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ADDITION THEOREMS AND BINARY EXPANSIONS

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ABSTRACT. Let an interval $I \subset \mathbb{R}$ and subsets $D_0, D_1 \subset I$ with $D_0 \cup D_1 = I$ and $D_0 \cap D_1 = \emptyset$ be given, as well as functions $r_0: D_0 \longrightarrow I$, $r_1: D_1 \longrightarrow I$. We investigate the system (S) of two functional equations for an unknown function $f: I \longrightarrow [0, 1]$:

$$(S) \qquad \begin{aligned} 2f(x) &= f(r_0(x)) \quad \text{if } x \in D_0, \\ 2f(x) - 1 &= f(r_1(x)) \quad \text{if } x \in D_1. \end{aligned}$$

We derive conditions for the existence, continuity and monotonicity of a solution. It turns out that the binary expansion of a solution can be computed in a simple recursive way. This recursion is algebraic for, *e.g.*, inverse trigonometric functions, but also for the elliptic integral of the first kind. Moreover, we use (S) to construct two kinds of peculiar functions: surjective functions whose intervals of constancy are residual in I, and strictly increasing functions whose derivative is 0 almost everywhere.

1. Introduction. We begin with an introductory example. Let $x \geq 0$. Set

$$a_0 = x \text{ and } a_{n+1} = \frac{2a_n}{1 - a_n^2} \quad (\text{with } a_{n+1} = -\infty \text{ if } a_n = \pm 1).$$

Then

$$\sum_{\substack{a_n < 0 \\ n \geq 0}} \frac{1}{2^{n+1}} = \frac{\arctan x}{\pi}.$$

Simon Plouffe (Université Bordeaux-I) detected this identity numerically; the purpose of the present paper is to explain why it is true and how the underlying method can be used to get similar recursions for other functions. Let us observe two things: First, the above series is just the binary expansion of $\arctan(x)/\pi$. Second, the recursion formula corresponds in a certain way to the addition theorem for $2 \arctan x$. In the present paper we will explore how addition theorems of this type can be used to compute similar binary expansions for a wide class of functions.

More precisely, we will deal with the following type of functional equations. Let an interval $I \subseteq \mathbb{R}$ and subsets $D_0, D_1 \subseteq I$ with $D_0 \cup D_1 = I$ and $D_0 \cap D_1 = \emptyset$ be given, as well as functions $r_0: D_0 \longrightarrow I$, $r_1: D_1 \longrightarrow I$. Then consider the system (S) of the following two functional equations for an unknown function $f: I \longrightarrow [0, 1]$.

$$(S_0) \qquad 2f(x) = f\big(r_0(x)\big) \quad \text{if } x \in D_0,$$

$$(S_1) \qquad 2f(x) - 1 = f\big(r_1(x)\big) \quad \text{if } x \in D_1.$$

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Stating the functional equations in this way implies that for each solution f the following must be true: $x \in D_0 \Rightarrow f(x) \in [0, 1/2]$ and $x \in D_1 \Rightarrow f(x) \in [1/2, 1]$.

Why are we interested in this system of functional equations? The reason is that on the one hand there are many interesting functions which can be shown to solve the system, and that on the other hand the system links these functions to Plouffe iterations like the one described for the arctan. In fact, as we will see in the next section of this paper, Plouffe's iteration for the arctan is proved upon the observation that

$$f(x) = \begin{cases} \arctan(x)/\pi & \text{if } x \in [0, \infty) \\ 1 + \arctan(x)/\pi & \text{if } x \in [-\infty, 0) \end{cases}$$

satisfies (S) on $I = \mathbb{R} \cup \{-\infty\}$ with $D_0 = [0, \infty)$, $D_1 = [-\infty, 0)$, $r_0(x) = \frac{2x}{1-x^2}$, $r_0(1) = -\infty$, $r_1(x) = \frac{2x}{1-x^2}$, $r_1(-1) = -\infty$. Other interesting functions which satisfy the system (S) are the logarithm, inverse trigonometric and hyperbolic functions and even the elliptic integral of the first kind. Moreover, it is also possible to describe quite peculiar functions as solutions of the system. If we choose $D_0 = [0, 1/2)$, $D_1 = [1/2, 1]$, $r_0(x) = 4x^2$, $r_1(x) = 1 - 4(x-1)^2$, then the solution of (S) is surjective but has intervals of constancy which are residual in I. If we choose $D_0 = [0, 3/4)$, $D_1 = [3/4, 1]$, $r_0(x) = 4x/3$, $r_1(x) = 4x - 3$, then the solution is strictly increasing but has derivative 0 almost everywhere (it is the inverse of a function with the same property constructed by G. de Rham in [2]). We will prove these two statements in somewhat greater generality in Section 3, after having given conditions for the existence, continuity and monotonicity of a solution in Section 2 of this paper.

We conclude the introduction with two basic observations about the functional equations. Firstly, we have that $f \equiv 0$ if and only if $D_1 = \emptyset$. (Indeed, assume $D_0 = I$ and $x_0 \in D_0$ with $f(x_0) \neq 0$. Iterate equation (S_0) until $2^n f(x_0) = f\big((r_0 \circ \cdots \circ r_0)(x_0)\big) = f\big(r_0^n(x_0)\big) > 1/2$. This contradicts $r_0^n(x_0) \in D_0 \Rightarrow f\big(r_0^n(x_0)\big) \in [0, 1/2]$. If, on the other hand, $f \equiv 0$, then $D_1 \neq \emptyset$ would yield a contradiction.) Similarly, $f \equiv 1$ if and only if $D_0 = \emptyset$.

