ON ONE-PARAMETER SUBGROUPS IN FINITE DIMENSIONAL LOCALLY COMPACT GROUP WITH NO SMALL SUBGROUPS

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Let G be a locally compact topological group and let U be a neighborhood of the identity in G. A curve $g(\lambda)$ ($|\lambda| \leqq 1$) in G, which satisfies the conditions,

$$g(s)g(t) = g(s+t) \quad (|s|, |t|, |s+t| \leqq 1),$$

is called a one-parameter subgroup of G. If there exists a neighborhood U_1 of the identity in G such that for every element x of U_1 there exists a unique one-parameter subgroup $g(\lambda)$ which is contained in U and $g(1) = x$, we shall call, for the sake of simplicity, that U has the property (S)*. It is well known that the neighborhoods of the identity in a Lie group have the property (S)*. More generally it is proved that if G is finite dimensional, locally connected, and is without small subgroups,[1] G has the same property.[2] In this note, these theorems will be generalized to the case when G is finite dimensional and without small subgroups.

The writer's proof is based on the theorems recently developed by D. Montgomery and A. Gleason.[3] Their theorems, which will be used in this note, are summarized in § 1. In § 2 it will be proved that the group G, which is finite dimensional and without small subgroups, is locally connected and our theorem is reduced to the known case.

§ 1. THEOREM 1 (*Montgomery*).[4] *Let G be a locally compact locally connected n-dimensional group* ($n < \infty$). *Then there exists a neighborhood V of the identity in G possessing the following properties:*

Let A and B, ($B \subset A$), *be compact subsets of V. Then the sufficient con-*

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1) G is called to be without small subgroups, if there exists a compact neighborhood of the identity in G which does not contain non-trivial subgroups of G.

2) Cf. Chevalley, C. [1]. C. Chevalley proved the case when G is locally euclidean and without small subgroups. K. Iwasawa communicated to the present author that D. Montgomery pointed out that the Chevalley's method may be applicable even when G is locally connected and without small subgroups. It is also informed that H. Yamabe obtained the same result.

3) D. Montgomery [7], [8], [9], [10], A. Gleason [2].

4) This Theorem and its Corollaries are valid when G is a locally connected finite dimensional homogeneous space, or more generally, G is a locally homogeneous space. See D. Montgomery [8].

ditions for $A-B$ to be an open subset of G are

1) *B carries an $(n-1)$-cycle z*[5] *which is not homologous to zero in B, and*

2) *A is minimal with respect to the properties*

a) $B \subset A$

b) *z is homologous to zero in A.*

COROLLARY 1 to THEOREM 1. (*Invariance theorem of domain*). *Let G_1 and G_2 be locally compact locally connceted groups. Suppose that* $\dim G_1 = \dim G_2 = n < \infty$. *Let M be an open subset of G_1 and f be a topological mapping of M into G_2. Then the image $f(M)$ of M under the mapping f is an open subset of G_2.*

Proof. Let V_i be the neighborhood of the identity in G_i pointed out in Theorom 1 $(i=1, 2)$. Let p_2 be a point of $f(M)$ and let p_1 be the ponit M such that $f(p_1)=p_2$. We can take a neighborhood V_1' of the identity in G_1 such that $\overline{V}_1' \subseteqq V_1$, $\overline{V}_1' p_1 \subseteqq M$, $f(\overline{V}_1' p_1) \subseteqq V_2 p_2$. Since the dimension of $\overline{V}_1'$ is n, there exist compact subsets A_1 and B_1 of $\overline{V}_1'$ satisfying the conditions *1)* and *2)* of the Theorem 1.[6] Moreover, we can assume that the identity in G_1 is contained in $A_1 - B_1$. Then $f(A_1 p_1)$ and $f(B_1 p_1)$ are subsets of $V_2 p_2$ and satisfy the conditions *1)* and *2)* of the Theorem 1. Hence by Theorem 1 $f(A_1 p_1) - f(B_1 p_1)$ is an open subset of G_2. Since $p_2 \in f(A_1 p_1) - f(B_1 p_1) \subseteqq f(M)$, $f(M)$ is an open subset of G_2.

