilon_{v,0}(\mathcal{Y}, V) \epsilon(\mathcal{Y}_v, V) \\ &= \epsilon_{\infty}(\mathcal{Y}, V) \prod_{v \text{ finite}} \epsilon_{v,0}(D', V) \epsilon(\mathcal{Y}_v, V) \\ &= \epsilon_{\infty}(\mathcal{Y}, V) \epsilon_{\infty,0}(D', V) \prod_{v \text{ finite}} \epsilon_{v,0}(D', V) \epsilon(D'_v, V) \epsilon(D'_v, V)^{-1} \epsilon(\mathcal{Y}_v, V) \\ &= \epsilon(D', V) \epsilon_{\infty,0}(\mathcal{Y}, V) \prod_{v \text{ finite}} \epsilon(D'_v, V)^{-1} \epsilon(\mathcal{Y}_v, V). \end{aligned} \tag{1}$$

In these calculations,  $D'_v = D' \otimes_{\mathbb{Z}} \mathbb{Z}/p(v)$  is the finite collection of closed points of  $D'$  lying above the finite place  $v$  of  $\mathbb{Q}$ .

LEMMA 3.3. *The epsilon constant  $\epsilon(D', V)$  is positive.*

*Proof.* We have that  $D'$  is a one-dimensional object, and the restriction of  $V$  to  $D'$  will still be an orthogonal representation. By applying the theorem of Fröhlich and Queyruet to the normalization of  $D'$  (which we denote by  $(D')^{\#}$ ), we get that  $\epsilon((D')^{\#}, V)$  is positive. Now, because the definition of local constants involves only the Galois action on general fibers,  $\epsilon_{v,0}((D')^{\#}, V) = \epsilon_{v,0}(D', V)$ . Thus, we are only concerned with the difference between the terms  $\epsilon((D')^{\#}_v, V)$  and  $\epsilon(D'_v, V)$ , all of which come about from the singular points  $z$  of  $D'$ . The action of  $G$  is étale at these points, and thus we can compute the local constants at these points as  $\epsilon(y, V) = \det(-F|(H^0(y, \mathbb{Q}_{\ell}) \otimes V)^{G_x}) = \det(V)(\pi_{\mathcal{Y}_v^{\text{red}}, y})$ , which is equal to one due to our hypotheses that  $V$  has trivial determinant.  $\square$

Thus, we have reduced the calculation of the sign of  $\epsilon(\mathcal{Y}, V)$ , which is an inherently two-dimensional calculation, to a collection of fibral computations  $\epsilon(D'_v, V)^{-1} \epsilon(\mathcal{Y}_v, V)$  for each finite place  $v$ , and a calculation for the archimedean component  $\epsilon_{\infty,0}(\mathcal{Y}, V)$ .

### 3.2 The one-component case

THEOREM 3.4. *Let  $\mathcal{X}, \mathcal{Y}, D'$  be as above and let  $V$  be an orthogonal representation of trivial determinant and zero dimension. Furthermore, assume  $v$  is a finite place of  $\mathbb{Q}$  such that  $\mathcal{Y}_v^{\text{red}}$  is irreducible. Then  $\epsilon(D'_v, V)^{-1} \epsilon(\mathcal{Y}_v, V) = 1$ .*

*Proof.* Assume that  $\mathcal{Y}_v^{\text{red}}$  consists of a single component. Then  $\mathcal{Y}_v^{\text{red}}$  is smooth by hypothesis. Let  $c$  be an irreducible component of  $\mathcal{X}_v$  with generic point  $\mu_c$ . Let  $G_{\mu_c}$  be the Galois group acting on the generic point of  $c$ , and  $I_{\mu_c}$  be the inertia group at the generic point of  $c$ . Then we have that  $I_{\mu_c} \subseteq G_{\mu_c} \subseteq G$ . We denote  $I_{\mu_c}$  by  $I$ . The tameness hypothesis implies that the order of  $I$  is relatively prime to  $v$ , and that  $I$  is a cyclic group. The specific structure of  $I$  is discussed in detail in the Appendix to [CEPT97a].

We begin by computing

$$\epsilon(\mathcal{Y}_v, V) = \prod_i \det \left( -F \left| \left( H^i \left( \overline{\mathbb{Z}/v\mathbb{Z}} \otimes_{\mathbb{Z}/v\mathbb{Z}} \mathcal{Y}_v, \mathbb{Q}_\ell \right) \otimes V \right)^G \right. \right)^{(-1)^{i+1}},$$

where  $F$  is the Frobenius element as described above. We know by our hypotheses that the cover  $\mathcal{X}_v^{\text{red}} \rightarrow \mathcal{Y}_v^{\text{red}}$  is a tame  $G_{\mu_c}/I$ -cover of smooth curves over  $\mathbb{Z}/p\mathbb{Z}$ . Furthermore, the action of  $G/I$  on  $\mathcal{X}_v^{\text{red}}$  is étale because  $I = I_{\mu_c} = I_{\mathcal{X},x}$ , the inertia group of the point  $x$ , for all points  $x \in \mathcal{X}_v^{\text{red}}$ . This implies that  $\epsilon(\mathcal{Y}_v, V) = \epsilon(\mathcal{Y}_v^{\text{red}}, V^I)$ . The group  $I$  acts trivially on the cohomology group  $H^*(\mathcal{X}_v^{\text{red}}, \mathbb{Q}_\ell)$ , so Saito’s formulae [Sai93] imply that  $\epsilon(\mathcal{Y}_v, V)$  can be calculated as  $\det(V^I)(K_{\mathcal{Y}_v^{\text{red}}})$ , where  $K_{\mathcal{Y}_v^{\text{red}}}$  is the canonical divisor on  $\mathcal{Y}_v^{\text{red}}$ . The terms  $K_{\mathcal{Y}_v^{\text{red}}}$  are well defined, as we have assumed that for all finite  $v$  the irreducible components of  $\mathcal{Y}_v^{\text{red}}$  are themselves smooth.