Secondly, the requirement that a given function f is the solution of a system of type (S) is not very restrictive. Indeed, the following is a necessary and sufficient condition for f to satisfy such a system: For each $x \in I$ with $f(x) < 1/2$, there must exist a $y(x) \in I$ with $2f(x) = f\big(y(x)\big)$; for each $x \in I$ with $f(x) > 1/2$ there must exist a $y(x) \in I$ with $2f(x) - 1 = f\big(y(x)\big)$; and for each $x \in I$ with $f(x) = 1/2$ there must exist a $y(x) \in I$ with $f\big(y(x)\big) = 0$ or $f\big(y(x)\big) = 1$. Necessity is obvious. On the other hand, if this condition is satisfied (with, say, $f\big(y(x)\big) = 0$ if $f(x) = 1/2$), then we can define $D_0 := f^{-1}\big([0, 1/2)\big)$, $D_1 := f^{-1}([1/2, 1])$, and the functions r_0, r_1 can be set equal to this y on I. So, each function from which this can be reasonably expected satisfies a system of type (S). The challenge is, of course, to find functions r_0, r_1 which are "simpler" than f (*e.g.*, f transcendental and r_0, r_1 algebraic).

For $x \in [0, 1]$, we will always denote the binary expansion of x by $(0, \nu_0\nu_1 \cdots)_2$; this means $x = \sum_{n=0}^{\infty} \nu_n / 2^{n+1}$ with $\nu_0, \nu_1, \ldots \in \{0, 1\}$.

2. Properties of solutions of (S).

THEOREM 1. *The system (S) has exactly one solution* $f\colon I \longrightarrow [0,1]$.

PROOF. First we show that there is at most one solution. For each solution f, there is only one possible value for $f(x)$ at a point $x \in I$. This value is determined by the following recursion: Set

$$(1) \qquad a_0 = x \quad \text{and} \quad a_{n+1} = \begin{cases} r_0(a_n) & \text{if } a_n \in D_0, \\ r_1(a_n) & \text{if } a_n \in D_1. \end{cases}$$

Then for each solution f we have

$$f(a_{n+1}) = \begin{cases} f\big(r_0(a_n)\big) = 2f(a_n) & \text{if } a_n \in D_0, \\ f\big(r_1(a_n)\big) = 2f(a_n) - 1 & \text{if } a_n \in D_1. \end{cases}$$

Compare this with how we compute the binary expansion of $y \in [0,1]$. With

$$y_0 := y, \quad y_{n+1} := \begin{cases} 2y_n & \text{if } y_n \in [0, \frac{1}{2}), \\ 2y_n - 1 & \text{if } y_n \in (\frac{1}{2}, 1], \\ 2y_n \text{ or } 2y_n - 1 & \text{if } y_n = \frac{1}{2}, \end{cases}$$

we get $y = \sum_{y_{n+1}=2y_n-1,\ n\geq 0} \frac{1}{2^{n+1}}$. Since y_n and $f(a_n)$ obey the same recursion (recall that $a_n \in D_0$ implies $f(a_n) \in [0, 1/2]$ while $f(a_n) \in [0, 1/2)$ implies $a_n \in D_0$, and similarly for $a_n \in D_1$), we can identify $y_n = f(a_n)$ and get

$$(2) \qquad f(a_0) = f(x) = \sum_{\substack{a_n \in D_1 \\ n \geq 0}} \frac{1}{2^{n+1}}.$$

On the other hand, define a function f by this recursion. Then f indeed maps I into $[0,1]$, and f satisfies the equations (S_0), (S_1): $x \in D_0$ implies, by (2), $f(x) \leq 1/2$ and

$$f\big(r_0(x)\big) = f(a_1) = \sum_{\substack{a_n \in D_1 \\ n \geq 1}} \frac{1}{2^n} = 2 \sum_{\substack{a_n \in D_1 \\ n \geq 1}} \frac{1}{2^{n+1}} = 2 \sum_{\substack{a_n \in D_1 \\ n \geq 0}} \frac{1}{2^{n+1}} = 2f(x).$$

Similarly, $x \in D_1$ implies $f\big(r_1(x)\big) = 2f(x) - 1$. ∎

The recursion (1), (2) is important throughout this paper; we will constantly make use of it. In the arctan example, it is exactly Plouffe's iteration. Note that in the statement of Theorem 1 no regularity other than the boundedness is required or asserted.

Under which conditions is the solution f continuous? First of all, continuity of f does *not* imply the existence of continuous r_0, r_1. Consider, *e.g.*,

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, 1/4], \\ 3/4 - x & \text{if } x \in [1/4, 1/2], \\ 3/2x - 1/2 & \text{if } x \in [1/2, 1]. \end{cases}$$

Then, necessarily, $r_0(x) = 2x$ if $x \in [0, 1/16)$ and $r_0(x) = 8/3x + 1/3$ if $x \in (1/8, 1/4]$, but between $1/16$ and $1/8$, there must be at least one point of discontinuity.

However, if we assume continuity of r_0, r_1, we can give a sufficient and necessary condition for f to be continuous. In the following Theorem 2, closure and interior of D_0 and D_1 are taken in I.

THEOREM 2. *Let r_0, r_1 be continuous. Then the solution f of (S) is continuous if and only if*

for each $x_0 \in D_0 \cap \overline{D_1}$: $\forall n \in \mathbb{N}_0 : r_1^n\big(r_0(x_0)\big) \in \mathring{D}_1$ and $\exists \delta(n) > 0 : \forall y \in D_1$ with $|x_0 - y| < \delta(n)$ we have $r_0^n\big(r_1(y)\big) \in \mathring{D}_0$;

for each $x_0 \in \overline{D_0} \cap D_1$: $\forall n \in \mathbb{N}_0 : r_0^n\big(r_1(x_0)\big) \in \mathring{D}_0$ and $\exists \delta(n) > 0 : \forall y \in D_0$ with $|x_0 - y| < \delta(n)$ we have $r_1^n\big(r_0(y)\big) \in \mathring{D}_1$.