COROLLARY 2 to THEOREM 1. *Under the same notations and assumptions as in the Corollary 1, let N be an open subset of M such that $\overline{N} \subseteqq M$, and let x be an arbitrary point of $f(N)$. Then*

$$C_x(G_2 - f(bdry\, N))^{7)} \subseteqq f(N).$$

Proof. From the Corollary 1, it is easy to prove that

$$bdry\, f(N) = f(bdry\, N).$$

Hence, $G_2 - f(bdry\, N) = G_2 - bdry\, f(N) = f(N) \cup (G_2 - \overline{f(N)})$, $f(N) \cap (G_2 - \overline{f(N)}) = \phi$,[8] and both $f(N)$ and $(G_2 - \overline{f(N)})$ are open subsets of G_2. Since $x \in f(N)$, it follows that

$$C_x(G_2 - f(bdry\, N) \subseteqq f(N)).$$

THEOREM 2[9] (*Montgomery*). *Let G be a locally compact n-dimensional*

5) Cycles are in the sense of Cech.

6) Cf. Hurewicz and Wallman [2], p. 151.

7) If x is a point of topological space A, $C_x(A)$ is the connected component of A which contains x.

8) ϕ denotes the empty set.

9) This is a part of Theorem 7 of D. Montgomery [9].

group $(n<\infty)$. *Then there exists a locally compact locally connected group* G^* *of dimension n and a continuous one-to-one mapping* α *of* G^* *into G satisfying the following conditions.*

Let C^* *be a neighborhood of the identity in* G^*, *then* $\alpha(C^*)=C$ *is an invariant local snbgroup of G and the factor local group of G by C is zero-dimensional.*

§2. A neighborhood U of the identity in a topological group G is called to have the property (S), if for every element x of U there exists an integer n such that $x^{2^n}\notin U$.

Lemma 1 (*Yamabe*).[10] *Let G be a locally compact group, and suppose that G is without small subgroups. Let U be a neighborhood of the identity e in G such that U contains no non-trivial subgroups. For every neighborhood V of e there exists a neighborhood* V^* *of e satisfying the following conditions.*

If x and x^k *are contained in* V^* *and if* x^i $(1\leqq i\leqq k)$ *are elements of U, then* x^i *is contained in V for* $i=1, 2, \ldots, k$.

Corollary to Lemma 1 (*Yamabe and Gotô*).[11] *If a locally compact group G is without small subgroups, G has the property* (S).

Lemma 2.[12] *Let G be a locally compact group which is without small subgroups. Then there exists a neighborhood U of the identity in G, in which the square root is unique. More strictly, if x and y are elements of U, and if* $x^2=y^2$, *it follows that* $x=y$.

In this case the mapping $\varphi(x)=x^2$ of U into G is one-to-one.

Lemma 3.[12] *Let G be a locally compact group which is without small subgroups. Then on a sufficiently small neighborhood U of the identity in G we can define a real valued continuous function f(x) satisfying the following conditions.*

(3) $$f(x^2)\geqq 2f(x) \quad \text{for} \quad x,\ x^2\in U,$$

(4) $$f(x)=0 \text{ if and only if } x \text{ is the identity.}$$

Now let U be a local group and let C be an invariant local subgroup of U. If we take a sufficiently small neighborhood W of the identity in U the factor local group W/C is defined as follows.[13]

(i) The element X of W/C is the coset $W\cap Cx$ for $x\in W$.

(ii) We shall consider that the product XY of a pair of elements X, Y of W/C is defined if and only if there exist elements $x\in X$ and $y\in Y$ such that

10) For the proof, see H. Yamabe [12].

11) H. Yamabe and M. Gotô [4].

12) See Kuranishi [5] and [6].

13) Pontrjagin [11], p. 83.

xy is contained in W. The product XY is equal to $W \cap Cxy$, which is independent of the choices of x and y.

(iii) The natural mapping $W \to W/C$ is continuous and open.

Let G be a locally compact finite dimensional group. Suppose that G is without small subgroups. Let G^* and α be the locally compact locally connected group and the continuous one-to-one mapping of G^* into G stated in Theorem 2. Let U be the sufficiently small neighborhood of the identity in G on which the function $f(x)$ of Lemma 2 is defined. U is naturally a local group. Take a sufficiently small open neighborhood C^* of the identity in G^* and let $C = \alpha(C^*)$. By Theorem 2 C is an invariant local subgroup of U. Take a sufficiently small neighborhood W_1 of the identity in U so that the factor local group W_1/C is defined. By Theorem 2 W_1/C is a zero-dimensional locally compact local group. Let β be the natural mapping $W_1 \to W_1/C$ and let φ be a mapping $\varphi(x) = x^2$. Take an open neighborhood W of the identity in U such that $\overline{W} \subseteqq W_1$. Let V_1 be the neighborhood of the identiy in U such that

(5) $$\varphi(bdry\ W) \cap V_1^2 = \phi,$$

(6) $$V_1^2 \subseteqq W,$$

(7) $$V_1 \cap C \text{ is connected.}$$

Let V be a neighborhood of the identity in U such that $V^4 \subseteqq V_1$, $V = V^{-1}$.