Next we look at the term  $\epsilon(D'_v, V)$ . Let  $D$  be the preimage of  $D'$  in  $\mathcal{X}$ , and let  $\ell$  be a prime different from  $v$ . Let  $I_{D,x}$  be the cyclic inertia group of a point  $x$  lying above the points in  $\mathcal{Y}_v \cap D'$  (note that this is independent of which point  $x$  we choose). Because  $D'_v$  is zero-dimensional, we know that  $\epsilon(D'_v, V) = \prod_{y \in D'_v} \epsilon(y, V)$ , where

$$\epsilon(y, V) = \det(-F | (H^0(\pi^{-1}(y)^{\text{red}}, \mathbb{Q}_\ell) \otimes V)^G),$$

if we view  $\pi$  as the cover  $D_v \rightarrow D'_v$ . We know that

$$\pi^{-1}(y)^{\text{red}} = (y \times_{D'_v} D_v)^{\text{red}} = x \times_{G_x} G.$$

In particular, this implies that

$$H^0(\pi^{-1}(y)^{\text{red}}, \mathbb{Q}_\ell) \otimes V = (\text{Ind}_{G_x}^G H^0(x, \mathbb{Q}_\ell)) \otimes V.$$

We know that  $I_{\mu_c} = I_{\mathcal{X},x}$  for all points  $x$ , and furthermore that  $I_{\mathcal{X},x}$  is a normal subgroup of  $G$  because there exists a unique irreducible component of  $\mathcal{X}_v^{\text{red}}$ . This allows us to compute that

$$\text{Ind}_{G_x}^G H^0(x, \mathbb{Q}_\ell) = \text{Infl}_{G/I_{\mathcal{X},x}}^G \text{Ind}_{G_x/I_{\mathcal{X},x}}^{G/I_{\mathcal{X},x}} H^0(x, \mathbb{Q}_\ell).$$

Recall that  $I = I_{\mathcal{X},x}$  acts trivially on  $H^0(x, \mathbb{Q}_\ell)$ . This allows us to compute that

$$\begin{aligned} \epsilon(y, V) &= \det(-F | (H^0(\pi^{-1}(y)^{\text{red}}, \mathbb{Q}_\ell) \otimes V)^G) \\ &= \det(-F | (\text{Infl}_{G/I}^G \text{Ind}_{G_x/I}^{G/I} H^0(x, \mathbb{Q}_\ell) \otimes V)^G) \\ &= \det(-F | (\text{Ind}_{G_x/I}^{G/I} H^0(x, \mathbb{Q}_\ell) \otimes V^I)^{G/I}) \\ &= \epsilon(y, V^I), \end{aligned}$$

where  $\epsilon(y, V^I)$  is the local constant associated to the  $G/I$ -cover  $\mathcal{X}_v^{\text{red}} \rightarrow \mathcal{Y}_v^{\text{red}}$ .

This last term is in turn equal to  $\det(V^I)(\pi_{\mathcal{Y}_v^{\text{red}},y})$ , where  $\pi_{\mathcal{Y}_v^{\text{red}},y}$  is the local uniformizer from class-field theory since  $\mathcal{X}_v^{\text{red}} \rightarrow \mathcal{Y}_v^{\text{red}}$  is an unramified  $G/I$ -cover. Finally, we can put these terms together to get that  $\epsilon(D'_v, V) = \det(V^I)(D' \cap \mathcal{Y}_v^{\text{red}})$ , where  $D' \cap \mathcal{Y}_v^{\text{red}}$  is viewed as a divisor on  $\mathcal{Y}_v^{\text{red}}$ .

LEMMA 3.5. *Under the above hypotheses,  $D' \cap \mathcal{Y}_v^{\text{red}}$  is a canonical divisor on  $\mathcal{Y}_v^{\text{red}}$ .*

Given this lemma, we will have shown that  $\epsilon(D'_v, V) = \det(V^I)(K) = \epsilon(\mathcal{Y}_v, V)$ , so in particular  $\epsilon(D'_v, V)^{-1} \epsilon(\mathcal{Y}_v, V) = 1$ , and Theorem 3.4 will be proven.

In order to prove Lemma 3.5, recall that we chose  $D'$  so that  $\mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T) = \omega_{\mathcal{Y}/\mathbb{Z}}(\mathcal{Y}_S^{\text{red}})$ . We note that if we look at the two exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_{\mathcal{Y}}(-\mathcal{Y}_v^{\text{red}}) \rightarrow \mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{O}_{\mathcal{Y}_v^{\text{red}}} \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}_{\mathcal{Y}}(D' - \mathcal{Y}_v^{\text{red}}) \rightarrow \mathcal{O}_{\mathcal{Y}}(D') \rightarrow \mathcal{O}_{\mathcal{Y}}(D')|_{\mathcal{Y}_v^{\text{red}}} \rightarrow 0, \end{aligned}$$

we get that, for all primes  $v$ ,  $\mathcal{O}_{\mathcal{Y}}(D')|_{\mathcal{Y}_v^{\text{red}}}$  is the same as  $\mathcal{O}_{\mathcal{Y}_v^{\text{red}}}(D' \cap \mathcal{Y}_v^{\text{red}})$ . Furthermore, for those primes  $v$  which are in  $S$  (and in particular are not in  $T$ ), we further get that  $\mathcal{O}_{\mathcal{Y}}(D')|_{\mathcal{Y}_v^{\text{red}}} = \mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T)|_{\mathcal{Y}_v^{\text{red}}}$ . We now are able to make the following computation for all  $v \in S$  such that  $\mathcal{Y}_v^{\text{red}}$  is irreducible:

$$\begin{aligned} \mathcal{O}_{\mathcal{Y}_v^{\text{red}}}(D' \cap \mathcal{Y}_v^{\text{red}}) &= \mathcal{O}_{\mathcal{Y}}(D')|_{\mathcal{Y}_v^{\text{red}}} \\ &= \mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T)|_{\mathcal{Y}_v^{\text{red}}} \\ &= \omega_{\mathcal{Y}/\mathbb{Z}}(\mathcal{Y}_S^{\text{red}})|_{\mathcal{Y}_v^{\text{red}}} \\ &= \omega_{\mathcal{Y}/\mathbb{Z}}(\mathcal{Y}_v^{\text{red}})|_{\mathcal{Y}_v^{\text{red}}} \\ &= \omega_{\mathcal{Y}_v^{\text{red}}}. \end{aligned}$$

In other words, for such  $v$ ,  $D' \cap \mathcal{Y}_v^{\text{red}}$  is a canonical divisor on  $\mathcal{Y}_v^{\text{red}}$  under these assumptions.

It remains to show that Lemma 3.5 holds for primes  $w$  outside of the set  $S$ . We know that, for such  $w$ , the fibers  $\mathcal{Y}_w$  are reduced and smooth and that the local equations have a nice form. This implies in particular that  $\mathcal{Y}_w^{\text{red}} = \mathcal{Y}_w$  is a principal divisor and thus that  $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}_w^{\text{red}})$  is isomorphic to  $\mathcal{O}_{\mathcal{Y}}$ .

