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ISOMETRIES OF H^p SPACES OF BOUNDED SYMMETRIC DOMAINS

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1. Introduction. The isometries of the Hardy spaces H^p $(0 < p < \infty, p \neq 2)$ of the unit disc were determined by Forelli in 1964 [**3**]. For $p = 1$ the result had been found earlier by deLeeuw, Rudin and Wermer [**2**]. For several variables the state of affairs at present is this: For the polydisc the isometries of H^p *onto* itself have been characterized by Schneider [**13**]. For the unit ball the same result was proved in the case $p > 2$ by Forelli [**4**]. Finally in [**12**] Rudin removed the restriction $p > 2$ and also established some results about isometries of H^p of the ball and the polydisc *into* itself.

The purpose of this note is to show that the methods developed by Forelli, Rudin and Schneider apply to bounded symmetric domains in general. We get a complete characterization of the isometries of H^p *onto* itself. For isometries *into*, we prove the same kind of reduction to a question about "inner maps" as Rudin [**12**]. In the final section we give some examples and a discussion of what can be said about such maps in general.

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2. Preliminaries. Let D be a bounded symmetric domain given in the canonical Harish-Chandra realization in the complex n-space $\mathbf{C}^n$. The following facts are well known [**6**; **9**]. Let G be the group of holomorphic automorphisms of D, K the isotropy group of O in G. Then K acts by complex linear transformations. Let S be the Bergman-Šilov boundary of D; S is a compact real-analytic submanifold of $\mathbf{C}^n$ to which the action of G extends. It is homogeneous under K and therefore has a unique normalized K-invariant measure μ; all our measure theoretic notions, in particular the spaces $L^p(S)$ will be relative to this measure. The domain D always has a realization as a generalized half-plane: This is a tube domain over a homogeneous self-dual cone if and only if $\dim_{\mathbf{R}} S = n$. In this case D is said to be of *tube type*. The Cartesian product of bounded symmetric domains is again in the same class and its Šilov boundary is the product of the Šilov boundaries of the factors. Every such domain arises in a natural way as the product of irreducible ones.

As proved in [**7**] D has a Szegö kernel, i.e. for all fixed $z \in D$ there exists a function S_z, holomorphic on $\bar{D}$ and such that

$$f(z) = \int_S f\bar{S}_z d\mu$$

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for all continuous holomorphic f. There is also a "Poisson kernel", i.e. for every $z \in D$ there exists a positive continuous function P_z on S such that

$$(1) \qquad \int_S (f \circ \tau) d\mu = \int_S f P_{\tau(0)} d\mu$$

for all automorphisms τ of D, and all integrable f on S. The two kernels are related by

$$(2) \qquad P_z = \frac{|S_z|^2}{S_z(z)}.$$

The sets D and S are circular, i.e. the group $\mathbf{T}$ of complex numbers of modulus one acts on them by multiplication. So $\mathbf{T}$ is contained in the center of K and consequently we have, for every $f \in L^1(S)$, and every measurable $f \geqq 0$, that

$$(3) \qquad \int_S f d\mu = \int_S \left(\int_{\mathbf{T}} f(\zeta w) d\zeta \right) d\mu(w),$$

where $d\zeta$ is the normalized Haar measure of $\mathbf{T}$. As first observed by Bochner [**1**], formula (3) is useful for reducing function-theoretic questions on D to the one-variable case.

It has been shown by R. Hermann [**10**, page 371] that D is convex, and therefore star-shaped relative to the origin. Consequently, if f is a (possibly vector-valued) function on D and $0 \leqq r < 1$, then we can define a function f_r on S by $f_r(w) = f(rw)$. If $f^*(w) = \lim f_r(w)$ exists for a.e. $w \in S$, as $r \to 1$, then we say that f^* is the boundary function of f. The Nevanlinna class $N(D)$ is defined [**11**; **14**] as the class of holomorphic functions f on D such that

