Proceedings of the Edinburgh Mathematical Society (1996) **39**, 285–289 ©

MINIMAL PERMUTATION REPRESENTATIONS OF THE TWO DOUBLE COVERS OF S_n

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(Received 6th June 1994)

Let G be a finite group and denote by $\mu(G)$ (see [2]) the least positive integer m such that G has a faithful permutation representation in the symmetric group of degree m. This note considers the value of $\mu(G)$ when G is a double cover of the symmetric group.

1991 *Mathematics subject classification*: 20B99.

1. Introduction

We will adhere to the notation of [2] and denote by $\mu(G)$ the least positive integer m such that a finite group G can be embedded in S_m, the symmetric group of degree m. It will also be convenient to distinguish a permutation in S_m by the number of non-trivial disjoint cycles it contains.

For $n \geqq 4$, S_n has two proper double covers, non-isomorphic if $n \neq 6$, see [1]. These have the following standard presentations.

The double cover of S_n which lifts a transposition of S_n to an element of order 4, will be denoted by $\tilde{S}_n$, so that $\tilde{S}_n$ is the group with generators $z, r_1, r_2, \ldots, r_{n-1}$ and relations

$$\begin{aligned} z^2 &= 1, \\ zr_i &= r_i z,\ r_i^2 = z \text{ for } 1 \leqq i \leqq n-1, \\ (r_j r_{j+1})^3 &= z \text{ for } 1 \leqq j \leqq n-2, \\ r_k r_h &= z r_h r_k \text{ for } |h-k| > 1 \text{ and } 1 \leqq h, k \leqq n-1. \end{aligned}$$

Denote by $\hat{S}_n$ the double cover of S_n which lifts a transposition of S_n to an element of order 2. Thus $\hat{S}_n$ is the group with generators $z, s_1, s_2, \ldots, s_{n-1}$ and relations

$$\begin{aligned} z^2 &= 1, \\ zs_i &= s_i z,\ s_i^2 = 1 \text{ for } 1 \leqq i \leqq n-1, \\ (s_j s_{j+1})^3 &= 1 \text{ for } 1 \leqq j \leqq n-2, \\ s_k s_h &= z s_h s_k \text{ for } |h-k| > 1 \text{ and } 1 \leqq h, k \leqq n-1. \end{aligned}$$

Note. Factoring out the central subgroup $Z = \langle 1, z \rangle$ recovers S_n in both cases.

2. $\mu(\tilde{S}_n)$

For computations in $\tilde{S}_n$ we will use the method outlined in a paper by David B. Wales [3]. In this elements are of the form $\pm[\sigma_1]\dots[\sigma_r]$, where the σ_i are disjoint cycles in S_n and $\pm[\sigma_i]$ are the corresponding lifts in $\tilde{S}_n$.

Definition 2.1. For $1 \leqq a_i \leqq m$ we define

$$[a_1 a_2 \dots a_m] = a_1 a_2 \dots a_m a_1.$$

We call $\pm[a_1 a_2 \dots a_k]$ *signed cycles* in $\tilde{S}_n$. Each is a lift of the cycles $(a_1 a_2 \dots a_k)$ in S_n.

In fact each a_i corresponds to an element of a subgroup of a Clifford algebra which is isomorphic to $\tilde{S}_n$. But the following rules are sufficient to enable the calculation of products of disjoint signed cycles, which we refer to as signed permutations, in $\tilde{S}_n$ (these appear as 2.3 and 2.4 in [3]).

1.A $[a_i] = -1$.

1.B $[a_1 a_2 \dots a_m] = (-1)^{m+1}[a_2 a_3 \dots a_m a_1]$.

1.C $[a_1 a_2 \dots a_{m-1}]a_m = (-1)^m a_m [a_1 a_2 \dots a_{m-1}]$.

Example.

$$\begin{aligned}[12](-[234]) &= -[12][234] = [21][234] = 2122342\\ &= 21[2]34 = -21342 = -[2134] = [1342].\end{aligned}$$

As a further example, note that by identifying r_i with the positive signed cycle $[i, i+1]$, and z with -1, we recover our presentation for $\tilde{S}_n$.