Recall that by definition we have that  $D' + \mathcal{Y}_T = K_{\mathcal{Y}} + \mathcal{Y}_S^{\text{red}}$ . This tells us that

$$D' + \mathcal{Y}_T - \mathcal{Y}_S^{\text{red}} + \mathcal{Y}_w^{\text{red}} = K_{\mathcal{Y}/\mathbb{Z}} + \mathcal{Y}_w^{\text{red}}$$

and therefore that

$$\mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T - \mathcal{Y}_S^{\text{red}} + \mathcal{Y}_w^{\text{red}})|_{\mathcal{Y}_w^{\text{red}}} = \omega_{\mathcal{Y}/\mathbb{Z}}(\mathcal{Y}_w^{\text{red}})|_{\mathcal{Y}_w^{\text{red}}}.$$

The right-hand side is equal to  $\omega_{\mathcal{Y}_w^{\text{red}}}$  by the adjunction formula. To calculate the left-hand side, we observe that  $w$  is not in  $S$  by hypothesis, although it may be in  $T$ . Thus there exists an integer  $m$  depending on the multiplicity of  $w$  in  $T$  for which the following calculations hold:

$$\begin{aligned} \mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T - \mathcal{Y}_S^{\text{red}} + \mathcal{Y}_w^{\text{red}})|_{\mathcal{Y}_w^{\text{red}}} &= \mathcal{O}_{\mathcal{Y}}(D' + m\mathcal{Y}_w^{\text{red}})|_{\mathcal{Y}_w^{\text{red}}} \\ &= \mathcal{O}_{\mathcal{Y}}(D')|_{\mathcal{Y}_w^{\text{red}}} \otimes \mathcal{O}_{\mathcal{Y}}((\mathcal{Y}_w^{\text{red}})^{\otimes m})|_{\mathcal{Y}_w^{\text{red}}} \\ &\cong \mathcal{O}_{\mathcal{Y}}(D')|_{\mathcal{Y}_w^{\text{red}}} \otimes \mathcal{O}_{\mathcal{Y}} \\ &\cong \mathcal{O}_{\mathcal{Y}}(D')|_{\mathcal{Y}_w^{\text{red}}} \\ &= \mathcal{O}_{\mathcal{Y}}(D' \cap \mathcal{Y}_w^{\text{red}}), \end{aligned}$$

which proves Lemma 3.5 and therefore Theorem 3.4. □

*Remark 3.6.* Note that we can identify  $\det(V^I)$  with a character of order 1 or 2 of  $\text{Pic}_{\text{Weil}}(\mathcal{Y}_v^{\text{red}})$ . Thus,  $\epsilon(\mathcal{Y}_v^{\text{red}}, \det(V^I)) = 1$ , as it is the ratio of epsilon constants associated to zeta functions. In particular we can show that both  $\epsilon(\mathcal{Y}_v, V)$  and  $\epsilon(D', V)$  are trivial, which would give us another way of proving Theorem 3.4. However, for what follows it is more illuminating instead to consider what their ratio is, and in particular how close each is to being of the form  $\det(V^I)(K)$ .

### 3.3 Partial trivializations and the canonical cycles

In this section we will describe in detail the relative canonical cycle associated to line bundles with partial trivializations, as defined by Saito [Sai93], as well as other machinery which we will need in order to compute the terms  $\epsilon(D'_v, V)^{-1}\epsilon(\mathcal{Y}_v, V)$  in the case where  $\mathcal{Y}_v^{\text{red}}$  consists of more than one component.

**DEFINITION 3.7.** Let  $\mathcal{D}$  be a divisor on a scheme  $X$  and let  $\{\mathcal{D}_i\}_{i \in I}$  be the set of irreducible components of  $\mathcal{D}$ . A locally free sheaf  $\mathcal{E}$  on  $X$  is said to be partially trivialized on  $\mathcal{D}$  if there exists a family  $\rho = (\rho_i)$  of surjective  $\mathcal{O}_{\mathcal{D}_i}$ -morphisms  $\rho_i : \mathcal{E}|_{\mathcal{D}_i} \rightarrow \mathcal{O}_{\mathcal{D}_i}$ .

Given a partial trivialization of the sheaf  $\mathcal{E}$  of rank  $n$  on  $X$ , Saito defines the relative top Chern class  $c_n(\mathcal{E}, \rho) \in H^{2n}(X \bmod \mathcal{D}, \mathbb{Z}_q(n))$  based on an idea of Anderson [And92]. In particular, Saito notes that there is a canonical isomorphism

$$\Phi : H^{2n}(X \bmod \mathcal{D}, \mathbb{Z}_q(n)) \rightarrow H^{2n}(\mathcal{V} \bmod \Delta, \mathbb{Z}_q(n)),$$

where  $\mathcal{V}$  is the covariant vector bundle associated to the dual of  $\mathcal{E}$  and  $\Delta = (\Delta_i) = (\rho_i^*(1))$  is the family of closed subschemes of  $\mathcal{V}$  obtained as the pullbacks of the global section under the maps of the partial trivialization. We also have a natural map from  $H^0(X, \mathbb{Z}_q) \rightarrow H^{2n}(\mathcal{V} \bmod \Delta, \mathbb{Z}_q(n))$ . Let  $[0]$  be the image of the class  $1 \in H^0(X, \mathbb{Z}_q)$  under this map, and then Saito defines the relative top Chern class to be the inverse image of  $[0]$  under the canonical isomorphism  $\Phi$  above. Relative top Chern classes satisfy nice functorial properties, and the relative top Chern class is mapped to the normal top Chern class under the canonical map  $H^{2n}(X \bmod \mathcal{D}, \mathbb{Z}_q(n)) \rightarrow H^{2n}(X, \mathbb{Z}_q(n))$ . Furthermore, the following corollary of Proposition 1 in [Sai93] gives us a way to compare the relative top Chern classes associated to two different partial trivializations.

