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FINITE SUBLATTICES OF THREE-GENERATED LATTICES

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Abstract

Every lattice generated by three unordered elements contains a finite sublattice generated by three unordered elements. A list $\mathscr{L}$ of twelve finite lattices, each generated by a three-element unordered set, is given. It is proved that every lattice generated by a three-element unordered set contains a sublattice isomorphic to one of the lattices in $\mathscr{L}$; moreover, $\mathscr{L}$ is the smallest such list.

Lattices generated by three elements abound. For example, Crawley and Dean (1959) have shown that there are uncountably many non-isomorphic three-generated lattices. It follows that there is no countable lattice into which every countable lattice can be embedded since such a lattice would, of necessity, contain only countably many finitely generated sublattices. On the other hand, three-generated lattices in a certain sense account for all countably generated lattices. In particular, Sorkin (1954) and independently Dean (1956) have shown that any countably generated lattice can be embedded in a lattice with three generators. In this connection we recall the well-known result of Whitman (1942) that the free lattice $FL(n)$ on n unordered generators is contained in $FL(3)$ as a sublattice, for every positive integer n; in fact, he showed that even $FL(\aleph_0)$ is embeddable in $FL(3)$.

Let L be a lattice with three generators. If the three generators are totally ordered then, of course, L is just a three-element chain. It is routine to verify that, if only two of the generators are comparable, then L is a homomorphic image of the nine-element lattice illustrated in Figure 1. Our interest here is with the case in which the generators of L are unordered. In particular, our motivation for the work reported in this paper stems from the question: *does every lattice generated by three unordered elements contain a finite sublattice generated by three unordered elements?*

Let $\mathscr{L} = \{L_i \mid i = 1, 2, \cdots, 9\} \cup \{L_7^d, L_8^d, L_9^d\}$ be the set of lattices illustrated in Figure 2, (L_i^d denotes the dual of L_i); every lattice in $\mathscr{L}$ is generated by three unordered elements. The purpose of this paper is to show that these lattices are characteristic of lattices generated by a three-element unordered set.

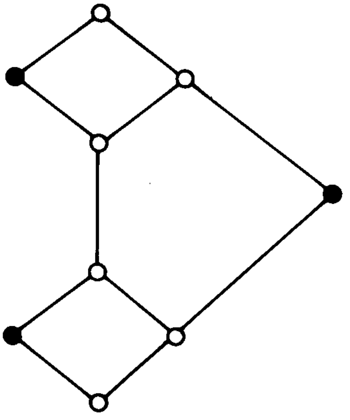

Figure 1.

