

Torsion theories for modules over a triangular matrix ring

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Our objects of study are the (hereditary) torsion theories which exist for the category $\Gamma\text{-mod}$ of left modules over a triangular matrix ring $\Gamma = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ where R and S are associative rings with identity and M is an R - S -bimodule. Our method is to prove and then exploit the fact that the torsion theories for $\Gamma\text{-mod}$ correspond bijectively to pairs of torsion theories, one for $R\text{-mod}$ and one for $S\text{-mod}$.

We develop a set of criteria for a torsion theory for $\Gamma\text{-mod}$ to be stable, and offer sufficient conditions for each stable torsion theory for $S\text{-mod}$ to induce a stable torsion theory for $\Gamma\text{-mod}$. We determine just when Γ is a stable ring.

Turning to the (Gabriel) filters associated with the torsion theories for $\Gamma\text{-mod}$, we show that each filter contains a basis of matrix ideals, and discuss necessary and sufficient conditions for a filter to contain a basis of finitely generated or principal left ideals, or to have a least element. We go on to examine the centrally splitting torsion theories for $\Gamma\text{-mod}$.

Turning next to torsion radicals, we characterise each torsion radical t for $\Gamma\text{-mod}$ in terms of a torsion radical t' for $R\text{-mod}$ and a torsion radical t'' for $S\text{-mod}$, and determine necessary and sufficient conditions for t to be the "product" of t' and t'' .

We continue with quotient functors, and show that each quotient functor Q for $\Gamma\text{-mod}$ may be described as a pullback. We use this

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description to compute Q in particular situations, and to determine when Q may be viewed as a "product" of quotient functors, one for $R\text{-mod}$ and one for $S\text{-mod}$.

We apply our work on quotient functors to obtain a description of the ring of quotients of Γ with respect to a torsion theory, and to compute the ring of quotients in particular situations. We describe the maximal ring of quotients of Γ in the case when M is faithful as a right S -module.

We then examine the questions of perfection and splitting with regard to torsion theories for $\Gamma\text{-mod}$.

We apply our earlier work to obtain all of the (Goldman) prime torsion theories for $\Gamma\text{-mod}$, their quotient functors, and the critical ideals of Γ .

We conclude with a list of problems and tentative suggestions for further research.