**COROLLARY 3.8.** *Let  $X$  be an  $\mathbb{F}_p$ -scheme, and let  $(\mathcal{E}, \rho)$  be a partially trivialized locally free sheaf on  $X$ . Let  $\sigma_i = f_i^{-1} \cdot \rho_i : \mathcal{E}|_{\mathcal{D}_i} \rightarrow \mathcal{O}_{\mathcal{D}_i}$  where  $f_i$  comes from  $\mathbb{F}_p^*$ , so that  $\sigma = (\sigma_i)$  is another partial trivialization of  $\mathcal{E}$ . Finally, let  $\mathcal{E}_i = \text{Ker}(\rho_i)$  so that  $\rho|_{\mathcal{D}_i}$  is a partial trivialization of  $\mathcal{E}_i$ . Then we can compute the difference between the relative top Chern classes as*

$$c_n(\mathcal{E}, \rho) - c_n(\mathcal{E}, \sigma) = \sum_i \{f_i\} \cup c_{n-1}(\mathcal{E}_i, \rho|_{\mathcal{D}_i}).$$

In § 2 of [Sai93], Saito uses the construction of relative top Chern classes to define the relative canonical cycle.

**DEFINITION 3.9.** Let  $\mathcal{D}$  be a divisor with simple normal crossings on a variety  $X$  of dimension  $n$  defined over a perfect field  $F$  of characteristic  $p$ , and let  $U = X - \mathcal{D}$ . Let  $\Omega_{X/F}^1(\log \mathcal{D})$  be the locally free  $\mathcal{O}_X$ -module of rank  $n$  of differential 1-forms on  $X$  with logarithmic poles along  $\mathcal{D}$ . Then the cycle

$$c_{X,U} = (-1)^n c_n(\Omega_{X/F}^1(\log \mathcal{D}), \text{res})$$

is called the relative canonical cycle. It lies inside the cohomology with compact support  $H_c^{2n}(X \bmod \mathcal{D}, \hat{\mathbb{Z}}'(n))$ , where  $\hat{\mathbb{Z}}' = \prod_{q \neq p} \mathbb{Z}_q$ . The relative canonical cycle has degree equal to  $\chi(U_{\overline{F}}) = \sum (-1)^q \dim H_c^q(U_{\overline{F}}, \overline{\mathbb{Q}}_l)$ . Note that this definition differs from that of S. Saito in [Sai84], but only up to a change in sign.

Saito observes that one can also define a relative top Chern class (and hence a relative canonical cycle) sitting inside of  $H^n(X \bmod \mathcal{D}, \mathbb{G}_m)$ , the divisor class group with modulus  $\mathcal{D}$ , and in particular we can define  $c_{X,U}$  as an element of  $H^n(X \bmod \mathcal{D}, \mathbb{G}_m)$  in the case when  $n = 1$ . For our work we will want to consider the case where  $X$  is one of the components of  $\mathcal{Y}_v^{\text{red}}$ , and therefore is of dimension one. Following Example 1 of [Sai93] we will look at the relative top Chern class  $c_{X,U}$  lying inside the generalized class group

$$H^1(X \bmod \mathcal{D}, \mathbb{G}_m) = \left[ \left( \bigoplus_{x \notin \mathcal{D}} \mathbb{Z} \right) \oplus \left( \bigoplus_{x \in \mathcal{D}} K^*/U_x^1 \right) \right] / K^*,$$

where  $K$  is the fraction field of  $X$  and  $U_x^1 = 1 + m_x$ . In particular, the class  $c_{X,U}$  can be computed in the following way. Let  $\omega$  be a non-trivial rational section of  $\Omega_X^1(\log \mathcal{D})$  such that, for all points  $x \in \mathcal{D}$ ,  $\text{ord}_x(\omega) = -1$  and  $\text{res}_x(\omega) = 1$ . Then the relative canonical cycle is represented by the class of the zero cycle which is supported off of  $\mathcal{D}$ , given by

$$-c_{X,U} = \sum_{x \in U} \text{ord}_x(\omega) + \sum_{x \in \mathcal{D}} \overline{\text{res}_x(\omega)}$$

in the above decomposition of the generalized class group.

PROPOSITION 3.10. Let  $\mathcal{Y}_v^{\text{red}}$  consist of two components  $F'$  and  $G'$ . Let  $D'$  be a horizontal divisor chosen as in the previous sections.

- 1) There is a canonical isomorphism  $\phi : \mathcal{O}_{F'}(D' \cap F') \rightarrow \omega_{F'}(F' \cap G')$  up to multiplication by a global unit.
- 2) The global section  $1 \in \Gamma(\mathcal{O}_{F'}(D' \cap F'))$  maps under  $\phi$  to an element  $\gamma \in \Gamma(\omega_{F'}(F' \cap G'))$  such that  $\text{ord}_x(\gamma) = 1$  if  $x \in F' \cap D'$ ,  $\text{ord}_x(\gamma) = -1$  if  $x \in F' \cap G'$ , and  $\text{ord}_x(\gamma) = 0$  otherwise.
- 3) Set  $a_x = \text{res}_x(\gamma)$  for all  $x \in F' \cap G'$ . Then  $c_{F',U_{F'}}$  is such that  $-c_{F',U_{F'}}$  is the class in  $[(\bigoplus_{x \in (F'-G')} \mathbb{Z}) \oplus (\bigoplus_{x \in F' \cap G'} K^*/U_x^1)]/K^* = H^1(F' \text{ mod } (F' \cap G'), K)$  of the element

$$c = \left( \bigoplus_{x \in D' \cap F'} 1 \in \mathbb{Z} \right) \oplus \left( \bigoplus_{x \in F' - D' - G'} 0 \in \mathbb{Z} \right) \oplus \left( \bigoplus_{x \in F' \cap G'} a_x \right).$$

The proof of part 1 follows from carrying through a series of calculations analogous to those in the proof of Lemma 3.5. In particular,

$$\begin{aligned} \mathcal{O}_{F'}(D' \cap F') &= \mathcal{O}_{\mathcal{Y}}(D')|_{F'} \\ &= \mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T)|_{F'} \\ &\cong \omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}})|_{F'} \\ &= \omega_{\mathcal{Y}}(\mathcal{Y}_v^{\text{red}})|_{F'} \\ &= [\omega_{\mathcal{Y}}(F') \otimes \mathcal{O}_{\mathcal{Y}}(G')]|_{F'} \\ &= \omega_{\mathcal{Y}}(F')|_{F'} \otimes \mathcal{O}_{\mathcal{Y}}(G')|_{F'} \\ &= \omega_{F'} \otimes \mathcal{O}_{F'}(F' \cap G') \\ &= \omega_{F'}(F' \cap G'). \end{aligned}$$

Proving part 2 is just a matter of calculating the various orders and residues of  $\gamma$  given that we know them for the element  $1 \in \Gamma(\mathcal{O}_{F'}(D' \cap F'))$ . Explicitly, one can compute these by following the residue map on elements of the sheaves through the equalities and congruences in the calculations above. Note that all of the isomorphisms are unique with the exception of  $\mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T) \cong \omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}})$ . This map, while not unique, is well defined up to multiplication by a global unit, and therefore when we look at classes mod  $K^*$  the discrepancy will not matter.