Hence if we consider the elements of $\tilde{S}_n$ as products of signed cycles, with -1 replacing the central element z, the following lemma is easily proved by induction.

Lemma 2.2. *A positive signed cycle of length k in $\tilde{S}_n$ has either*

1. order k if $k \equiv 5$, 6, 7 or 0 modulo 8, or

2. order $2k$ if $k \equiv 1$, 2, 3 or 4 modulo 8.

While a negative signed cycle of length k in $\tilde{S}_n$ has either

1. order k if $k \equiv 0$, 1, 3 or 6 modulo 8, or

2. order $2k$ if $k \equiv 2$, 4, 5 or 7 modulo 8.

Example.

$$\begin{aligned}(+[123])^3 &= [123][123][123] = 123112311231\\ &= 123[1]23[1]231 = 12323231\\ &= 1[23][32]1 = -1[23][23]1\\ &= -12322321 = -123[2]321\\ &= 123321 = 12[3]21 = -1221\\ &= -1[2]1 = 11 = [1] = -1.\end{aligned}$$

$$\text{Thus } (+[123])^6 = (-1)^2 = 1.$$

While

$$(-[123])^3 = -(+[123])^3 = -(-1) = 1.$$

Proposition 2.3. *Let H be a subgroup of $\tilde{S}_n$. If H contains a positive signed cycle of length k with $k \equiv 1$, 2, 3 or 4 modulo 8, or a negative signed cycle of length k with $k \equiv 2$, 4, 5 or 7 modulo 8, then $-1 \in H$.*

Proof. Clear by the previous lemma, for if H contains a singed cycle τ of length k satisfying the above conditions, then $\tau^k = -1 \in H$. □

Proposition 2.4. *Let H be a subgroup of $\tilde{S}_n$, $n \geqq 2$. A permutation representation, ρ say, from $\tilde{S}_n/H$ to S_n is not a faithful representation of $\tilde{S}_n$ if $-1 \in H$.*

Proof. First note that ρ is faithful only if for all $g \in \tilde{S}_n$,

$$\mathrm{Core}_{\tilde{S}_n}(H) = \bigcap g^{-1}Hg = \{1\}.$$

So if $-1 \in H$, since -1 is central, then $\mathrm{Core}_{\tilde{S}_n}(H) \neq \{1\}$ and ρ is not a faithful representation. □

Ths proposition is sometimes of help in determining the value of $\mu(\tilde{S}_n)$.

Example. We consider the value of $\mu(\tilde{S}_4)$. First note that S_4 has permutations whose cycle types are 1, 2, 2^2, 3 and 4, thus it follows that $\tilde{S}_4$ has signed permutations of cycle type 1, 2, 2^2, 3 and 4.

By the previous proposition we require subgroups of $\tilde{S}_4$ not having positive signed cycles of length 2, 3 or 4, or negative signed cycles of length 2 or 4. So the only possible non-trivial subgroups we may consider can contain only positive or negative signed cycles type 2^2 and negative signed cycles of type 3. But any signed cycle of the form $\pm[ab][cd] \in \tilde{S}_4$ squares to give -1 as

$$\begin{aligned}(\pm[ab][cd])^2 &= [ab][cd][ab][cd] \\ &= [ab][cd]aba[cd] \\ &= -[ab]aba[cd][cd] \\ &= -[ab][ab][cd][cd] \\ &= -(-1)(-1) \\ &= -1.\end{aligned}$$

Hence the only possible non-trivial subgroups which do not contain -1 are isomorphic to

$$H = \{1, -[123], -[132]\} \text{ which has core} = \{1\}.$$

Thus

$$\mu(\tilde{S}_4) = |\tilde{S}_4 \colon H| = 48/3 = 16.$$

e.g. $\tilde{S}_4$ is minimally embedded in S_{16}.