PROOF. Assume that f is continuous. Without loss of generality (wlog), take any $x_0 \in D_0 \cap \overline{D_1}$. Then $2f(x_0) = f\big(r_0(x_0)\big)$ and

$$\lim_{\substack{x \to x_0 \\ x \in D_1}} f\big(r_1(x)\big) = 2 \lim_{\substack{x \to x_0 \\ x \in D_1}} f(x) - 1 = 2f(x_0) - 1 = f\big(r_0(x_0)\big) - 1.$$

Since $f(I) \subseteq [0, 1]$, we must have $f\big(r_0(x_0)\big) = 1$ and $\lim_{\substack{x \to x_0 \\ x \in D_1}} f\big(r_1(x)\big) = 0$. Thus, $r_0(x_0) \in D_1$. Since $2f\big(r_0(x_0)\big) - 1 = 1 = f\Big(r_1\big(r_0(x_0)\big)\Big)$, we get $r_1\big(r_0(x_0)\big) \in D_1$, and, inductively, $r_1^n\big(r_0(x_0)\big) \in D_1$.

Observe that for any $x \in \partial D_0$ (which equals $\partial D_1 = (D_0 \cap \overline{D_1}) \cup (\overline{D_0} \cap D_1)$), $f(x) = 1/2$. Choose $\delta > 0$ such that $f(y) - f(x_0) = f(y) - 1/2 < 1/2^{n+2}$ for $y \in D_1$ with $|x_0 - y| < \delta$. Then we have $f(y) < 1/2 + 1/2^{n+2}$, therefore $2f(y) - 1 = f\big(r_1(y)\big) < 1/2^{n+1}$, thus $2^n f\big(r_1(y)\big) = f\Big(r_0^n\big(r_1(y)\big)\Big) < 1/2$; and that means $r_0^n\big(r_1(y)\big) \in D_0$.

Lastly, since $f\Big(r_1^n\big(r_0(x_0)\big)\Big) = 1$ and $f\Big(r_0^n\big(r_1(y)\big)\Big) < 1/2$, it follows that $r_1^n\big(r_0(x_0)\big) \in \mathring{D}_1$ and $r_0^n\big(r_1(y)\big) \in \mathring{D}_0$. (Recall that $f(\partial D_0) = f(\partial D_1) = \{1/2\}$.)

On the other hand, assume that the condition holds. Take any $x \in I$, $n \in \mathbb{N}$, and set $\varepsilon := 1/2^{n-1}$. Consider the numbers $a_0, a_1, \ldots, a_n$ where $a_k = (r_{\nu_{k-1}} \circ \cdots \circ r_{\nu_0})(x)$ and $\nu_k = \begin{cases} 0 & \text{if } a_k \in D_0 \\ 1 & \text{if } a_k \in D_1 \end{cases}$. Take

$$k_0 := \begin{cases} \min\big\{k \in \{0, \ldots, n\} : a_k \in \partial D_0 (= \partial D_1)\big\} & \text{if this set is not empty,} \\ n + 1 & \text{otherwise.} \end{cases}$$

Since r_0, r_1 are continuous and $a_k \in \mathring{D}_{\nu_k}$ for $k = 0, \ldots, k_0 - 1$, there is a $\delta_1 > 0$ such that $|x - y| \leq \delta_1$ implies that $(r_{\nu_{k-1}} \circ \cdots \circ r_{\nu_0})(y) \in \mathring{D}_{\nu_k}$ for $k = 0, \ldots, k_0 - 1$.

Assume (wlog) $a_{k_0} \in D_0$; then $f(x) = (0, \nu_0 \cdots \nu_{k_0-1} 011 \cdots)_2$. Again, since r_0, r_1 are continuous and $a_{k_0+1}, \ldots, a_n \in \mathring{D}_1$, there is a $\delta_1 \geq \delta_2 > 0$ such that $|x - y| \leq \delta_2$ and $(r_{\nu_{k_0-1}} \circ \cdots \circ r_{\nu_0})(y) \in D_0$ imply that $(r_1^m \circ r_0 \circ r_{\nu_{k_0-1}} \circ \cdots \circ r_{\nu_0})(y) \in D_1$ for $m = 0, \ldots, n - k_0 - 1$. This means that $f(y) = (0, \nu_0 \cdots \nu_{k_0-1} 01 \cdots 1 \mu_{n+1} \mu_{n+2} \cdots)_2$.

Lastly, we have a $\delta_1 \geq \delta_3 > 0$ such that $|x-y| \leq \delta_3$ and $(r_{\nu_{k_0-1}} \circ \cdots \circ r_{\nu_0})(y) \in D_1$ imply that $|(r_{\nu_{k_0-1}} \circ \cdots \circ r_{\nu_0})(x) - (r_{\nu_{k_0-1}} \circ \cdots \circ r_{\nu_0})(y)| < \delta(n-k_0-1)$, which in turn means that $(r_0^m \circ r_1 \circ r_{\nu_{k_0-1}} \circ \cdots \circ r_{\nu_0})(y) \in D_0$ for $m = 0, \ldots, n-k_0-1$. This means that $f(y) = (0, \nu_0 \cdots \nu_{k_0-1} 10 \cdots 0 \mu_{n+1}\mu_{n+2}\cdots)_2$.

Now, if we choose $\delta := \min\{\delta_2, \delta_3\}$, then $|x-y| \leq \delta$ implies that $|f(x)-f(y)| \leq \varepsilon$. ∎

OBSERVATION. If the solution f of (S) is continuous and neither $f \equiv 0$ nor $f \equiv 1$ then it is "almost surjective": $f(I) \supseteq [0,1] \setminus \{p\}$ where $p = 0$ or $p = 1$.

PROOF. As has been seen in section 1, we have $D_0, D_1 \neq \emptyset$. Moreover, there is an $x \in I$ with $f(x) \in (0,1)$. Assume (wlog) $x \in D_0$. As above, iterate equation (S_0) until $2^n f(x) = f\big((r_0 \circ \cdots \circ r_0)(x)\big) \in [1/2, 1]$. Since f is continuous, there is an $x_0 \in I$ with $f(x_0) = 1/2$. If $x_0 \in D_0$, then $2f(x_0) = 1 = f\big(r_0(x_0)\big)$, and there is an $x_1 \in \overline{D_1}$ with $f(x_1) = 1/2$. It follows that

$$2 \lim_{\substack{x \to x_1 \\ x \in D_1}} f(x) - 1 = 0 = \lim_{\substack{x \to x_1 \\ x \in D_1}} f\big(r_1(x)\big).$$

If $x_0 \in D_1$, then the same reasoning leads to $f\big(r_1(x_0)\big) = 0$ and the existence of an $x_1 \in \overline{D_0}$ with $\lim_{\substack{x \to x_1 \\ x \in D_1}} f\big(r_0(x)\big) = 1$. ∎

We are now interested in monotonic solutions.