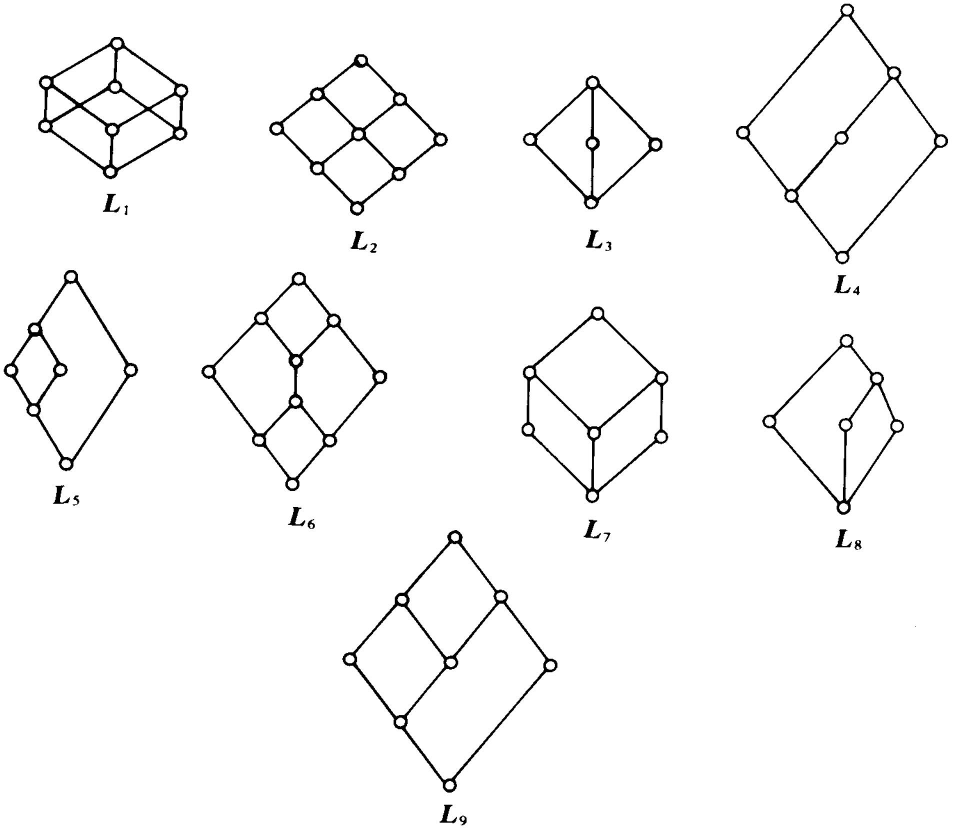

Figure 2.

THEOREM. *Let L be a lattice generated by a three-element unordered set. Then L contains a sublattice, generated by a three-element unordered set, isomorphic to one of the lattices in $\mathscr{L}$.*

For a careful reading of the proof it is instructive and, indeed, indispensable to actually draw, at each stage, that part of the lattice in question. Therefore it is noteworthy that, apart from $\mathbf{L}_1$, every lattice in $\mathscr{L}$ is planar. It is this property of the Hasse diagrams of the lattices in $\mathscr{L}$ that makes the problem studied in this paper feasible and the combinatorial arguments straightforward.

Before proceeding to the proof we dispose of certain preliminaries.

Let L be a lattice. For $x, y \in L$ we write $x \parallel y$ if x is incomparable with y. Recall that $x \in L$ is *join-reducible* (*meet-reducible*) if there exist $y, z \in L$ with $y \parallel z$ and $x = y \vee z$ ($x = y \wedge z$); x is *join-irreducible* (*meet-irreducible*) if it is not join-reducible (meet-reducible); x is *doubly irreducible* if it is join-irreducible and meet-irreducible. A subset A of L is an *antichain* if $x \parallel y$ for each pair of distinct elements x, y in A. As usual, for $a, b \in L$ we write $(b, a) = \{x \in L \mid b < x < a\}$, $(a] = \{x \in L \mid x \leqq a\}$, and $[a) = \{x \in L \mid x \geqq a\}$. We define a *pentagon* to be a quintuple $\langle a, b, c, u, v\rangle$ such that $a, b, c, u, v \in L$ and

$$u > a > b > v, \quad c \wedge a = v, \quad c \vee b = u.$$

For all further terminology we refer to Crawley and Dilworth (1973).

The case in which L is modular is easily handled. Any lattice which is modular but nondistributive contains a sublattice isomorphic to $\mathbf{L}_3$. If L is distributive and generated by a three-element antichain, then L is a homomorphic image of the free distributive lattice $FD(3)$ on three generators such that the images of the free generators form a three-element antichain in L. It is routine to verify that either L has a sublattice isomorphic to $\mathbf{L}_1$ or L is itself isomorphic to $\mathbf{L}_2$.

We commence our attack on the nonmodular case with a sequence of technical lemmas; the first lemma is part of the folklore.