Part 3 of the proposition now follows from combining part 2 with Saito’s definition of the relative canonical cycle given above, setting  $X = F'$  and  $\mathcal{D} = F' \cap G'$ .

This proposition gives us an explicit way to construct the relative canonical class in our situation. In particular, the  $a_x$  terms come about because of the difference in natural partial trivializations on the sheaves  $\mathcal{O}_{F'}(D' \cap F')$  and  $\omega_{F'}(F' \cap G')$  associated to the restriction map  $\mathcal{O}_{F'}(D' \cap F') \rightarrow \mathcal{O}_{F'}(D' \cap F')|_{F' \cap G'} = \mathcal{O}_{F' \cap G'}$  and the residue maps  $\text{res}_x$ .

For the computations in the next section, we will need the following definitions.

DEFINITION 3.11. Define the following classes which lie in the generalized class group  $H^1(F' \text{ mod } (F' \cap G'), K)$ .

- a) Define an element  $\lambda \in (\bigoplus_{x \in (F'-G')} \mathbb{Z}) \oplus (\bigoplus_{x \in F' \cap G'} K^*/U_x^1)$  which has components equal to  $1 \in \mathbb{Z}$  at all points  $x \in D' \cap F'$ , equal to  $0 \in \mathbb{Z}$  at all points  $x$  in  $F' - D' - G'$ , and equal to the identity in  $K^*/U_x^1$  for all points  $x \in F' \cap G'$ . We then look at the class  $[\lambda]_{F'} \in H^1(F' \text{ mod } (F' \cap G'), K)$ , which is the first relative Chern class of the line bundle  $\mathcal{O}_{F'}(D' \cap F')$  with partial trivializations. One can define  $[\lambda]_{G'}$  in a similar way.
- b) Let  $\delta_{F'}$  be the class in  $H^1(F' \text{ mod } (F' \cap G'), K)$  which corresponds to the element  $\delta = (\bigoplus 0) \oplus (\bigoplus a_x) \in (\bigoplus_{x \in F' - G'} \mathbb{Z}) \oplus (\bigoplus_{x \in F' \cap G'} K^*/U_x^1)$ . In other words, this element is trivial at all places



corresponding to  $x \notin F' \cap G'$ , and for those places which correspond to points  $x \in F' \cap G'$  consists of the terms  $a_x$  coming about as the difference between the partial trivializations of  $\mathcal{O}_{\mathcal{Y}}(D' + \mathcal{Y}_T)$  and  $\omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}})$ , as found in the above characterization of  $c_{F',U_{F'}}$ . Note that  $\delta_{F'}$  can be thought of as the quotient of  $c_{F',U_{F'}}$  and  $[\lambda]_{F'}$ . One can define  $\delta_{G'}$  in a similar way.

### 3.4 The general case

We have shown that in the situation where  $\mathcal{Y}_v^{\text{red}}$  consists of a single component  $F'$  then the fibral contribution to the root number is positive. Now we will consider the next case, where  $\mathcal{Y}_v^{\text{red}}$  consists of two irreducible components, say  $F'$  and  $G'$ . Note that in particular this implies that  $v \in S$ . Recall that from Equation (1) we are interested in comparing  $\epsilon(D'_v, V)$  and  $\epsilon(\mathcal{Y}_v, V)$ . By our initial assumptions,  $D'$  intersects  $\mathcal{Y}_v^{\text{red}}$  in smooth points of  $\mathcal{Y}_v^{\text{red}}$ , so in particular we get that the set  $D' \cap F' \cap G' = \emptyset$ .

Define  $I_1 = I_{\mu(F')}$  and  $I_2 = I_{\mu(G')}$ , where  $F$  and  $G$  are components of  $\mathcal{X}_v^{\text{red}}$  lying above  $F'$  and  $G'$  respectively. Then  $\det(V^{I_1})$  is a character of the Galois group of the cover  $F \rightarrow F'$ , which will be tame with respect to the divisor  $F' \cap G'$ . Class-field theory says that we can therefore view  $\det(V^{I_1})$  as a character of the ray class group of  $F'$  with conductor  $F' \cap G'$ . We wish to define the term  $\det(V^{I_1})(\pi_{D',y})$  for points  $y \in F'$ . In order to do so, we view  $\det(V^{I_1})$  as a character of the idèles  $J_{F'}$  of  $F'$ . In other words, it is an idèle class character modulo the conductor, which will be supported on  $F' \cap G'$ . We then define  $\det(V^{I_1})(\pi_{D',y})$  to be the value of  $\det(V^{I_1})$  on the idèle  $(1, \dots, 1, \pi_{D',y}, 1, \dots)$  which is trivial away from  $y$ . This is well defined as the conductor of  $\det(V^{I_1})$  does not involve  $y$  and the difference between two local uniformizers is a unit.

If we define  $\det(V^{I_1})(D' \cap F')$  to be equal to the product

$$\prod_{y \in D' \cap F'} \det(V^{I_1})(\pi_{D',y}),$$

then this term will be independent of the choices of uniformizers as all components are unramified, and we are able to make the following calculation:

$$\begin{aligned} \epsilon(D'_v, V) &= \prod_{y \in D' \cap \mathcal{Y}_v^{\text{red}}} \epsilon(y, V) \\ &= \prod_{y \in D' \cap F'} \epsilon(y, V) \prod_{y \in D' \cap G'} \epsilon(y, V) \\ &= \prod_{y \in D' \cap F'} \det(V^{I_1})(\pi_{D',y}) \prod_{y \in D' \cap G'} \det(V^{I_2})(\pi_{D',y}) \\ &= \det(V^{I_1})(D' \cap F') \cdot \det(V^{I_2})(D' \cap G'). \end{aligned} \tag{2}$$

Recall that in the case where  $\mathcal{Y}_v^{\text{red}}$  consisted of a single component  $F'$ , we were able to show that  $\epsilon(D'_v, V) = \det(V^I)(D' \cap F')$ . In that case Lemma 3.5 showed that our hypothesis on  $D'$  implied that  $D' \cap F'$  was a canonical divisor on  $F'$ . The preceding subsection showed that in this more complicated case, while  $D' \cap F'$  is not a canonical divisor on  $F'$ , it is close to being one. To make this precise requires the results of the previous subsection. In particular, when viewed as an idèle class character,  $\det(V^{I_1})$  breaks into components  $\det(V^{I_1})_x$  which are unramified for all  $x \notin F' \cap G'$ , and therefore we get

$$\det(V^{I_1})(D' \cap F') = \prod_{y \in D' \cap F'} \det(V^{I_1})_y(\pi_{D',y}) = \det(V^{I_1})([\lambda]_{F'}).$$

