CHARACTERISTIC MULTIPLIERS AND STATIONARY INTEGRALS

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In this note some rudimentary results about the characteristic multipliers of periodic solutions of differential equations are given which supplement those given by Poincaré [2], Chapitre IV, and by Wintner [4].

Motivation was supplied by some recent numerical computations of Bartlett [1] who found many periodic solutions of the restricted 3-body problem with the following property: at the periodic solution the energy integral assumes a value which is an extremum with respect to the values which it assumes at nearby periodic solutions of the same family.

1. Preliminaries

Our results will refer to an autonomous differential equation

$$\dot{x} = f(x) \tag{1}$$

in which the given function f is assumed to be of class $C^{(1)}$ on an open set S lying in R^n or C^n (the spaces of n-tuples of real or complex numbers respectively). A solution x is to be defined on some interval of the real line R and its range is to be a subset of S.

We shall use the letter Φ throughout to denote a family $\{\phi_\gamma : \gamma \in J\}$ of periodic solutions of (1) where J is an open interval of R. We shall suppose that no ϕ_γ is an equilibrium (i.e. constant) solution of (1) and we shall use $\tau(\gamma)$ to denote the primitive period (> 0) of ϕ_γ.

By saying that the family Φ of periodic solutions is *smooth* we shall mean that

(a) the mapping $(\gamma, t) \to \phi_\gamma(t)$ is of class $C^{(1)}$ on the product space $J \times R$;

(b) the function τ, the period along the family, is differentiable on J.

(Mr. W. A. Coppel remarks that, in view of the implicit function theorem, if (a) is satisfied then continuity of the function τ is sufficient to ensure its differentiability.)

As a notational convenience we shall suppose that $0 \in J$ and shall single out the member ϕ_0 of the family for a distinguished rôle. Relative to ϕ_0 the equations of variation of (1) are

$$\dot{x} = \partial f(\phi_0)x \tag{2}$$

where ∂f denotes the Jacobian matrix of the function f.

The *monodromy matrix* Γ of ϕ_0 is defined as $X(\tau(0))$ where X is the fundamental matrix of solutions of (2) which satisfies the initial condition $X(0) = I$, I being the unit matrix. The eigenvalues of Γ are called the *characteristic multipliers* of the periodic solution ϕ_0.

Let g be a function from the region S, on which f is defined, into the reals and let g be of class $C^{(1)}$. If for each solution x of (1) the composite function $g(x)$ is constant we shall say that g is a (*conservative*) *integral of* (1). For any family Φ of periodic solutions of (1) we shall denote the function

$$\gamma \to g(\phi_\gamma) \text{ by } g_\Phi$$

and we shall refer to it as the *integral g along the family* Φ. By saying that the integral g is *nontrivially stationary along the family* Φ *at* ϕ_0 we shall mean that

$$\nabla g(\xi) \neq 0 \text{ and } \dot{g}_\Phi(0) = 0 \tag{3}$$

where ∇ is the gradient operator, $\cdot$ denotes differentiation as usual and where ξ is some point in the range of the function ϕ_0, say $\xi = \phi_0(0)$.

Note that if (1) is a Hamiltonian differential equation and g is its energy integral then the first condition of (3) follows from the fact that ϕ_0 is not an equilibrium solution.

If $n = 1$ the first condition in (3) is inconsistent with the assumption that ϕ_0 is not an equilibrium solution. The results which follow are therefore significant only when $n \geqq 2$.

2. Statement and discussion of results

THEOREM 1. *If the differential equation* (1) *admits an integral g which is nontrivially stationary along a smooth family* Φ *of periodic solutions of* (1) *at* ϕ_0 *then*

(a) *the period* τ *along* Φ *is stationary at* ϕ_0,

or

(b) ϕ_0 *has at least* 3 *of its characteristic multipliers equal to* 1.

In the case $n = 2$ the alternative (a) is the only possible conclusion, of course.

Note that if (1) is a *Hamiltonian* differential equation, each of its periodic solutions has characteristic multipliers which occur in reciprocal pairs (see, e.g., Wintner [5], § 151) so that the alternative (b) in the conclusion of Theorem 1 may be replaced by:

ϕ_0 *has at least* 4 *characteristic multipliers equal to* 1.

Next, a slightly weakened version of Theorem 1:

THEOREM 2. *Let Φ be a smooth family of periodic solutions of* (1) *and suppose that*

(a) *the period along Φ is nowhere stationary,*
(b) *every member of Φ has at most* 2 *characteristic multipliers equal to* 1.