LEMMA 1. *If $\{x, y, z\}$ is a three-element antichain in a lattice L, then the sublattice generated by $\{x \vee y, y \vee z, x \vee z\}$ is a homomorphic image of $\mathbf{L}_1$.*

PROOF. This is an immediate consequence of the fact that $\{x \vee y, y \vee z, x \vee z\}$ generates a sublattice isomorphic to $\mathbf{L}_1$ in the free lattice $FL(3)$ generated by x, y and z. □

COROLLARY 2. *Let $\{x, y, z\}$ be a three-element antichain in a lattice L which has no sublattice isomorphic to $\mathbf{L}_1$. If $x \vee y$ is incomparable with $y \vee z$ then $x \vee z = x \vee y \vee z$.* □

LEMMA 3. *Let $\langle a, b, c, u, v\rangle$ be a pentagon in a lattice L and let $s \in (c, u)$. If L contains no sublattice isomorphic to $\mathbf{L}_7$ or $\mathbf{L}_9$ then $a \wedge s = b \wedge s$.*

PROOF. Let us suppose that $a \wedge s > b \wedge s$. If $b \wedge s = v$, then

$$K = \{u, b \vee (a \wedge s), c \vee (a \wedge s), b, a \wedge s, c, v\} \cong \mathbf{L}_7$$

(that is, K is a sublattice of L isomorphic to $\mathbf{L}_7$). Hence we may assume that $b \wedge s > v$. If $c \vee (b \wedge s) = c \vee (a \wedge s)$, then

$$\{u, b \vee (a \wedge s), c \vee (a \wedge s), a \wedge s, b, b \wedge s, c, v\} \cong \mathbf{L}_9.$$

Otherwise, $c \vee (b \wedge s) < c \vee (a \wedge s)$. If $(a \wedge s) \wedge [c \vee (b \wedge s)] = b \wedge s$, then

$$\{u, b \vee (a \wedge s), c \vee (a \wedge s), a \wedge s, b, c \vee (b \wedge s), b \wedge s\} \cong \mathbf{L}_7;$$

if $(a \wedge s) \wedge [c \vee (b \wedge s)] > b \wedge s$, then

$$\{u, b \vee ((a \wedge s) \wedge [c \vee (b \wedge s)]), b, (a \wedge s) \wedge [c \vee (b \wedge s)], \\ c \vee (b \wedge s), b \wedge s, c, v\} \cong \mathbf{L}_9. \qquad \square$$

LEMMA 4. *Let $\langle a, b, c, u, v\rangle$ be a pentagon in a lattice L, let $s \in (v, b)$ and $t \in (v, c)$. If L contains no sublattice isomorphic to $\mathbf{L}_2$, $\mathbf{L}_6$, $\mathbf{L}_7$, $\mathbf{L}_7^d$, $\mathbf{L}_9$ or $\mathbf{L}_9^d$, then either $c \vee s = u$ or $\langle c, t, b \wedge (c \vee s), c \vee s, v\rangle$ is a pentagon in L.*

PROOF. Let us suppose that $c \vee s < u$. If, in addition, $b \vee t < u$, then by Lemma 3 and its dual we have that $a \wedge (c \vee s) = b \wedge (c \vee s)$, $b \vee t = a \vee t$ and $b \vee [c \wedge (b \vee t)] = a \vee [c \wedge (b \vee t)]$. If $(b \vee t) \wedge (c \vee s) = [b \wedge (c \vee s)] \vee [c \wedge (b \vee t)]$, then

$$\{u, b \vee t, c \vee s, b, c, (b \vee t) \wedge (c \vee s), b \wedge (c \vee s), c \wedge (b \vee t), v\} \cong \mathbf{L}_2;$$

otherwise, $(b \vee t) \wedge (c \vee s) > [b \wedge (c \vee s)] \vee [c \wedge (b \vee t)]$ and

$$\{u, b \vee t, c \vee s, b, c, (b \vee t) \wedge (c \vee s), [b \wedge (c \vee s)] \vee [c \wedge (b \vee t)], \\ b \wedge (c \vee s), c \wedge (b \vee t), v\} \cong \mathbf{L}_6.$$

Hence, $b \vee t = u$.