Therefore Equation (2) states that, in the case where  $V$  is an orthogonal virtual representation of dimension 0 and trivial determinant,  $\mathcal{Y}_v^{\text{red}}$  consists of two components  $F'$  and  $G'$ , and  $D'$  is chosen

as above, then we have that

$$\epsilon(D'_v, V) = \det(V^{I_1})([\lambda]_{F'}) \det(V^{I_2})([\lambda]_{G'}). \tag{3}$$

Switching gears, we now want to take a look at the term  $\epsilon(\mathcal{Y}_v, V)$ . For the moment, we will assume that  $F' \cap G'$  consists of a single point  $z$ . We begin by looking at the two exact sequences

$$\begin{aligned} 0 \rightarrow U &= \mathcal{Y}_v^{\text{red}} - z \rightarrow \mathcal{Y}_v^{\text{red}} \rightarrow z \rightarrow 0, \\ 0 \rightarrow U &\rightarrow \widehat{\mathcal{Y}}_v^{\text{red}} = F' \amalg G' \rightarrow \{z_{F'}, z_{G'}\} \rightarrow 0, \end{aligned}$$

where  $z_{F'}$  (respectively  $z_{G'}$ ) is the point  $z$  thought of as sitting just on  $F'$  (respectively  $G'$ ). Epsilon factors are multiplicative within exact sequences as well as in disjoint unions, so these sequences imply that

$$\begin{aligned} \epsilon(\mathcal{Y}_v^{\text{red}}, V) &= \epsilon(U, V)\epsilon(z, V) \\ &= \frac{\epsilon(F', V)\epsilon(G', V)}{\epsilon(z_{F'}, V)\epsilon(z_{G'}, V)}\epsilon(z, V). \end{aligned} \tag{4}$$

To continue, we must consider the  $\epsilon(F', V)$  term. In order to compute this term we use the following result proven by Saito [Sai93].

LEMMA 3.12. *Let  $X, U$  be as in Definition 3.9, and let the action of  $G$  be étale on  $U$ . Then  $\prod_{y \in U} \epsilon_y(X, V) = \det(V)(c_{X,U})$ .*

Applying this lemma to our situation, we are able to make the following computation:

$$\begin{aligned} \epsilon(F', V) &= \epsilon(F', V^{I_1}) \\ &= \prod_{y \in (F')^0} \epsilon_y(F', V^{I_1}) \\ &= \epsilon_z(F', V^{I_1}) \prod_{y \neq z} (\epsilon_y(F', V^{I_1})) \\ &= \epsilon_z(F', V^{I_1}) \det(V^{I_1})(c_{F', U_{F'}}) \\ &= \epsilon_{0,z}(F', V^{I_1})\epsilon(z_{F'}, V^{I_1}) \det(V^{I_1})(c_{F', U_{F'}}). \end{aligned} \tag{5}$$

Plugging Equation (5) (and the analogous formula for  $\epsilon(G', V)$ ) into Equation (4) gives that

$$\epsilon(\mathcal{Y}_v^{\text{red}}, V) = \epsilon_{0,z}(F', V^{I_1})\epsilon_{0,z}(G', V^{I_2}) \det(V^{I_1})(c_{F', U_{F'}}) \det(V^{I_2})(c_{G', U_{G'}})\epsilon(z, V),$$

which we can combine with Equation (3) to get that

$$\frac{\epsilon(\mathcal{Y}_v, V)}{\epsilon(D'_v, V)} = \frac{\det(V^{I_1})(c_{F', U_{F'}}) \det(V^{I_2})(c_{G', U_{G'}})}{\det(V^{I_1})([\lambda]_{F'}) \det(V^{I_2})([\lambda]_{G'})} \epsilon_{0,z}(F', V^{I_1})\epsilon_{0,z}(G', V^{I_2})\epsilon(z, V).$$

Note that

$$\frac{\det(V^{I_1})(c_{F', U_{F'}})}{\det(V^{I_1})([\lambda]_{F'})} = \det(V^{I_1})(\delta_{F'}),$$

where  $\delta$  is the class defined in Definition 3.11.

Considering a slightly more general case, in which we still only have two components, but where  $F' \cap G'$  consists of more than one point, it is clear that all of the calculations will follow through and we will get that

$$\frac{\epsilon(\mathcal{Y}_v, V)}{\epsilon(D'_v, V)} = \det(V^{I_1})(\delta_{F'}) \det(V^{I_2})(\delta_{G'}) \prod_{z \in F' \cap G'} \epsilon_{0,z}(F', V^{I_1})\epsilon_{0,z}(G', V^{I_2})\epsilon(z, V).$$

If we have more than two components in  $\mathcal{Y}_v^{\text{red}}$  then the bookkeeping becomes more complicated but the mathematics does not. We first set up the necessary notation. Let  $C_i$  be the components of  $\mathcal{Y}_v^{\text{red}}$ . Furthermore, let  $C_{i,j} = C_i \cap C_j$ , let  $Z = \bigcup_{i \neq j} C_{i,j}$  be the collection of all intersection points, and let  $U_{C_i}$  be the open set consisting of  $C_i - Z$ . Finally, let  $I_i$  be the inertia group associated to  $C_i$  as above. We are still interested in computing  $\epsilon(\mathcal{Y}_v, V)$  and  $\epsilon(D'_v, V)$ . Let  $\lambda_v$  and  $\delta_{v,C_i}$  be the classes  $\lambda$  and  $\delta$  defined above for a particular class  $v$  and a particular component  $C_i$ . In particular, recall that  $\delta_{v,C_i}$  can be calculated purely from looking at points  $z \in Z$ .