Then for every integral g of (1) *such that ∇g is nowhere zero, the function g_Φ (the corresponding integral along the family) is one-one.*

Theorem 2 is weaker, for conservative Lagrangian systems, than the following result which is stated in § 100 of Wintner's book [5]:

(*) *Along a (suitably smooth) family of periodic solutions of a conservative Lagrangian system, the period is a (single-valued) function of the energy.*

At the crux of the proof of (*) given by Wintner, however, there is a fallacy and (*) itself is false — except perhaps for systems with one degree of freedom — a counterexample being given below in Section 4. Theorem 2 is proposed as a replacement for (*).

Our final theorem provides a partial converse of Theorem 1:

THEOREM 3. *Let ϕ_0 be a periodic solution of* (1) *with at least* 3 *of its characteristic multipliers equal to* 1. *If the matrix $\Gamma - I$ (Γ the monodromy matrix of ϕ_0) has rank $n-1$, then every integral g of* (1) *is stationary at ϕ_0 along any smooth family of periodic solutions to which ϕ_0 belongs.*

Some interest attaches to the conditions imposed on Γ in the above theorem as they are sufficient to ensure that ϕ_0 does in fact belong to a (locally unique) smooth family of periodic solutions. This follows by a straightforward application of Poincaré's "continuity method" (as expounded for example in Siegel [3], § 19) to ϕ_0 as generating solution.

Theorem 3 may be of some use in the numerical search for periodic solutions, for example by providing information about the Jordan normal forms of monodromy matrices — information which would probably be unobtainable by the standard methods of numerical analysis.

3. Proofs

Two lemmas will be used. The first gives some more or less classical results which relate the monodromy matrix of a periodic solution to various derivatives, while the second is a result of linear algebra concerning the orthogonality of eigenvectors and a generalization of it may be published later.

We use the dash $'$ to denote transposition of matrices.

LEMMA 1. *Let $\{\phi_\gamma : \gamma \in J\}$ be a smooth family of periodic solutions of* (1) *and for each real t let $u(t)$ be the derivative at* 0 *of the function $\gamma \to \phi_\gamma(t)$. The monodromy matrix Γ of ϕ_0 then satisfies the relations*

$$\begin{aligned} \Gamma\dot{\phi}_0(0) &= \dot{\phi}_0(0) \\ \Gamma u(0) &= u(0) - \dot{\tau}(0)\dot{\phi}_0(0) \\ [\nabla g(\phi_0(0))]'\Gamma &= [\nabla g(\phi_0(0))]' \end{aligned} \tag{4}$$

where, in the last equation, g is assumed to be an integral of (1).

PROOF. By the superposition principle, if x is a solution of the equations of variation (2) then

$$\Gamma x(0) = x(\tau(0)). \tag{5}$$

Now each of the functions $\dot{\phi}_0$ and u is a solution of the equations of variation (Wintner [5], §§ 148, 149) and moreover

$$u(t) = \psi(t) - t\dot{\tau}(0)\tau(0)^{-1}\dot{\phi}_0(t)$$

where ψ as well as $\dot{\phi}_0$ has the period $\tau(0)$. If in turn $\dot{\phi}_0$ and u are substituted in (5) in place of x and periodicity is used, then the first and second equations in (4) are obtained, respectively.

Finally note that the function $(\xi, t) \to [\nabla g(\phi_0(t))]'\xi$ is a (nonconservative) integral of (2) (Wintner [5], § 87) and from this fact and the periodicity of ϕ_0 follows the last equation in (4).

LEMMA 2. *Let A be a complex matrix of order n ($\geqq 2$) with p of its eigenvalues equal to* 0 *and let the Jordan normal form of A be represented either by a block diagonal matrix*

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \tag{6}$$

where C_2 is nonsingular and where C_1 is the p-th order matrix

$$\begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & 0 & \\ & 1 & 0 & & & \\ & & \cdot & \cdot & & \\ & 0 & & \cdot & \cdot & \\ & & & & 1 & 0 \end{bmatrix} \tag{6'}$$

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or by C_1 itself when $p = n$. If x, y, z are column vectors satisfying the equations

$$(7) \qquad Ax = 0, \quad Ay = x, \quad A'z = 0$$

then $y'z = 0$ when $p > 2$; but $y'z \neq 0$ when $p = 2$ provided that $x \neq 0$ and $z \neq 0$.