Now, if $t \vee [b \wedge (c \vee s)] < c \vee s$ then

$$\{u, b, c \vee s, c, t \vee [b \wedge (c \vee s)], b \wedge (c \vee s), c \wedge (t \vee [b \wedge (c \vee s)]), v\} \cong \mathbf{L}_9^d;$$

thus, $t \vee [b \wedge (c \vee s)] = c \vee s$. If $s \vee t < c \vee s$, then

$$\{c \vee s, b \wedge (c \vee s), c, s \vee t, c \wedge (s \vee t), (s \vee t) \wedge b \wedge (c \vee s), v\} \cong \mathbf{L}_7^d.$$

Therefore, $s \vee t = c \vee s$ and $\{c, t, b \wedge (c \vee s), c \vee s, v\}$ is a pentagon in L. $\square$

LEMMA 5. *Let L be a nonmodular lattice. In L there exists a pentagon $\langle a, b, c, u, v\rangle$ such that*

(i) *a is join-irreducible, b is meet-irreducible, and s is doubly irreducible for all $s \in (b, a)$,*

or there exists an element $d \in L$ such that

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(ii) $a = b \vee d$ *and* $\{b, c, d\}$ *is a three-element antichain, or dually*
(iii) $b = a \wedge d$ *and* $\{a, c, d\}$ *is a three-element antichain.*

PROOF. Since L is nonmodular it contains at least one pentagon. Note that if $\langle a, b, c, u, v\rangle$ is a pentagon and $s \in (b, a)$, then $\langle s, b, c, u, v\rangle$ and $\langle a, s, c, u, v\rangle$ are also pentagons. Also observe that if $b \parallel d$ and $d \leqq a$, then $c \parallel d$ and consequently $\{b, c, d\}$ is a three-element antichain.

Assume that (i) fails for every pentagon in L and let $\langle a, b, c, u, v\rangle$ be a fixed pentagon in L. Consider the case in which a is join-reducible, say $a = p \vee q$ with $p \parallel q$. If $p \leqq b$, then $a = b \vee q$ and hence (ii) is satisfied by the pentagon $\langle a, b, c, u, v\rangle$ and the element $d = q$. If $p \geqq b$, then the pentagon $\langle a, p, c, u, v\rangle$ and $d = q$ satisfy (ii). If $p \parallel b$, then the pentagon $\langle b \vee p, b, c, u, v\rangle$ and $d = p$ satisfy (ii). When b is meet-reducible the dual argument applies and gives rise to a pentagon satisfying (iii). Finally, if $s \in (b, a)$ is either join-reducible or meet-reducible we simply apply the above argument to one of the pentagons $\langle s, b, c, u, v\rangle$ or $\langle a, s, c, u, v\rangle$. □

LEMMA 6. *Let L be a nonmodular lattice containing a pentagon $\langle a, b, c, u, v\rangle$ and an element d such that $a = b \vee d$ and $\{b, c, d\}$ is a three-element antichain. Then L contains a sublattice isomorphic to $\mathbf{L}_4, \mathbf{L}_5, \mathbf{L}_7, \mathbf{L}_8, \mathbf{L}_9$ or $\mathbf{L}_9^d$.*

PROOF. If $d \vee c < u$ then, by Lemma 3, $d \leqq a \wedge (d \vee c) = b \wedge (d \vee c) \leqq b$, which contradicts $d \parallel b$; hence $d \vee c = u$.

Let us suppose that $d \wedge b \geqq v$. It follows that $d \wedge c = d \wedge a \wedge c = d \wedge v = v$. If $d \wedge b = v$, then $\{u, a, d, b, c, v\} \cong \mathbf{L}_8$; thus $d \wedge b > v$. If $c \vee (d \wedge b) = u$, then $\{u, a, d, b, c, d \wedge b, v\} \cong \mathbf{L}_5$. If $c \vee (d \wedge b) < u$, then, by Lemma 3, $a \wedge [c \vee (d \wedge b)] = b \wedge [c \vee (d \wedge b)]$. Moreover, applying Lemma 3 to the pentagon $\langle a, d, c, u, v\rangle$ we obtain $a \wedge [c \vee (d \wedge b)] = d \wedge [c \vee (d \wedge b)]$ and consequently,

$$d \wedge [c \vee (d \wedge b)] = d \wedge a \wedge [c \vee (d \wedge b)] = d \wedge b \wedge [c \vee (d \wedge b)] = d \wedge b.$$

Therefore, $\{u, a, d, b, c \vee (d \wedge b), d \wedge b\} \cong \mathbf{L}_8$.