OBSERVATION. If f is injective, then so are r_0, r_1. The converse is not true, not even when assuming continuity and strict monotonicity for r_0, r_1.

PROOF. If $r_0(x) = r_0(y)$, then $f\big(r_0(x)\big) = f\big(r_0(y)\big)$ and, by (S_0), $2f(x) = 2f(y)$. If f is injective, then $x = y$ follows. The same for r_1.

On the other hand, consider $f(x) = x^2$ on $I = [-1,1]$ which is not injective. But f satisfies the system (S), with $D_0 = [-1/\sqrt{2}, 1/\sqrt{2}]$, $D_1 = I \setminus D_0$, and $r_0(x) = \sqrt{2}x$, $r_1(x) = \begin{cases} \sqrt{2x^2-1} & \text{if } x > 0 \\ -\sqrt{2x^2-1} & \text{if } x < 0 \end{cases}$ which are continuous and strictly monotonic. ∎

THEOREM 3. *Assume that r_0, r_1 are increasing and that $D_0 < D_1$ ($x \in D_0$, $y \in D_1$ implies $x < y$). Then f is increasing. Similarly, f is decreasing if r_0, r_1 are increasing and $D_1 < D_0$.*

PROOF. We only prove the first part here. Take $x, y \in I$ with $x \leq y$. As in the proof of Theorem 2, consider the numbers $a_k = (r_{\nu_{k-1}} \circ \cdots \circ r_{\nu_0})(x)$, $b_k = (r_{\mu_{k-1}} \circ \cdots \circ r_{\mu_0})(y)$. There are two possibilities: Either $a_k \in D_0 \Leftrightarrow b_k \in D_0$ for all $k \in \mathbb{N}$, which implies $f(x) = f(y)$. Or there exists a k_0 such that for all $k < k_0$, $a_k \in D_0 \Leftrightarrow b_k \in D_0$, and $a_{k_0} \in D_0$, $b_{k_0} \in D_1$. (We have $a_{k_0} \leq b_{k_0}$ since r_0, r_1 are increasing.) This implies $f(x) = (0, \nu_0 \cdots \nu_{k_0-1} 0 \nu_{k_0+1} \cdots)_2 \leq (0, \nu_0 \cdots \nu_{k_0-1} 1 \mu_{k_0+1} \cdots)_2 = f(y)$. ∎

In Section 3, we will see that strict monotonicity of r_0, r_1 does *not* imply strict monotonicity of f.

By inspecting the recursion formula (1), (2), one would guess that it allows statements on the points in I which are mapped by f to a rational value. The last theorem in this section addresses this possibility. The interiors of D_0, D_1 and I are to be taken in $\mathbb{R}$.

Define a function $r: I \longrightarrow I$ by $r(x) = \begin{cases} r_0(x) & \text{if } x \in D_0 \\ r_1(x) & \text{if } x \in D_1 \end{cases}$.

THEOREM 4. *Assume that I is closed (possibly, $I = [-\infty, \infty]$), that r_0, r_1 are continuous and increasing, $D_0 < D_1$ or $D_1 < D_0$, and that r_0 maps $\mathring{D}_0$ onto $\mathring{I}$ and r_1 maps $\mathring{D}_1$ onto $\mathring{I}$. (Then f is continuous and monotonic.) Lastly, assume that f is strictly monotonic. Then*

$$\{x \in I : f(x) \in \mathbb{Q}\}$$
$$= \{x \in I : x \textit{ is in an orbit under } r \textit{ of a fixed point of } r^k = r \circ \cdots \circ r \ \ (k \geq 1)\}.$$

PROOF. Call the right-hand set in the assertion M.

(i) We show that $f(x) \in \mathbb{Q}$ for $x \in M$.

Indeed, $x \in M$ implies that there exist $\nu_0, \ldots, \nu_{n-1}$ and $\mu_1, \ldots, \mu_k$ such that

$$(r_{\mu_k} \circ \cdots \circ r_{\mu_1} \circ r_{\nu_{n-1}} \circ \cdots \circ r_{\nu_0})(x) = (r_{\nu_{n-1}} \circ \cdots \circ r_{\nu_0})(x),$$

where all of these maps are defined, *i.e.*, $x \in D_{\nu_0}$, $r_{\nu_0}(x) \in D_{\nu_1}$, and so on. By the recursion formula (1), (2), this means that

$$f(x) = (0, \nu_0 \cdots \nu_{n-1}\mu_1 \cdots \mu_k\mu_1 \cdots \mu_k \cdots)_2,$$

that is, $f(x) \in \mathbb{Q}$.

(ii) Assume $f(x) \in \mathbb{Q}$, say, $f(x) = (0, \nu_0 \cdots \nu_{n-1}\mu_1 \cdots \mu_k\mu_1 \cdots \mu_k \cdots)_2$. Let $I = [a, b]$ and assume wlog $D_0 < D_1$. It follows that $r_0(a) = a$ and $r_1(b) = b$. Now observe that r^k has a fixed point x_0 in $(r_{\mu_1}^{-1} \circ \cdots \circ r_{\mu_k}^{-1})(I)$: If $(\mu_1 \cdots \mu_k) \neq (0 \cdots 0)$ and $\neq (1 \cdots 1)$, then for $S := (r_{\mu_1}^{-1} \circ \cdots \circ r_{\mu_k}^{-1})(\mathring{I})$, we have $\bar{S} \subseteq \mathring{I}$, and since r_0 and r_1 are onto $\mathring{I}$, we have

$$r^k(S) = (r^{k-1} \circ r_{\mu_1} \circ r_{\mu_1}^{-1} \circ \cdots \circ r_{\mu_k}^{-1})(\mathring{I}) = (r^{k-1} \circ r_{\mu_2}^{-1} \circ \cdots \circ r_{\mu_k}^{-1})(\mathring{I}) = \cdots = \mathring{I},$$

therefore, r^k has a fixed point in S. If $(\mu_1 \cdots \mu_k) = (0 \cdots 0)$, then the fixed point is a, for $(\mu_1 \cdots \mu_k) = (1 \cdots 1)$, it is b.