$$\int_S \log^+ |f_r| d\mu$$

is bounded for $0 \leqq r < 1$. The subclass $N_*(D)$ of $N(D)$ consists of those f for which $\{\log^+ |f_r|\}$ is a uniformly integrable family. Any strongly convex function φ (cf. [**11**, p. 37] for the definition of this concept) gives a further subclass $H_\varphi(D)$ of $N_*(D)$ which consists of the functions satisfying

$$\sup_{0 \leqq r < 1} \int_S \varphi(\log |f_r|) d\mu < \infty.$$

The special case $\varphi(t) = \exp pt$ $(0 < p < \infty)$ gives the spaces $H^p(D)$. Finally, $H^\infty(D)$ is defined as the space of bounded holomorphic functions on D. A function is called *inner* if it is in $H^\infty(D)$, and $|f^*| = 1$ a.e. A holomorphic map $F : D \to D$ is called an inner map if $F^*(w) \in S$ for a.e. $w \in S$.

A number of properties of these function spaces have been proved in [**5**] and [**14**] by the use of subharmonicity and of formula (3). Thus, for each $f \in N(D)$, f^* exists, and $\log |f^*| \in L^1(S)$ unless $f = 0$. This implies that $f^* \neq 0$ a.e., and hence each $f \in N(D)$ is uniquely determined by f^*. One can define in an obvious way the spaces of boundary functions $N(S)$, $N_*(S)$, $H_\varphi(S)$,

$H^p(S)$; then $H^p(S)$ is a closed subspace of $L^p(S)$. If $g \in N(S)$, then the unique function $F \in N(D)$ such that $F^* = g$ will be denoted by $\tilde{g}$.

We also need the following result, a simple extension of Theorem 3.4.2 of [**11**]. We omit the proof.

LEMMA 1. *Let $f \in N_*(D)$ and let φ be strongly convex.*

(i) *Then $f \in H_\varphi(D)$ if and only if $\varphi(\log |f^*|) \in L^1(S)$. If this is the case, then for $0 \leqq r < 1$,*

$$(4) \qquad \int_S \varphi(\log |f_r|)d\mu \leqq \int_S \varphi(\log |f^*|)d\mu,$$

moreover the left hand side of (4) converges to the right hand side as $r \to 1$.

(ii) *$f \in H^\infty(D)$ if and only if $f^* \in L^\infty(S)$.*

3. The main results. The following lemma is a rather trivial generalization of a result of Schneider [**13**]; we give a simple self-contained proof.

LEMMA 2. *Let $g \in N_*(D)$, $g \neq 0$, and $h \in L^\infty(S)$. If $g^*h^k \in N_*(S)$ for $k = 1, 2, 3, \ldots$, then $h \in H^\infty(S)$.*

Proof. One may assume that $||h||_\infty \leqq 1$. Define $f_k = (g^*h^k)^\sim$ for $k = 1, 2, 3, \ldots$ and $H = f_1/g$. It suffices to prove that $H \in H^\infty(D)$, since $H^* = h$.

By considering boundary values one sees that

$$(5) \qquad f_1^k = g^{k-1} f_k,$$

$$(6) \qquad g H^k = f_k.$$

To prove that H is holomorphic, following [**13**], we consider (5) in the local ring of holomorphic functions at z, $z \in D$. Let p be an irreducible factor of g and let p^λ, p^γ, p^{λ_k} be the highest powers of p occurring in f_1, g, and f_k, respectively. Then (5) shows that $k\lambda \geqq (k-1)\gamma + \lambda_k$ for $k = 1, 2, 3, \ldots$, whence $\lambda \geqq \gamma$ and g divides f_1 at z. Since z is arbitrary, H is holomorphic in D.