Hence $d \wedge b \ngeqq v$. If $d \vee v < a$, then choosing $d' = d \vee v$ we have that $b \parallel d' \parallel c$, $d' \vee b = a$ and $d' \wedge b \geqq v$, whence the preceding argument applies. Consequently, $d \vee v = a$. If $d \wedge b = d \wedge v$, then $\{u, a, c, d, b, v, d \wedge v\} \cong \mathbf{L}_4$ so that $d \wedge b < d \wedge v$. If $c \vee (d \wedge b) = u$, then $\{u, a, c, d, v \vee (d \wedge b), d \wedge b, v, d \wedge v\} \cong \mathbf{L}_9^d$ and we get that $c \vee (d \wedge b) < u$. By Lemma 3, $a \wedge [c \vee (d \wedge b)] = b \wedge [c \vee (d \wedge b)]$ so that

$$\{u, a, c \vee (d \wedge b), b, d, b \wedge [c \vee (d \wedge b)], d \wedge b\} \cong \mathbf{L}_4. \qquad \square$$

Now to complete the nonmodular case. Let L be a nonmodular lattice generated by a three-element antichain $\{x, y, z\}$, and by way of contradiction let us assume that L has no sublattice isomorphic to one of the lattices in $\mathscr{L}$. In the light of Lemma 5 and Lemma 6 and its dual we may assume that there is a pentagon $\langle a, b, c, u, v\rangle$ in L which satisfies condition (i) of Lemma 5. If there is an

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element $d \in L$ such that $\{a, c, d\}$ is a three-element antichain, then $\{b, c, d\}$ is also a three-element antichain, and conversely; furthermore, $a \wedge d = b \wedge d$ and $a \vee d = b \vee d$. Indeed, if $\{a, c, d\}$ is a three-element antichain, then either $\{b, c, d\}$ is a three-element antichain or $b \leqq d$; but in the latter case $a \wedge d$ is a meet-reducible element with $b \leqq a \wedge d < a$, contrary to assumption. The dual argument establishes the converse. Likewise, if $a \wedge d < b \wedge d$, then $b \vee (a \wedge d)$ is a join-reducible element with $b < b \vee (a \wedge d) \leqq a$, contrary to assumption; hence $a \wedge d = b \wedge d$ and, dually, $a \vee d = b \vee d$. We now verify that among the pentagons in L which satisfy condition (i) of Lemma 5, there is at least one for which such an element, d, exists.

Let $\langle a, b, c, u, v\rangle$ be a pentagon in L which satisfies condition (i) of Lemma 5. If one of the generators, say x, lies in the interval (b, a), then we can restrict our attention to the pentagon $\langle x, b, c, u, v\rangle$ which also satisfies condition (i) of Lemma 5; hence we may assume that no generator lies in the interval (b, a). Let us suppose that, for every $d \in L$, neither $\{a, c, d\}$ nor $\{b, c, d\}$ is a three-element antichain. Since the generators x, y and z form an antichain there are, up to duality, only two possible cases: (a) $x, y, z \in [a) \cup (c]$, and (b) $x, y, z \in [a) \cup [c)$. We now show that Case (a) is impossible, and in Case (b) we construct the desired pentagon.

CASE (a). It is a simple matter to verify that $[a) \cup (c]$ is a sublattice of L. Since this sublattice contains the generators of L, $[a) \cup (c] = L$ which is impossible in view of the fact that $b \notin [a) \cup (c]$.