PROOF. Let B be the nonsingular matrix which reduces A to Jordan normal form $C = B^{-1}AB$ and put $u = B^{-1}x$, $v = B^{-1}y$, $w = B'z$. The equations (7) are then equivalent to

$$(8) \qquad Cu = 0, \quad Cv = u, \quad C'w = 0,$$

and moreover

$$(9) \qquad y'z = v'B'(B')^{-1}w = v'w.$$

Since the matrix C has the form (6) or (6′) it is possible to solve the equations (8) for the components u_i, v_i, w_i $(1 \leqq i \leqq n)$ of the vectors u, v, w to get the solutions:

$$\begin{aligned} u_i &= 0 \quad \text{for} \quad i \neq p \quad (1 \leqq i \leqq n); \\ v_{p-1} = u_p \quad \text{and} \quad v_i &= 0 \quad \text{for} \quad i \neq p, \quad i \neq p-1 \quad (1 \leqq i \leqq n); \\ w_i &= 0 \quad (2 \leqq i \leqq n). \end{aligned}$$

From (9) and the solutions just obtained it is clear that

$$(10) \qquad y'z = v'w = v_1 w_1$$

and hence that $y'z = 0$ unless $p = 2$.

On the other hand if $p = 2$ and $x \neq 0$, and $z \neq 0$ then $u \neq 0$, $w \neq 0$ and so $v_{p-1} \neq 0$, $w_1 \neq 0$. From (10) it now follows that $y'z \neq 0$.

PROOF OF THEOREM 1. Suppose, contrary to the conclusion of Theorem 1, that the monodromy matrix Γ of ϕ_0 has less than 3 eigenvalues equal to 1 and that $\dot{t}(0) \neq 0$. Lemma 1 then shows that Γ has exactly 2 eigenvalues equal to 1 and that $\Gamma - I$ has as Jordan normal form (6) or (6′) with $p = 2$.

The first hypothesis of Lemma 2 is therefore satisfied with $A = \Gamma - I$. The remaining hypotheses of Lemma 2 are satisfied by the choice

$$x = -\dot{t}(0)\dot{\phi}_0(0), \quad y = u(0), \quad z = \nabla g(\phi_0(0)),$$

where $u(0)$ is the derivative at 0 of the function $\gamma \to \phi_\gamma(0)$. Since $\dot{t}(0) \neq 0$ and ϕ_0 is not an equilibrium solution, $x \neq 0$; and from the first of (3) follows $z \neq 0$. Lemma 2 now gives $y'z \neq 0$.

Now the integral along the family, g_Φ, is the composite function $\gamma \to g(\phi_\gamma(0))$. The chain rule gives therefore

$$(11) \qquad \dot{g}_\Phi(0) = [\nabla g(\phi_0(0))]'u(0) = z'y \neq 0.$$

This contradicts the hypothesis of Theorem 1.

Proof of Theorem 3. The conditions imposed ensure that $\Gamma - I$ has Jordan normal form (6) or (6′) with $p > 2$. Now let Φ be a smooth family of periodic solutions of (1) containing ϕ_0 and let x, y, z be defined as in the proof of Theorem 1. Application of Lemma 2 gives $y'z = 0$ and hence, by use of the chain rule as in (11), $\dot{g}_\Phi(0) = 0$ as required.

4. Counterexample for (*)

To obtain a counterexample for the statement (*) mentioned in Section 2 consider the system with Lagrangian function defined by

$$L(\dot{x}, \dot{y}, x, y) = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+2(x^2+y^2)^{-1}$$

which corresponds to the equations of motion

$$\ddot{x} = -4x(x^2+y^2)^{-2}, \quad \ddot{y} = -4y(x^2+y^2)^{-2}.$$

For each $\gamma > 0$ this system admits the periodic solution given by

$$x(t) = \gamma \cos (2t\gamma^{-2}), \quad y(t) = \gamma \sin (2t\gamma^{-2}).$$

The period of this solution is clearly $\pi\gamma^2$ while its energy is easily verified to be 0.

The family of periodic solutions obtained by variation of γ now gives the desired counterexample, the period and energy along the family being strictly monotonic and constant, respectively.

References

[1] J. H. Bartlett, 'The restricted problem of three bodies', *Mat. Fys. Skr. Dan. Selsk.* 2 (1964).

[2] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste* (*Tome* I) (Gauthier-Villars, Paris, 1892).

[3] C. L. Siegel, *Vorlesungen über Himmelsmechanik* (Springer, 1956).

[4] A. Wintner, 'Three notes on characteristic exponents and equations of variation in celestial mechanics,' *Amer. J. Math.* 53 (1931), 605–625.

[5] A. Wintner, *The analytical foundations of celestial mechanics* (Princeton, 1947).

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