To prove that H is bounded note that since $g \in N_*(D)$ we have $g \in H_\varphi(D)$ for some strongly convex φ. Since $||h||_\infty \leqq 1$, $|f_k^*| \leqq |g^*|$ a.e., and Lemma 1 implies that $f_k \in H_\varphi(D)$. Also by Lemma 1,

$$\int_S \varphi(\log |(f_k)_r|)d\mu \leqq \int_S \varphi(\log |g^*|)d\mu.$$

Denote the right hand side of this inequality by A. Let $0 \leqq r < 1$ be fixed and $E_r = \{w \in S : |H_r(w)| > 1\}$. Using (6) and Jensen's inequality we have

$$\varphi\left(\int_{E_r} \log |H_r^k g_r| d\mu\right) \leqq \int_{E_r} \varphi(\log |H_r^k g_r|)d\mu \leqq A,$$

and hence for any fixed real a such that $\varphi(a) \geqq A$,

$$k \int_{E_r} \log |H_r| d\mu + \int_{E_r} \log |g_r| d\mu \leqq a.$$

Since this holds for all $k = 1, 2, 3, \ldots$ and since $\log |H_r| > 0$ on the open set E_r, it follows that E_r is empty. Thus $|H_r| \leqq 1$ on S, whence $|\tilde{H}_r| \leqq 1$ on D, and, letting r tend to 1, $|H| \leqq 1$ on D.

The following theorem is the principal result of this paper. For a discussion of inner maps and the condition (8) see Section 4.

THEOREM. *Let D be a bounded symmetric domain in the canonical realization and let $0 < p < \infty$, $p \neq 2$.*

(i) *Let $T : H^p(D) \to H^p(D)$ be a linear isometry, and denote $T1$ by g. Then there exists an inner map τ of D such that, for all $f \in H^p(D)$,*

$$(7) \qquad Tf = g\,(f \circ \tau),$$

and

$$(8) \qquad \int_S (h \circ \tau^*)|g^*|^p d\mu = \int_S h d\mu$$

for every bounded Borel function h on S.

(ii) *If τ is an inner map of D and $g \in H^p(D)$ is such that* (8) *holds with every continuous function h on S, then* (7) *defines an isometry of $H^p(D)$.*

(iii) *The linear isometry T is onto $H^p(D)$ if and only if τ is an automorphism of D and*

$$(9) \qquad g = \alpha\left(\frac{S_u{}^2}{S_u(u)}\right)^{1/p},$$

where α is a complex number of modulus one, S is the Szegö kernel of D, and $u = \tau^{-1}(0)$. With this g, (8) *is automatically satisfied.*

Proof. Let T be an isometry of $H^p(D)$ and set $g = T1$. Then $g \in H^p(D)$, so $g^* \neq 0$ a.e. Define the measure $d\nu$ on S by $d\nu = |g^*|^p d\mu$. Define, for all $f \in H^\infty(S)$, $Af = Tf/g^*$. Then $A1 = 1$ and

$$\int |Af|^p d\nu = \int |f|^p d\mu.$$

Thus, the hypotheses of Rudin's Theorem II [**12**] are satisfied, and it follows that $A(fh) = (Af)(Ah)$ for all f, h in $H^\infty(S)$. It follows further that for every m and $f_1, f_2, \ldots f_m \in H^\infty(S)$ the m-tuples $F = (f_1, \ldots f_m)$ and $G = (Af_1, \ldots Af_m)$ are equimeasurable. (This means that $\mu(F^{-1}(E)) = \nu(G^{-1})(E))$ for all Borel sets $E \subset \mathbf{C}^n$. It implies that F and G have the same essential range. In particular $||Af||_\infty = ||f||_\infty$ for all $f \in H^\infty(S)$. Note also that $||f||_\infty$ is the same relative to μ and to ν because these measures are mutually absolutely continuous.) We have furthermore that

$$g^*(Af)^k = g^*A(f^k) = T(f^k) \in H^p(S) \quad \text{for } k = 1, 2, 3, \ldots,$$

because A is multiplicative. Lemma 2 now implies that $Af \in H^\infty(S)$, and we have shown that A is an isometric algebra endomorphism of $H^\infty(S)$.