Set $y := (r_{\nu_0}^{-1} \circ \cdots \circ r_{\nu_{n-1}}^{-1})(x_0)$, then y is in an orbit of x_0 and $f(y) = (0, \nu_0 \cdots \nu_{n-1}\mu_1 \cdots \mu_k \mu_1 \cdots \mu_k \cdots)_2$ by (1), (2). Since f is injective, $x = y$. ∎

3. **Several examples.** We begin with a short list of transcendental functions f which satisfy a system of type (S) where the functions r_0, r_1 are algebraic.

1.

$$f(x) = \ln x / \ln 2, \quad I = [1, 2],$$
$$D_0 = [1, \sqrt{2}), \quad D_1 = [\sqrt{2}, 2],$$
$$r_0(x) = x^2, \quad r_1(x) = x^2/2.$$

Of course, the recursion (1), (2) can be used to compute the binary expansion of the function f. Take for example $x = a_0 = 3/2$. Then $a_1 = 9/8$, $a_2 = 81/64$, $a_3 = 6561/4096, a_4 = 3^{16}/2^{25}$, and so on. Here are the first 20 binary digits of $\ln 3/\ln 2$: $(1.10010101110000000001)_2$.

An application of Theorem 4 would yield that $f(x)$ is rational if and only if $x = 2^q$ with rational q. But we know that already.

2.

$$f(x) = \arccos(x)/\pi, \quad I = [-1, 1],$$
$$D_0 = (0, 1], \quad D_1 = [-1, 0],$$
$$r_0(x) = 2x^2 - 1, \quad r_1(x) = 1 - 2x^2.$$

Theorem 4 says that the points whose value is rational are the fixed points of $r^k = \pm(r_0 \circ \cdots \circ r_0)$ (the sign on each set $(r_{\mu_1}^{-1} \circ \cdots \circ r_{\mu_k}^{-1})(I)$ must be chosen so that r^k is increasing) and their orbits. Or put the other way round, these are the values of $\cos(\pi x)$ if x is rational. Note that $r_0^k = r_0 \circ \cdots \circ r_0$ is the Chebychev polynomial of the first kind T_{2^k} for $[-1, 1]$.

3.

$$f(x) = 2\arcsin(x)/\pi, \quad I = [0, 1],$$
$$D_0 = [0, 1/\sqrt{2}), \quad D_1 = [1/\sqrt{2}, 1],$$
$$r_0(x) = 2x\sqrt{1 - x^2}, \quad r_1(x) = 2x^2 - 1.$$

4.

$$f(x) = \begin{cases} \arctan(x)/\pi & \text{if } x \in [0, \infty) \\ 1 + \arctan(x)/\pi & \text{if } x \in [-\infty, 0) \end{cases}, \quad I = \mathbb{R} \cup \{-\infty\}$$
$$D_0 = [0, \infty), \quad D_1 = [-\infty, 0),$$
$$r_0(x) = \frac{2x}{1 - x^2}, \; r_0(1) = -\infty, \quad r_1(x) = \frac{2x}{1 - x^2}, \; r_1(-1) = -\infty.$$

5.

$$f(x) = \operatorname{arccot}(x)/\pi, \quad I = \mathbb{R} \cup \{-\infty\},$$
$$D_0 = [0, \infty), \quad D_1 = [-\infty, 0),$$
$$r_0(x) = \frac{x^2 - 1}{2x}, \; r_0(0) = -\infty, \quad r_1(x) = \frac{x^2 - 1}{2x}.$$

6.

$$f(x) = \operatorname{arsinh}(x)/\ln 2, \quad I = [0, 3/4],$$
$$D_0 = [0, 1/2\sqrt{2}), \quad D_1 = [1/2\sqrt{2}, 3/4],$$
$$r_0(x) = 2x\sqrt{1 + x^2}, \quad r_1(x) = 5/2x\sqrt{1 + x^2} - 3/2x^2 - 3/4.$$

7. Denote by F and K the incomplete (resp. complete) elliptic integral of the first kind, take $k \in [0,1)$.

$$f(x) = \begin{cases} F(\sqrt{1-x^2}, k)/2K(k) & \text{if } x \geq 0 \\ 1 - F(\sqrt{1-x^2}, k)/2K(k) & \text{if } x < 0 \end{cases}, \quad I = [-1,1],$$

$$D_0 = (0,1], \quad D_1 = [-1,0],$$

$$r_0(x) = \frac{x^2 - (1-x^2)(k^2x^2 + 1 - k^2)}{x^2 + (1-x^2)(k^2x^2+1-k^2)}, \quad r_1(x) = -\frac{x^2 - (1-x^2)(k^2x^2+1-k^2)}{x^2+(1-x^2)(k^2x^2+1-k^2)}.$$

(Another way to write this is the following, with $x' := \sqrt{1-x^2}$: $r_0(x) = \frac{x^2+k^2x'^4-x'^2}{1-k^2x'^4}$, $r_1(x) = -\frac{x^2+k^2x'^4-x'^2}{1-k^2x'^4}$.)

Set, *e.g.*, $k = 1/\sqrt{2}$ and $x = 1/\sqrt{2}$. Then the recursion (1), (2) yields

$$2F(1/\sqrt{2}, 1/\sqrt{2})/\beta(1/4, 1/4) = (0.001110010000011\cdots)_2$$

where $\beta(x,y) = \int_0^1 (1-u)^{x-1}u^{y-1}du$ for $x, y > 0$ (beta function); note that $K(1/\sqrt{2}) = 2^{-2}\beta(1/4, 1/4)$ (*cf.* [1]). As a second example, set $k = \sqrt{2}-1$ and $x = 1/\sqrt{2}$. Then we get

$$2^{9/4}F(1/\sqrt{2}, \sqrt{2}-1)/\beta(1/8, 1/8) = (0.001111100001010\cdots)_2;$$

note that $K(\sqrt{2}-1) = 2^{-13/4}\beta(1/8, 1/8)$ (*cf.* [1]).