Lemma 4. *Let X be an element of $\beta(V)$ such that X^2 is contained in $\beta(V)$. Then for every element y of $X^2 \cap \overline{V}$, there exists an element x of X such that $y = x^2$.*

Proof. Let
$$X = W_1 \cap Cx_0, \quad x_0 \in V,$$
and
$$M^* = \alpha^{-1}((W_1 \cap Cx_0)x_0^{-1}).$$

We define the topological mapping $\psi(a)$ of M^* into G^* by
$$\psi(a) = \alpha^{-1}((\varphi((\alpha(a))x_0))x_0^{-2}).^{14)}$$

Since $N^* = \alpha^{-1}((W \cap Cx_0)x_0^{-1})$ is an open set containing the identity e^* in G^* and $\overline{N}^* \subseteqq M^*$, by Corollary 2 to Theorem 1,

(8) $$C_{e^*}(G^* - \psi(bdry\ N^*)) \subseteqq \psi(N^*).$$

Since
$$\alpha(\alpha^{-1}(V_1 \cap C) \cap \psi(bdry\ N^*))$$
$$\subseteqq (V_1 \cap C) \cap (\varphi(bdry\ (W \cap Cx_0)))x_0^{-2}$$
$$\subseteqq [V_1 x_0^2 \cap \varphi(bdry\ W)]x_0^{-2}$$
$$\subseteqq [V_1^2 \cap \varphi(bdry\ W)]x_0^{-2} = \phi \qquad \text{(by condition (5))}$$

14) α is the injection of G^* into G.

and since $V_1 \cap C$ is connected, it follows that

$$\alpha^{-1}(V_1 \cap C) \subseteqq C_{e^*}(G^* - \psi(bdry\, N^*)). \tag{9}$$

If $cx_0^2 \in X^2 \cap \bar{V} = (W_1 \cap Cx_0^2) \cap \bar{V} = \bar{V} \cap Cx_0^2$, it follows that

$$c \in \bar{V}x_0^{-2} \cap C \subseteqq \bar{V}V^{-2} \cap C \subseteqq V_1 \cup C,$$

that is,

$$x^2 \cap \bar{V} \subseteqq (V_1 \cap C)x_0^2$$

$$\alpha^{-1}[(X^2 \cap \bar{V})x_0^{-2}] \subseteqq \alpha^{-1}(V_1 \cap C). \tag{10}$$

From (8), (9) and (10), it follows that

$$\alpha^{-1}[(X^2 \cap \bar{V})x_0^{-2}] \subseteqq \psi(N^*) = \alpha^{-1}[\varphi(W \cap Cx_0)x_0^{-2}],$$

that is,

$$X^2 \cap V \subseteqq \varphi(W \cap Cx_0).$$

Hence the lemma is proved.

We now define the function $F(X)$ on $\beta(\bar{\bar{\bar{W}}})$ by

$$F(X) = \inf_{x \in X \cap \bar{V}} f(x).^{15)} \tag{11}$$

LEMMA 5. *Let V be the neighborhood of the identity in G stated in Lemma 4. We can assume without loss of generality that $V = \{x | f(x) \leqq \delta\}$, where $f(x)$ is the function of Lemma 3. Then*

$$F(X^2) \geqq 2F(X) \quad \textit{if } X,\ X^2 \in \beta(V), \tag{12}$$

$$F(X) = 0 \quad \textit{if and only if } X \textit{ is the identity}, \tag{13}$$

$$F(X) \quad \textit{is continuous.} \tag{14}$$

Proof. Continuity of $F(X)$: Let $X_n \in \beta(V)$, and $X_n \to X \in \beta(V)$. There exists a sequence x_n $(n = 1, 2, \ldots)$ of V such that $F(X_n) = f(x_n)$. We can assume without loss of generality that $x_n \to x \in V \cap X$. Then

$$F(X) \leqq f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} F(X_n). \tag{15}$$

Let x be the element of X such that $F(X) = f(x)$. For arbitrary positive number ε, there exists a neighborhood V_2 of the identity in G such that

$$f(y) \leqq f(x) + \varepsilon \quad \text{for} \quad y \in V_2 x.$$

Since β is an open mapping, there exists an integer N' such that

$$X_n \in \beta(V_2 x) \quad \text{for} \quad n > N'.$$

15) $f(x)$ is the function of Lemma 3.