For the latter, the computation works just as it did before, as we know that if  $i < j$  the  $C_{i,j}$  are disjoint from each other as well as from  $D'$ . We obtain that

$$\begin{aligned} \epsilon(D'_v, V) &= \prod_i \det(V^{I_i})(C_i \cap D') \\ &= \prod_i \det(V^{I_i})([\lambda_v]_{C_i}). \end{aligned}$$

To compute  $\epsilon(\mathcal{Y}_v, V)$  we need to use the exact sequences

$$\begin{aligned} 0 \rightarrow U = \mathcal{Y}_v^{\text{red}} - Z \rightarrow \mathcal{Y}_v^{\text{red}} \rightarrow Z \rightarrow 0, \\ 0 \rightarrow U \rightarrow \widehat{\mathcal{Y}_v^{\text{red}}} = \prod_i C_i \rightarrow \prod_{i \neq j} C_{i,j} \rightarrow 0, \end{aligned}$$

where  $\prod_{i \neq j} C_{i,j}$  can be thought of as the set consisting of two copies of  $Z$ , with each point considered as sitting once on each of the two  $C_i$  which it comes from originally. We can now use these sequences as well as the above calculations of  $\epsilon(C_i, V)$  to get that

$$\begin{aligned} \epsilon(\mathcal{Y}_v^{\text{red}}, V) &= \epsilon(U, V)\epsilon(Z, V) \\ &= \prod_i \epsilon(C_i, V) \prod_{z \in Z} \frac{\epsilon(z, V)}{\epsilon(z_{C_{i_1}}, V)\epsilon(z_{C_{i_2}}, V)} \\ &= \prod_i \det(V^{I_i})(c_{C_i, U_{C_i}}) \prod_{z \in Z} \epsilon_{0,z}(C_{i_1}, V^{I_{i_1}})\epsilon_{0,z}(C_{i_2}, V^{I_{i_2}})\epsilon(z, V), \end{aligned}$$

where we think of  $z \in Z$  as lying on  $C_{i_1} \cap C_{i_2}$ . If we put all of these calculations together we get the following result.

**THEOREM 3.13.** *Under all of the above hypotheses and notation, we get that, for all  $v$ ,*

$$\frac{\epsilon(\mathcal{Y}_v, V)}{\epsilon(D'_v, V)} = \prod_i \det(V^{I_i})(\delta_{v,C_i}) \prod_{z \in Z} \epsilon_{0,z}(C_{i_1}, V^{I_{i_1}})\epsilon_{0,z}(C_{i_2}, V^{I_{i_2}})\epsilon(z, V),$$

where both of these products are equal to one if the set  $Z$  is empty.

Combining Equation (1), Lemma 3.3 and Theorem 3.13 gives a precise form of Theorem 1.1. Note that, other than the term  $\epsilon_\infty(\mathcal{Y}, V)$ , all of the terms depend only on the crossing points of the components of fibers  $\mathcal{Y}_v^{\text{red}}$ .

### 4. Examples

In this section we wish to show several examples that apply Theorem 1.1. In order to do this, we must first have concrete examples of finite groups acting tamely on arithmetic surfaces. We find one class of such examples by using the following result from the thesis of Kwon [Kwo95].

**THEOREM 4.1.** *Let  $X$  be an elliptic curve over  $K$  and let  $\mathcal{X}$  be the minimal model of  $X$  over  $\mathcal{O}_K$ . Consider the action of a group*

$$G \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \subset X(K)$$

of torsion points on  $\mathcal{X}$ . Then the action of  $G$  on  $\mathcal{X}$  is numerically tame if and only if, for each place  $v$  of  $\mathcal{O}_K$  whose residue characteristic  $p$  divides the order of  $G$ , the following conditions are satisfied:

- i) The minimal model  $\mathcal{X}$  has good or multiplicative reduction at  $v$ .
- ii) The Zariski closure in  $\mathcal{X}$  of the  $p$ -Sylow subgroup  $G_p$  of  $G$  is smooth over  $\text{Spec } \mathcal{O}_K$ .

In particular, these conditions imply that  $\text{gcd}(n, m) = 1$ .

This theorem provides us with a set of concrete criteria for checking when a finite group acts tamely on the integral model of an elliptic curve, as numerical tameness implies tameness. In particular, the second condition asks us to compute the  $p$ -torsion points of the minimal model over  $p$ , and check that they do not coalesce when we reduce mod  $p$ .

We will now show an example of a computation of the orthogonal  $\epsilon$ -constants associated to the tame action of a finite group on a surface. First, we must calculate terms  $\epsilon(z, V)$ , where  $z \in \mathcal{Y}$  is a closed point defined over a finite field. Section 2.5 of [CEPT97a] gives us the following way of making this computation.

LEMMA 4.2. *Let  $x$  be a point of  $\mathcal{X}$  over a point  $y \in \mathcal{Y}$  which has finite residue field. Furthermore, let  $F_x$  be the arithmetic Frobenius element lying in  $G$ . Then  $\epsilon(y, V) = \det(V^{I_x})(-F_x)$ , where  $I_x \subseteq G$  is the inertia group of the point  $x$ .*

Let  $X$  be the elliptic curve given by the equation  $y^2 + xy + y = x^3$ . This equation is minimal over every prime  $p \in \text{Spec}(\mathbb{Z})$ , and thus it defines  $\mathcal{X}$ , the minimal model over  $\mathbb{Z}$ . The torsion subgroup of  $X$  is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ , and the torsion points of order three are  $(0, 0)$  and  $(0, -1)$ . We wish to check to see whether or not the action of  $G \cong \mathbb{Z}/3\mathbb{Z}$  is numerically tame by the criteria in Theorem 4.1. In order to do this, we first note that the discriminant of  $\mathcal{X}$  is  $-26 = -1 \times 2 \times 13$ , and thus  $\mathcal{X}$  has good reduction at 3. Furthermore  $\mathcal{X}$  has multiplicative reduction at 2 and at 13 (and in particular the fibers have Kodaira type  $I1$ ).

Next we check condition ii. We have  $G_3 = G = \{(0, 0), (0, -1), \mathbf{0}\}$ , and these points clearly do not coalesce mod  $q$  for any prime  $q$  (and in particular for  $q = 3$ ). Thus, this action of  $G$  on  $\mathcal{X}$  does satisfy the appropriate conditions.

Therefore the action of  $G$  is in fact numerically tame on  $\mathcal{X}$ . Furthermore, it follows from formulae of Velu [Vel71] that  $\mathcal{Y} = \mathcal{X}/G$  is the integral model of the elliptic curve defined by the equation  $y^2 + xy + y = x^3 - 5x - 8$ . The fibers of  $\mathcal{Y}$  are also non-singular with the exceptions of the fibers at  $v = 2, 12$ , which are both of Kodaira type  $I3$ . However,  $\mathcal{Y}$  is not regular, and thus the results of § 3 do not apply and in fact the  $\epsilon$ -constants are not well defined. However, due to Theorem 3.8 of Kwon’s dissertation [Kwo95], we know that after a finite number of blow-ups on the singular fibers we can blow up  $\mathcal{X}$  in such a way that the action of the group  $G$  extends to a tame action of  $G$  on the blow-up  $\tilde{\mathcal{X}}$ , and the quotient  $\tilde{\mathcal{Y}} = \tilde{\mathcal{X}}/G$  is in fact regular. This theorem applies because all of the local decomposition groups must be subgroups of  $\mathbb{Z}/3\mathbb{Z}$  and in particular must be cyclic of degree  $n \leq 3$ .