Theorem 4 says that the points whose value is rational are the fixed points of $r^k = \pm(r_0 \circ \cdots \circ r_0)$ (the sign must on each set $(r_{\mu_1}^{-1} \circ \cdots \circ r_{\mu_k}^{-1})(I)$ be chosen so that r^k is increasing) and their orbits. Or put the other way round, these are the values of $\mathrm{cn}\big(2K(k)x\big)$ if x is rational.

The common denominator in these examples is an algebraic addition theorem (*cf.* [1]). Assume that $g\colon \mathbb{R} \longrightarrow \mathbb{R}$, injective on $[0,1]$ and $g([0,1]) = I$, satisfies $P\big(g(x), g(y), g(x+y)\big) = 0$ for all $x, y \in \mathbb{R}$ with a polynomial P. Set $D_0 := g\big([0,1/2)\big)$ and $D_1 := g([1/2,1])$. Then $f := (g|_{[0,1]})^{-1}$ satisfies (S) where r_0 and r_1 are determined by solving $P\big(g(x), g(x), g(2x)\big) = 0$ for $g(2x)$ and $P\big(g(2x), g(-1), g(2x-1)\big) = 0$ for $g(2x-1)$ (with the use of r_0 here). For examples 4 and 5, the interval $[0,1]$ would have to be replaced by $[0,1)$.

A bit more explicit is the following approach which does not work for examples 2 and 7. Assume $g\colon [0,2] \longrightarrow \mathbb{R}$ is injective on $[0,1]$ and on $[1,2]$ and $g([0,1]) = I$. (Again, replace these by halfopen intervals for examples 4 and 5.) Assume that g is a solution of the functional equation $g(x+y) = a\big(g(x), g(y)\big)$ for $x, y \in [0,1]$ with an algebraic function $a\colon I \times I \longrightarrow g([0,2])$. Let $f := (g|_{[0,1]})^{-1}$, therefore $f\colon I \longrightarrow [0,1]$. For $u, v \in I$ with $f(u) + f(v) \in [0,1]$, we have

$$g\big(f(u)+f(v)\big) = a\Big(g\big(f(u)\big), g\big(f(v)\big)\Big) = a(u,v),$$

or $f(u) + f(v) = f\big(a(u,v)\big)$. If we set $r_0(u) := a(u,u)$ for $u \in g\big([0,1/2)\big) =: D_0$, then for each $u \in D_0$ we have $f\big(r_0(u)\big) = 2f(u)$.

For any $x \in [0,1]$ and for any $u, v \in I$ with $f(u)+f(v) \in [x, x+1]$, there exists a $b = b(u,v) \in I$ with $a\big(b, g(x)\big) = a(u,v)$. Indeed, we can write $f(u)+f(v) = y+x$ with $y \in [0,1]$. Set $b := g(y)$. Then

$$\begin{aligned} a(u,v) &= a\Big(g\big(f(u)\big), g\big(f(v)\big)\Big) = g\big(f(u)+f(v)\big) \\ &= g(y+x) = a\big(g(y), g(x)\big) = a\big(b, g(x)\big). \end{aligned}$$

Now choose $x = 1$. Then, if $u, v \in I$ with $f(u)+f(v) \in [1,2]$, we have

$$g\big(f(u)+f(v)\big) = a(u,v) = a\big(b, g(1)\big) = g\big(f(b)+1\big).$$

Since g is injective on $[1,2]$, it follows that $f(u)+f(v) = f(b)+1$. If we set $r_1(u) := b(u,u)$ for $u \in g([1/2,1]) =: D_1$, then for each $u \in D_1$ we have $f\big(r_1(u)\big) = 2f(u) - 1$.

Let us see how this works for Example 3. Here, $g(x) = \sin(\pi x/2)$, $I = g([0,1]) = [0,1]$, $D_0 = g\big([0,1/2)\big) = [0, 1/\sqrt{2})$, $D_1 = g([1/2,1]) = [1/\sqrt{2}, 1]$ and $f(u) = 2\arcsin(u)/\pi$ for $u \in [0,1]$. g satisfies $g(x+y) = g(x)\sqrt{1-g^2(y)} + g(y)\sqrt{1-g^2(x)}$, therefore $a(u,v) = u\sqrt{1-v^2}+v\sqrt{1-u^2}$ for $u, v \in [0,1]$. To determine $b(u,v)$, we have to solve $a\big(b, g(1)\big) = a(u,v)$ for $b \in [0,1]$ if $u, v \in [0,1]$ and $f(u)+f(v) \in [1,2]$. This yields $b(u,v) = uv - \sqrt{1-u^2}\sqrt{1-v^2}$ for $u, v \in [0,1]$ and $u^2+v^2 \geq 1$. Thus, $r_0(u) = a(u,u) = 2u\sqrt{1-u^2}$ if $u \in D_0$ and $r_1(u) = b(u,u) = 2u^2 - 1$ if $u \in D_1$.

However, this kind of transcendental functions is not the only interesting class of solutions of (S) with "simple" r_0, r_1. In fact, (S) can be used to construct rather peculiar functions. We will give two examples here: First, we construct a continuous, "almost" surjective function f for which the set $\{x \in I : f$ is not constant in a neighbourhood of $x\}$ is nowhere dense in I. Second, we construct a strictly monotone function which has derivative 0 almost everywhere.

In the following Theorem 5, the interior of D_0, D_1 and I is taken in $\mathbb{R}$.