Let $e_1, \ldots, e_n$ be a basis of $\mathbf{C}^n$; $\zeta_1, \ldots, \zeta_n$ its dual basis. The functions $\zeta_1, \ldots, \zeta_n$ are coordinate functions on $\mathbf{C}^n$. Let $\zeta_1^*, \ldots, \zeta_n^*$ be their restrictions to S. Define $\tau(z)$ by

$$\tau(z) = \sum_{i=1}^{n} (A\zeta_i^*)^\sim(z)e_i,$$

then τ is a bounded holomorphic map of D into $\mathbf{C}^n$. It follows immediately that for every complex linear function $\lambda = \sum \lambda_i \zeta_i$ one has

$$(10) \quad \lambda(\tau(z)) = (A\lambda^*)^\sim(z).$$

As it was pointed out in Section 2, D is convex. Since it is also circular, it is the unit ball for a Banach space structure on $\mathbf{C}^n$. Let D' be the unit ball of the dual Banach space. It is well known that

$$D = \{z \in \mathbf{C}^n : |\lambda(z)| < 1 \text{ for all } \lambda \in D'\}.$$

By the defining property of S,

$$\sup \{|\lambda(z)| : z \in D\} = ||\lambda^*||_\infty,$$

and therefore $\lambda \in D'$ if and only if $||\lambda^*||_\infty < 1$. Since A is an isometry it follows by (10) that for every $z \in D$ and $\lambda \in D'$,

$$|\lambda(\tau(z))| \leqq ||A\lambda^*||_\infty = ||\lambda^*||_\infty < 1.$$

We have shown that τ maps D into itself. Also, by Rudin's theorem $(\zeta_1^*, \ldots, \zeta_n^*)$ and $(A\zeta_1^*, \ldots, A\zeta_n^*)$ have the same essential range, viz. S (this proves that τ is inner); they are equi-measurable; this implies (8) because every Borel function h on S is the restriction of some Borel function on $\mathbf{C}^n$.

Formula (10) and the multiplicativity of A imply $AP^* = P^* \circ \tau^*$ for every polynomial P on $\mathbf{C}^n$. Let now $f \in H^p(D)$ and let P_ν be a sequence of polynomials converging to f in H^p. Let $\rho(p)$ be 1 if $p \geqq 1$, and p if $p < 1$. Then

$$||Tf_r - g^*f_r \circ \tau^*||_p^{\rho(p)} \leqq ||Tf_r - TP_{\nu r}||_p^{\rho(p)} + ||g^*P_{\nu r} \circ \tau^* - g^*f_r \circ \tau^*||_p^{\rho(p)}$$
$$= ||f_r - P_{\nu r}||_p^{\rho(p)} + ||P_{\nu r} - f_r||_p^{\rho(p)}.$$

The equality of the second terms follows by (8). Letting ν tend to ∞, we have $Tf_r = g^*(f_r \circ \tau^*)$. Letting $r \to 1$ we have $Tf^* = g^*(f^* \circ \tau^*)$ a.e., finishing the proof of (i).

The proof of (ii) is a straightforward verification. To prove (iii) suppose T is onto, so it has an inverse given by $T^{-1}f = hf \circ \sigma$, with $h = T^{-1}1$. Then $1 = TT^{-1}1 = g(h \circ \tau)$, and for any linear function λ, $\lambda = TT^{-1}\lambda = g(h \circ \tau)(\lambda \circ \sigma \circ \tau) = \lambda \circ \sigma \circ \tau$. Hence τ is an automorphism and $\sigma = \tau^{-1}$. To prove (9), let F be continuous on S. Applying (8) to $h = F \circ \tau^{-1}$ and using

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(1) we have

$$\int_S F|g^*|^p d\mu = \int_S F \circ \tau^{-1} d\mu = \int_S F P_u d\mu$$

with $u = \tau^{-1}(0)$. Since this holds for any F, it follows that $|g^*|^p = P_u$. Now the equation

$$(11) \quad g = \alpha \left(\frac{S_u^{\ 2}}{S_u(u)} \right)^{1/p}$$

defines a function α on D which belongs to $H^p(D)$ because g does, and because S_u is nowhere zero on D [**7**]. From $|g^*|^p = P_u$ and (2) we conclude that $|\alpha^*| = 1$ a.e. Lemma 1 now implies that $\alpha \in H^\infty(D)$. Using again $1 = g(h \circ \tau)$ and (11) it follows that $\alpha^{-1} = (S_u(u)^{-1}S_u^{\ 2})^{1/p} h \circ \tau$, so $\alpha^{-1} \in H^p(D)$. But $(\alpha^{-1})^* = \bar{\alpha}^*$ by $|\alpha^*| = 1$. Hence both α and $\bar{\alpha}$ are in $H^\infty(D)$ and α must be constant. This concludes the proof of the "only if" part of (iii). The proof of the converse is an elementary verification.