Next we need to define a representation  $V$  of  $\mathbb{Z}/3\mathbb{Z}$  satisfying certain properties. We know from representation theory that there are two distinct non-trivial one-dimensional characters of  $\mathbb{Z}/3\mathbb{Z}$  of order three. Let us define  $V_1$  to be the sum of these characters and  $V_2$  to be  $2\chi_0$ , where  $\chi_0$  is the trivial character. We then define  $V$  to be  $V_1 - V_2$ . Sums of characters are always orthogonal representations, so  $V$  will be orthogonal. It also is not hard to see that  $V$  has dimension 0 and trivial determinant.

In general, computing  $\epsilon(\tilde{\mathcal{Y}}, V)$  might be difficult, but in light of Theorem 1.1, the computation simplifies greatly. In particular, we only need to compute  $\epsilon_{\infty,0}(\tilde{\mathcal{Y}}, V)$ ,  $\det(V^{I_j})(\delta_{v,C_i})$  for  $v = 2, 13$ , and the terms

$$\epsilon_{0,z}(C_{i_1}, V^{I_{i_1}})\epsilon_{0,z}(C_{i_2}, V^{I_{i_2}})\epsilon(z, V)$$

at the singular points above the primes  $p = 2$  and  $p = 13$ . For the above choice of the representation  $V$ , we can see that  $\det(V^I)$  is trivial for all possible inertia groups  $I$ . More precisely,  $\det(V_j^I)$  will be trivial for  $j = 1, 2$ . If  $I$  acts trivially on  $V_j$  this is obvious, as the  $\det(V_j)$  are both in fact trivial. On the other hand, if  $I$  acts non-trivially on  $V_j$ , then  $V_j^I$  will be trivial as the kernels of both characters which make up  $V_j$  are the same, and thus  $\det(V_j^I)$  will be trivial as well.

Let us first look at the part of the calculation of  $\epsilon(\tilde{\mathcal{Y}}, V)$  coming from the fiber of  $\tilde{\mathcal{Y}}$  above the prime 2. Denote the three components of  $\tilde{\mathcal{Y}}_2$  by  $C_1, C_2$ , and  $C_3$ . Let  $I_i$  be the inertia group associated to  $C_i$ . In particular,  $\det(V^{I_i})$  is trivial in each of these cases for the reasons described above. Thus, the  $\det(V^{I_j})(\delta_{2,C_i})$  terms are equal to one. Many of the  $\epsilon_{0,z}(C_i, V^{I_i})$  terms will also immediately be equal to one as many of the  $V^{I_i}$  terms are themselves trivial. To compute the others, we use Theorem 2 of Saito [Sai93]. Because we are looking at cases where  $\det(V_i^I)$  is trivial, these formulae reduce the computation of  $\epsilon_{0,z}(C_i, V^{I_i})$  to the computation of a Gauss sum  $\tau_{C_i}(V^{I_i})$ . We can now use the fact that our representation  $V$  is the sum of a representation and its complex conjugate to get that  $\tau_{C_i}(V^{I_i}) = 1$ .

The above paragraph holds for the points above  $p = 13$  as well, so we can ignore those terms. We can now use Lemma 4.2 in order to compute the  $\epsilon(z, V)$  terms. In particular, the fact that all of the  $\det(V^I)$  terms are trivial tells us that these terms are also equal to one. To summarize, we have that  $\epsilon(\tilde{\mathcal{Y}}, V) = \epsilon_{\infty,0}(\tilde{\mathcal{Y}}, V)$ .

In Theorem 4.0.1 of [CPT02] the authors show that the Euler characteristics and the character functions  $\zeta_S$  associated to the action of a finite group on a minimal model over  $\mathbb{Z}$  of an elliptic curve over  $\mathbb{Q}$  satisfying certain properties (which our example does satisfy) are trivial. This is because the group must act trivially on the various  $H^{p,q}$  pieces of the Hodge structure. But this in turn shows that  $\epsilon_{\infty,0}(\tilde{\mathcal{Y}}, V)$  is trivial, and thus  $\epsilon(\tilde{\mathcal{Y}}, V)$  is positive.

We note that many of the computations in the last example will hold whenever we are in the case of one of Kwon’s examples. The most difficult step seems to be ensuring the regularity of  $\mathcal{Y}$ , as Theorem 3.8 of [Kwo95] only allows us to do so for very small  $G$ . We conclude by stating a general theorem generalizing this example in the case  $G \cong \mathbb{Z}/3\mathbb{Z}$ .

**PROPOSITION 4.3.** *Let  $X$  be an elliptic curve over a number field  $K$  and let  $\mathcal{X}$  be the minimal model of  $X$  over  $\mathcal{O}_K$ . Let  $G \cong \mathbb{Z}/3\mathbb{Z}$  be a subgroup of torsion points on  $\mathcal{X}$ . Furthermore, assume that the minimal model  $\mathcal{X}$  has good or multiplicative reduction at all places above  $p = 3$  and that the Zariski closure in  $\mathcal{X}$  of  $G$  is smooth over  $\text{Spec } \mathcal{O}_K$ . We can blow up  $\mathcal{X}$  in such a way that the action of the group  $G$  extends to a tame action of  $G$  on the blow-up  $\tilde{\mathcal{X}}$ , and the quotient  $\tilde{\mathcal{Y}} = \tilde{\mathcal{X}}/G$  is regular. Let  $V$  be an orthogonal virtual representation of trivial dimension and determinant. If we also assume that the reductions of the finite fibers of  $\tilde{\mathcal{Y}}$  have strictly normal crossings, then the sign of  $\epsilon(\tilde{\mathcal{Y}}, V)$  will be the same as the sign of  $\prod \tau_{C_i}(V^{I_i})$  where the product runs over all components of fibers of bad reduction. Furthermore, if  $V$  can be expressed as the sum of a character and its complex conjugate then we get that  $\epsilon(\tilde{\mathcal{Y}}, V) > 0$ .*

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