CASE (b). Since $x, y, z \geqq v$, it follows that v is the zero of L. We have assumed that the element b is meet-irreducible and, since b is not a generator, it follows that b is join-reducible. If $b = p \vee q$, where $p \parallel q$, then $p, q \in (v, b)$. If $p \vee c = u$, then $\langle b, p, c, u, v\rangle$ is a pentagon, $b = p \vee q$ and $\{p, c, q\}$ is a three-element antichain; by Lemma 6, this contradicts our assumption that L has no sublattice isomorphic to one of the lattices in $\mathscr{L}$. Hence $p \vee c < u$, and similarly $q \vee c < u$. Moreover, $p \vee c \parallel q \vee c$ (since otherwise, if $p \vee c \leqq q \vee c$, say, then $u = b \vee c = p \vee q \vee c = q \vee c < u$). If $a \wedge (p \vee c) > b \wedge (p \vee c)$ then $b \vee [a \wedge (p \vee c)]$ is a join-reducible element with $b < b \vee [a \wedge (p \vee c)] \leqq a$; thus $a \wedge (p \vee c) = b \wedge (p \vee c)$. Consequently $\langle a, b, p \vee c, u, b \wedge (p \vee c)\rangle$ is a pentagon and $\{b, p \vee c, q \vee c\}$ is the required three-element antichain.

Now let $\langle a, b, c, u, v\rangle$ be a pentagon in L satisfying condition (i) of Lemma 5, such that there exists an element d in L with $\{a, c, d\}$ a three-element antichain. By duality the six cases described below are exhaustive.

A. $d \vee b \leqq u$ *and* $d \wedge a \geqq v$; B. $d \vee b > u$ *and* $d \wedge a \geqq v$;
C. $d \vee b \parallel u$ *and* $d \wedge a \geqq v$;

D. $d \vee b \parallel u$ *and* $d \wedge a < v$; E. $d \vee b > u$ *and* $d \wedge a < v$;
F. $d \vee b \parallel u$ *and* $d \wedge a \parallel v$.

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In each case we obtain either a contradiction (by constructing a sublattice of L isomorphic to one of the lattices in $\mathscr{L}$), a reduction to a previous case, or a reduction to a situation to which Lemma 6, or its dual, applies.

CASE A. First, we consider the subcase in which $d \vee b = u$ and $d \wedge a = v$. Let us suppose that $d \vee c < u$. If $b \wedge (d \vee c) = v$, then apply Lemma 6 to the pentagon $\langle d \vee c, d, b, u, v\rangle$; hence we assume that $b \wedge (d \vee c) > v$. If $c \vee [b \wedge (d \vee c)] < d \vee c$, then $\langle d \vee c, c \vee [b \wedge (d \vee c)], b, u, b \wedge (d \vee c)\rangle$ is a pentagon to which Lemma 6 again applies; hence we assume that $c \vee [b \wedge (d \vee c)] = d \vee c$. Applying a similar argument to the pentagon $\langle d \vee c, d \vee [b \wedge (d \vee c)], b, u, b \wedge (d \vee c)\rangle$ yields $d \vee [b \wedge (d \vee c)] = d \vee c$. If $d \wedge c > v$, then, in view of Lemma 4, $(d \wedge c) \vee [b \wedge (d \vee c)] = d \vee c$ so that $\{d \vee c, d, c, b \wedge (d \vee c), d \wedge c, v\} \cong \boldsymbol{L}_8^d$. If $d \wedge c = v$, then $\{d \vee c, b \wedge (d \vee c), d, c, v\} \cong \boldsymbol{L}_3$. If $d \vee c = u$ and, dually, $d \wedge c = v$, then $\{u, a, d, c, v\} \cong \boldsymbol{L}_3$. This completes the subcase in which $d \vee b = u$ and $d \wedge a = v$.

We now suppose that $d \vee b < u$. If $c \wedge (d \vee b) = v$, then apply Lemma 6 to the pentagon $\langle d \vee b, b, c, u, v\rangle$. Therefore, we may assume that $c \wedge (d \vee b) > v$ and similarly, that $b \vee [c \wedge (d \vee b)] = d \vee b$ (for otherwise apply Lemma 6, again, to the pentagon $\langle d \vee b, b \vee [c \wedge (d \vee b)], c, u, c \wedge (d \vee b)\rangle$). If $d \wedge a = v$, then d and the pentagon $\langle a, b, c \wedge (d \vee b), d \vee b, v\rangle$ satisfy the conditions of the first subcase. Finally, if $d \wedge a = d \wedge b > v$, then we apply the dual of the argument just given to d and the pentagon $\langle a, b, c \wedge (d \vee b), d \vee b, v\rangle$ from which it follows that d and the pentagon $\langle a, b, (d \wedge b) \vee [c \wedge (d \vee b)], d \vee b, d \wedge b\rangle$ satisfy the conditions of the first subcase.