THEOREM 5. *Let r_0, r_1 be continuous and increasing, $D_0 < D_1$, and r_0 mapping $\mathring{D}_0$ onto $\mathring{I}$, r_1 mapping $\mathring{D}_1$ onto $\mathring{I}$. (Then f is continuous and increasing.) If there exists an x_1 such that $x_1 \in \mathring{D}_0$ with $r_0(x_1) \leq x_1$ or $x_1 \in \mathring{D}_1$ with $r_1(x_1) \geq x_1$, then for each $x, y \in I$ with $x < y$, there exists a non-degenerate interval $J \subseteq [x,y]$ with $f|_J =$* const.

PROOF. Assume that $x_1 \in D_0$ with $r_0(x_1) \leq x_1$ (wlog). Let $\bar{x} := \max\{x \in D_0 : r_0(x) \leq x\}$. We have $\bar{x} \in \mathring{D}_0$ and, since r_0 is increasing, $r_0(x) \leq x \Leftrightarrow x \leq \bar{x}$, which means by (1), (2) that $f(x) = 0 \Leftrightarrow x \leq \bar{x}$. Inductively, $f(x) = (0, \nu_0 \cdots \nu_{n-1} 00 \cdots)_2 \Leftrightarrow (r_{\nu_{n-1}} \circ \cdots \circ r_{\nu_0})(x) \leq \bar{x}$. Since r_0, r_1 map $\mathring{D}_0$ resp. $\mathring{D}_1$ onto $\mathring{I}$, the set

$$(r_{\nu_0}^{-1} \circ \cdots \circ r_{\nu_{n-1}}^{-1})\big(\mathring{D}_0 \cap (-\infty, \bar{x})\big) = \operatorname{int}\{x \in I : f(x) = (0, \nu_0 \cdots \nu_{n-1} 00 \cdots)_2\}$$

is non-empty; since f is increasing, this set is a non-degenerate interval. Since the dyadic rationals are dense in $f(I)$, the assertion follows. ∎

At the end of Section 2, we asserted that there are non-strictly monotonic solutions f of (S) with strictly increasing r_0, r_1. Theorem 5 provides us with an example: Set $I = [0,1]$,

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$D_0 = [0, 1/2)$, $D_1 = [1/2, 1]$, $r_0(x) = 4x^2$ and $r_1(x) = 1 - 4(x-1)^2$. Then the solution of this system (S) has the desired property.

However, functions which have this property can also be constructed as solutions of systems (S) which do not satisfy the assumptions of Theorem 5. It is enough that r_0, r_1 map $\mathring{D}_0$ resp. $\mathring{D}_1$ onto $\mathring{I}$ and that the solution f has at least one non-degenerate interval of constancy. Here is another example: Set $I = [0, 1]$, $D_0 = [0, 1/2)$, $D_1 = [1/2, 1]$ and $r_0(x) = \begin{cases} 5x & \text{if } x \leq 1/8 \\ x + 1/2 & \text{if } x \geq 1/8 \end{cases}$, $r_1(x) = \begin{cases} x - 1/2 & \text{if } x \leq 7/8 \\ 5x - 4 & \text{if } x \geq 7/8 \end{cases}$.

We turn now to the construction of a continuous, strictly increasing function f which has derivative 0 almost everywhere. As mentioned, these functions, constructed in the following Theorem 6, are the inverses of de Rham's functions with the same property. Note that the set of measure 1 on which the derivative of f vanishes is mapped by f onto a set of measure 0 on which the derivative of f^{-1} does not exist, and vice versa.

THEOREM 6. *Take any $t \in (0, 1) \setminus \{1/2\}$. Set $D_0 = [0, t)$, $D_1 = [t, 1]$, $r_0(x) = x/t$, $r_1(x) = (x - t)/(1 - t)$. Then the solution f of this system (S) is strictly increasing and its derivative is* 0 *wherever it exists.*

PROOF. First, according to the conditions in Theorems 2 and 3, f is continuous and increasing. In fact, f is strictly increasing: Set $L := \min\{1/t, 1/(1-t)\}$, then $L > 1$. For each $x, y \in I$ for which there exists an $n \in \mathbb{N}_0$ such that $a_k(x) \in D_0 \Leftrightarrow a_k(y) \in D_0$ for all $k = 0, \ldots, n$ is true, we have $|a_n(x) - a_n(y)| \geq L^n|x - y|$ (with $a_n(x) := (r_{\nu_{n-1}} \circ \cdots \circ r_{\nu_0})(x)$, as in the proof of Theorem 2). Therefore, $f(x) = f(y)$, which means $a_n(x) \in D_0 \Leftrightarrow a_n(y) \in D_0$ for all $n \in \mathbb{N}_0$, implies $x = y$.

Since f is strictly monotonic, $f'(x)$ exists almost everywhere. Take any x where $f'(x)$ exists. Assume $f(x) = (0, \nu_0\nu_1 \cdots)_2$. Set $t_0 := t$ and $t_1 := 1 - t$. For each $n \in \mathbb{N}_0$, choose $x_n, y_n \in [0, 1]$ with $f(x_n) = (0, \nu_0 \cdots \nu_{n-1}00 \cdots)_2$ and $f(y_n) = f(x_n) + 1/2^n$. Since f is strictly increasing, we have $x_n \leq x < y_n$. It can be proved by induction (see below) that $y_n - x_n = t_{\nu_0} \cdots t_{\nu_{n-1}}$.

Now assume $f'(x) \neq 0$. Then

$$\frac{f(y_n) - f(x_n)}{y_n - x_n} \cdot \frac{y_{n+1} - x_{n+1}}{f(y_{n+1}) - f(x_{n+1})} \longrightarrow 1 \quad (n \longrightarrow \infty).$$

But we have

$$\frac{f(y_n) - f(x_n)}{y_n - x_n} \cdot \frac{y_{n+1} - x_{n+1}}{f(y_{n+1}) - f(x_{n+1})} = \frac{1/2^n}{t_{\nu_0} \cdots t_{\nu_{n-1}}} \cdot \frac{t_{\nu_0} \cdots t_{\nu_{n-1}} \cdot t_{\nu_n}}{1/2^{n+1}} = 2t_{\nu_n}.$$

The latter sequence has at least one partial sequence which is constant, but its value ($2t$ or $2(1-t)$) is unequal to 1 if $t \neq 1/2$. That is a contradiction. ∎

We close this section with the induction we omitted in the proof of Theorem 6. Notations are as in that proof.