Let x_n be a point of $X_n \cap V_2 x$, $(n = N' + 1, N' + 2, \ldots)$. Then

(16) $$F(X_n) \leqq f(x_n) \leqq f(x) + \varepsilon = F(X) + \varepsilon \quad \text{for} \quad n \leqq N',$$

from (15) and (16) it follows that $F(X)$ is a continuous function on $\beta(\overline{W})$.

(13) is obvious. We shall prove (12). Suppose that X and X^2 are elements of $\beta(V)$. There exists an element y of $X^2 \cap V$ such that

$$F(X^2) = f(y).$$

From Lemma 4 and the fact that $V = \{x | f(x) \leqq \delta\}$, there exists an element x of $X \cap V$ such that $x^2 = y$.
Hence

$$F(X^2) = f(y) = f(x^2) \geqq 2f(x) \geqq 2F(X).$$

LEMMA 6. *Let G be a locally compact finite dimensional group. Suppose that G is witout small subgroups. Then G is locally connected.*

Proof. Let V be a sufficiently small neighborhood of the identity in G. Since W/C is a zero-dimensional local group, $\beta(V)$ contains an open and compact subgroup H of W/C. We can take H so that H is the group in the large, i.e., the product is defined for every pair of elements of H and is contained in H.[17] By Lemma 5 there is defined the function $F(X)$ on the compact group H and satisfies the conditions

(12)′ $$F(X^2) \geqq 2F(X) \quad \text{for every element } X \text{ of } H.$$

(13), and (14). Hence H must be the group consisting of the identity element only. Since H is an open subset of W/C, W/C must be a discrete space. Thus W is locally connected.

THEOREM 3. *Let G be a finite dimensional locally compact group. Suppose that G is without small subgroups. Then for every neighborhood U of the identity in G there exists a neighborhood U_1 satisfying the following conditions.*

"For every element x of U_1, there exists a unique one-parameter subgroup $g(\lambda)$ $(0 \leqq \lambda \leqq 1)$ contained in U such that $g(1) = x$."

Proof. We can suppose without loss of generality that

(18) the function $f(x)$ of Lemma 3 is defined on U, and that

(19) the mapping $\varphi(x) = x^2$ of U into G is one-to-one. (Lemma 2.)

Take a neighborhood V of the identity in G such that $V^2 \subseteqq U$ and let V^* be an open neighborhood of the identity in G of the Lemma 1 with respect to V. By Lemma 6, G is locally connected. Hence from the condition (19) and

[17] This can be proved in the same way as in the case of the locally compact zero-dimensional groups.

the Corollary 1 to Theorem 1, $\varphi(V^*)$ is an open subset G and contains the identity. Choose a sufficiently small positive number δ such that

$$U_1 = \{x | f(x) < \delta\} \subseqq V^* \cap \varphi(V^*). \tag{20}$$

For every element x of U_1, there exists an element x_1 of V^* such that $x = x_1^2$. Since $f(x_1) \leqq \frac{1}{2} f(x_1^2) = \frac{1}{2} f(x) < \delta$, x_1 is contained in U_1. Thus there exists a suquence x_n $(n = 1, 2, \ldots)$ of elements of U_1 such that

$$x = x_n^{2^n}.$$

Since the square root is unique (Lemma 2),

$$x_n x_m = x_m x_n$$

and

$$x_n = x_m^{2^{m-n}} \quad \text{for} \quad m \geqq n.$$

Then there exists a unique one-parameter subgroup $g(\lambda)$ such that $g\left(\frac{1}{2^n}\right) = x_n$ for $n = 1, 2, \ldots$.[17] Suppose that

$$g\left(\frac{m}{2^n}\right) \subseqq V \quad \text{for} \quad m = 1, 2, \ldots, 2^n.$$

Put $y = g\left(\frac{1}{2^{n+1}}\right) \in U_1$. For $m = 2m' + 1$,

$$y^m = g\left(\frac{m}{2^{n+1}}\right) = g\left(\frac{m'}{2^n}\right) g\left(\frac{1}{2^{n+1}}\right) \in V^2 \subseqq U.$$

Hence

$$y^m \in U \quad \text{for} \quad m = 1, 2, \ldots, 2^{n+1},$$

and

$$y, y^{2^{n+1}} \in U_1 \subseqq V^*.$$

By Lemma 1, $\quad y^m \in V \quad \text{for} \quad m = 1, 2, \ldots 2^{n+1}.$

Hence

$$g\left(\frac{m}{2^n}\right) \in V \subseqq U \quad \text{for} \quad m = 1, 2, \ldots, 2^n, \quad n = 1, 2, \ldots.$$

Thus

$$g(\lambda) \in V \subseqq U \quad \text{for} \quad 0 \leqq \lambda \leqq 1.$$

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