CASE B. We may assume that $d \wedge u = d \wedge a$, for otherwise, we apply case A to $d \wedge u$ and the pentagon $\langle a, b, c, u, v\rangle$. Hence $\langle u, b, d, d \vee b, d \wedge a\rangle$ is a pentagon to which Lemma 6 applies.

CASE C. If $c \vee (d \wedge a) = u$, then the dual of Lemma 6 applies to the pentagon $\langle a, d \wedge a, c, u, v\rangle$; hence we may assume that $c \vee (d \wedge a) < u$. In addition, $b \wedge [c \vee (d \wedge a)] = d \wedge a$, for otherwise the dual of Lemma 6 again applies to the pentagon $\langle b \wedge [c \vee (d \wedge a)], d \wedge a, c, c \vee (d \wedge a), v\rangle$. If $(d \vee b) \wedge [c \vee (d \wedge a)] > d \wedge a$, then case B applies to d and the pentagon

$$\langle a, b, (d \vee b) \wedge [c \vee (d \wedge a)], b \vee ((d \vee b) \wedge [c \vee (d \wedge a)]), d \wedge a\rangle;$$

thus, $(d \vee b) \wedge [c \vee (d \wedge a)] = d \wedge a$. Consequently $d \wedge [c \vee (d \wedge a)] = d \wedge (d \vee b) \wedge [c \vee (d \wedge a)] = d \wedge a$. Applying Corollary 2 to the three-element antichain $\{b, c, d\}$ yields $d \vee c = d \vee b \vee c$.

Finally, apply Lemma 6 to the pentagon $\langle d \vee b, d, c \vee (d \wedge a), d \vee c, d \wedge a\rangle$.

CASE D. It suffices to apply case C to c and the pentagon $\langle a, b, d, d \vee b, d \wedge a\rangle$.

CASE E. Here we may apply case A to c and the pentagon $\langle a, b, d, d \vee b, d \wedge a\rangle$.

CASE F. In view of Corollary 2 and its dual we have that $d \vee c = d \vee b \vee c$ and $d \wedge c = d \wedge b \wedge c$. We may assume that $c \vee (d \wedge a) = u$ (for otherwise, apply case D to d and the pentagon $\langle a, b, c \vee (d \wedge a), u, b \wedge [c \vee (d \wedge a)]\rangle$) and, dually, that $c \wedge (d \vee b) = v$. Furthermore, we may assume that $d \vee v = d \vee b$ (for otherwise, apply case C to $d \vee v$ and the pentagon $\langle a, b, c, u, v\rangle$) and, dually, that $d \wedge u = d \wedge a$. Hence

$$\{d \vee c, d \vee b, u, d, u \wedge (d \vee b), c, v \vee (d \wedge a), d \wedge a, v, d \wedge v\} \cong \boldsymbol{L}_6.$$

The proof of the Theorem is now complete.

The general problem that naturally arises in connection with the main result of this paper is the following: *given a partially ordered set P of order n, does every lattice generated by a subset isomorphic to P contain a finite sublattice generated by a subset isomorphic to P?* The results of this paper, of course, settle this question for the case $n = 3$. Moreover, observe that the following is an immediate corollary of the Theorem: *every lattice of width greater than or equal to three contains a sublattice of width three isomorphic to one of the lattices in* $\mathscr{L}$. For other results related to the general problem we refer the reader to Dean (1961) and Wille (1974).

Elsewhere Davey, Poguntke and Rival (1975) have shown: *a lattice of finite length is semi-distributive if and only if it contains no sublattice isomorphic to one of* $\boldsymbol{L}_3$, $\boldsymbol{L}_4$, $\boldsymbol{L}_7$, $\boldsymbol{L}_7^d$, $\boldsymbol{L}_8$ *or* $\boldsymbol{L}_8^d$.

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