LEMMA. *For each $n \in \mathbb{N}_0$ and $i = (\nu_0 \cdots \nu_{n-1})_2 \in \{0, \ldots, 2^n - 1\}$, let $x_{i,n}, y_{i,n} \in [0, 1]$ such that $f(x_{i,n}) = i/2^n$, $f(y_{i,n}) = (i+1)/2^n$. Then $y_{i,n} - x_{i,n} = t_{\nu_0} \cdots t_{\nu_{n-1}}$.*

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PROOF. First, we define $r_0(t) := 1$, such that $2f(x) = f\big(r_0(x)\big)$ for $x \in D_0 \cup \{t\}$.

For $n = 0$, we have $i = 0$, $x_{0,0} = 0$, $y_{0,0} = 1$, and $y_{0,0} - x_{0,0} = 1$, whereas $t_{\nu_0} \cdots t_{\nu_{n-1}}$ is the empty product.

Assume now that the assertion has been proved for an $n \in \mathbb{N}_0$. To prove it for $n+1$, we distinguish two cases; first, assume $i \in \{0, \ldots, 2^n - 1\}$. Then $(i+1)/2^{n+1} \le 1/2$, which means that $x_{i,n+1}, y_{i,n+1} \in D_0 \cup \{t\}$ and that $x_{i,n}, y_{i,n}$ exist. Moreover, we have $f(x_{i,n+1}) = (0, 0\nu_1 \cdots \nu_n 00 \cdots)_2$ and $f(x_{i,n}) = (0, \nu_1 \cdots \nu_n 00 \cdots)_2$. Now,

$$f(x_{i,n+1}/t) = f\big(r_0(x_{i,n+1})\big) = 2f(x_{i,n+1}) = 2 \cdot i/2^{n+1} = i/2^n = f(x_{i,n});$$

since f is strictly increasing, that means that $x_{i,n+1} = tx_{i,n}$. Similarly, $y_{i,n+1} = ty_{i,n}$. Therefore,

$$t_0 \cdot t_{\nu_1} \cdots t_{\nu_n} = t \cdot (y_{i,n} - x_{i,n}) = ty_{i,n} - tx_{i,n} = y_{i,n+1} - x_{i,n+1}.$$

On the other hand, if $i \in \{2^n, \ldots, 2^{n+1} - 1\}$, then $i/2^{n+1} \ge 1/2$, which means that $x_{i,n+1}, y_{i,n+1} \in D_1$ and that $x_{i-2^n,n}, y_{i-2^n,n}$ exist. Moreover, we have $f(x_{i,n+1}) = (0, 1\nu_1 \cdots \nu_n 00 \cdots)_2$ and $f(x_{i-2^n,n}) = (0, \nu_1 \cdots \nu_n 00 \cdots)_2$. Now,

$$\begin{aligned} f\big((x_{i,n+1} - t)/(1-t)\big) &= f\big(r_1(x_{i,n+1})\big) = 2f(x_{i,n+1}) - 1 = 2 \cdot i/2^{n+1} - 1 \\ &= (i - 2^n)/2^n = f(x_{i-2^n,n}); \end{aligned}$$

since f is strictly increasing, that means that $x_{i,n+1} = (1-t)x_{i-2^n,n} + t$. Similarly, $y_{i,n+1} = (1-t)y_{i-2^n,n} + t$. Therefore,

$$\begin{aligned} t_1 \cdot t_{\nu_1} \cdots t_{\nu_n} &= (1-t) \cdot (y_{i,n} - x_{i,n}) \\ &= (1-t)y_{i,n} + t - \big((1-t)x_{i,n} + t\big) = y_{i,n+1} - x_{i,n+1}. \end{aligned}$$

■

4. **Closing remarks.** We conclude the paper with two remarks concerning possible generalizations.

First, the same type of reasoning as we did here would also work for systems of more than two equations. We give here only two examples.

1. $f(x) = \arccos(x)/\pi$ satisfies

$$\begin{aligned} 3f(x) &= f(4x^3 - 3x) && \text{if } x \in (1/2, 1], \\ 3f(x) - 1 &= f(-4x^3 + 3x) && \text{if } x \in (-1/2, 1/2], \\ 3f(x) - 2 &= f(4x^3 - 3x) && \text{if } x \in [-1, -1/2]. \end{aligned}$$

That means that f can be computed by the following recursion:

$$\text{Set } a_0 = x, \ a_{n+1} = \begin{cases} 4a_n^3 - 3a_n & \text{if } a_n \in (1/2, 1] \cup [-1, -1/2] \\ -4a_n^3 + 3a_n & \text{if } a_n \in (-1/2, 1/2] \end{cases}.$$

Then $\arccos(x)/\pi = \sum_{a_n \in (-1/2, 1/2]} \frac{1}{3^{n+1}} + \sum_{a_n \in [-1, -1/2]} \frac{2}{3^{n+1}}$.

2. Pick any $b \in \mathbb{N} \setminus \{1\}$. Then $f(x) = \ln x$ satisfies

$$bf(x) - \nu = f(e^{-\nu}x^b) \quad \text{if } x \in \left[\exp\left(\frac{\nu}{b}\right), \exp\left(\frac{\nu+1}{b}\right)\right],$$

for $\nu = 0, \ldots, b-1$ on $I = [1, e]$.

Second, there is no particular reason for the set I to be an interval. In fact, Theorems 1 and 2 seem to work in any topological space I. Possibly, functional equations of type (S) can be used to construct "counterexamples", *i.e.*, functions with peculiar properties, in such spaces. However, it might sometimes be necessary to take discontinuous functions r_0, r_1 to describe a continuous function f, as we saw in the example before Theorem 2. So, the construction of a counterexample by use of (S) might be more difficult than a straight-forward construction.

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