4. Inner maps. If D is of tube type, then it has many inner maps. In fact the Cayley transform [**9**] carries D to a tube D^c over a homogeneous self-dual cone Ω, and S to the real subspace S^c lying over the vertex of the cone. One can assume that $S^c = \mathbf{R}^n$, then $D^c = \mathbf{R}^n + i\Omega$. One can find cones Ω_1 and Ω_2 affinely equivalent to "octants" such that $\Omega_1 \subset \Omega \subset \Omega_2$. The tube domains $P_j = \mathbf{R}^n + i\Omega_j$, $j = 1, 2$ are affinely equivalent to the Cartesian product of n copies of the one-dimensional upper half-plane. One has $P_1 \subset D^c \subset P_2$, and all three domains have the same Šilov boundary, viz. $\mathbf{R}^n$. The restriction to D^c of any holomorphic mapping from P_2 to P_1 which carries $\mathbf{R}^n$ to $\mathbf{R}^n$ is an inner map of D^c.

Even if D is not of tube type but has an irreducible factor of tube type this construction gives a multitude of non trivial inner maps. However, in the case where no irreducible factors are of tube type, it may be that the only inner maps are the automorphisms of D. When D is the unit ball in $\mathbf{C}^n$ $(n \geqq 2)$, this is expressly conjectured by Rudin [**12**].

Finding all isometries of $H^p(D)$ amounts to finding all pairs (τ, g) satisfying (8). In case τ is an automorphism of D, the compatible g's are exactly the functions of form (9) where α now is an aribitrary inner function. If D has a factor of tube type, then D has non-constant inner functions (e.g. by the same construction that produces inner maps), so $H^p(D)$ has isometries *into* itself. If however, D has no tube type factors it is reasonable to conjecture, following Rudin, that there are no non-constant inner functions, and no inner maps different from automorphisms. This would imply that in this case every isometry of $H^p(D)$ is onto itself.

If τ is not an automorphism, an example of Rudin [**12**] shows that there may be no corresponding g. A complete description of all possibilities seems hopeless: at any rate, we have the following positive result.

LEMMA 3. *If D has an irreducible factor of tube type, then there exist inner maps τ of D which are not automorphisms, and elements g of $H^p(D)$ such that $f \mapsto g\,(f \circ \tau)$ is an isometry of $H^p(D)$ into itself.*

Proof. Assume first that D is of tube type. It is shown in [**8**] that there exist non-constant homogeneous polynomials on D which are inner functions. Let P be such, and define $\tau(z) = P(z)z$ for all $z \in \bar{D}$. Clearly τ is inner; it is not an automorphism because $P(0) = 0$ and P must have other zeroes in D as well. Apply (3) to calculate $\int_S h \circ \tau^* d\mu$ for a continuous h on S. Using the homogeneity of P and the fact that $|P^*| = 1$ one readily obtains that it is equal to $\int_S h d\mu$. Hence (8) holds with any inner function g on D and the assertion follows.

In the general case let $D = D_1 \times D_2$ with D_1 of tube type. Construct τ and g for D_1 as above and extend their definition to D by making them independent of the D_2-variables.

Remark. In the case where it is known that D is not irreducible, there is an even simpler construction: Let $D = D_1 \times D_2$ with D_1 of tube type and let g_1 be a non constant inner function on D_1, continuous on $\bar{D}_1$. Define $\tau : D \to D$ by $\tau(z_1, z_2) = (z_1, g_1(z_1)z_2)$ and let g be any inner function on D.

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