Following the notation of D.L. Johnson [**2**], let ρ along with $\{G_1, G_2, \ldots, G_m\}$ be a permutation representation of $\tilde{S}_n$, where each G_i is a subgroup of $\tilde{S}_n$ and the degree of ρ equals the sum of $|\tilde{S}_n \colon G_i|$. Also ρ is faithful if $\mathrm{Core}_{\tilde{S}_n}(\bigcap G_i) = \{1\}$. Note that by Proposition 2.3, if ρ is a faithful representation of $\tilde{S}_n$, then $\{G_1, G_2, \ldots, G_m\}$ must contain at least one subgroup which does not contain -1.

Lemma 2.5. *Let N be a normal non-trivial subgroup of $\tilde{S}_n$, $n \geqq 5$. Then either $N = Z = \{1, -1\}$, or $N = A_n$, the double cover of the alternating group.*

Proof. Clearly Z and $\tilde{A}_n$ are normal in $\tilde{S}_n$. So let N be any other normal subgroup of $\tilde{S}_n$. Then NZ/Z is a normal subgroup of $\tilde{S}_n/Z$, which is isomorphic to S_n. It follows that NZ is either $\tilde{S}_n$, $\tilde{A}_n$ or Z, so that N is $\tilde{S}_n$. □

Theorem 2.6. *For $n \geqq 4$, and H a subgroup of $\tilde{S}_n$ that does not contain -1,*

$$\mu(\tilde{S}_n) = \min_{H \leqq \tilde{S}_n} |\tilde{S}_n \colon H|$$

Proof. Clear for $n = 4$ by the previous example, so consider when $n \geqq 5$.

Let $\Omega = \{G_1, G_2, \ldots, G_m\}$ and ρ be a minimal permutation representation of $\tilde{S}_n$. Now suppose that, without any loss of generality, that G_1 does not contain -1 and let M be a maximal subgroup of $\tilde{S}_n$ which does not contain -1 but does contain G_1. Recall by Proposition 2.4 that we require $-1 \notin M$ for a faithful representation of $\tilde{S}_n$.

The core of M is the largest normal subgroup of $\tilde{S}_n$ contained in M, thus by the previous lemma this is $\{1\}$ as $\tilde{A}_n$ and Z both contain -1. Hence $\tilde{S}_n$ is isomorphic to a subgroup of S_y where $y = |\tilde{S}_n \colon M|$. Thus the degree of ρ is equal to

$$\sum_{i=1}^{m} |\tilde{S}_n \colon G_i| \leqq |\tilde{S}_n \colon M|,$$

so we have

$$|\tilde{S}_n \colon G_1| - |\tilde{S}_n \colon M| + \sum_{i=2}^{m} |\tilde{S}_n \colon G_i| \leqq 0$$

which as

$$|\tilde{S}_n \colon G_1| = |\tilde{S}_n \colon M| |M \colon G_1|$$

gives

$$|\tilde{S}_n \colon M| (|M \colon G_1| - 1) + \sum_{i=1}^{m} |\tilde{S}_n \colon G_i| \leqq 0$$

implying that $m = 1$ and $M = G_1$. Hence the degree of ρ is equal to

$$|\tilde{S}_n: G_1| = |\tilde{S}_n: M|,$$

where M is a maximal subgroup of $\tilde{S}_n$ not containing -1. □

3. $\mu(\hat{S}_n)$

By identifying -1 with z in the presentation for $\tilde{S}_n$ it is again possible to consider elements of $\tilde{S}_n$ as signed cycles, although different rules are required when considering the products of these signed cycles. However the following theorem follows readily from the previous section.

Theorem 3.1. *For $n \geqq 5$, and H a subgroup of $\tilde{S}_n$ that does not contain -1,*

$$\mu(\hat{S}_n) = \min_{H \leqq \hat{S}_n} |\hat{S}_n: H|$$

Proof. This follows by replacing $\tilde{S}_n$ by $\hat{S}_n$ in Proposition 2.4, Lemma 2.5 and Theorem 2.6. □

Although Theorem 2.6, along with Lemma 2.2, can go some way towards finding the value of μ, it would be of interest to find a formula for $\mu(\tilde{S}_n)$ (respectively $\mu(\tilde{S}_n)$) in terms of n.

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