Can. J. Math., Vol. XXXIV, No. 4, 1982, pp. 921–931

INTERSECTION PATTERNS OF FAMILIES OF CONVEX SETS

M. KATCHALSKI AND A. LIU

0. Introduction. In this paper, we study the intersection pattern of families of convex sets. Since we only consider finite families, we may assume that the sets are also compact.

As an example, we consider families of 5 convex sets in $\mathbf{R}^2$ such that every two intersect and no three intersect. One such family that comes immediately to mind is that of 5 lines in general position. However, this is not the only family which exhibits this intersection pattern. Fig. 1 shows a family of 3 lines (sides of the large triangle) and 2 triangles (inscribed in the large triangle) that has this property.

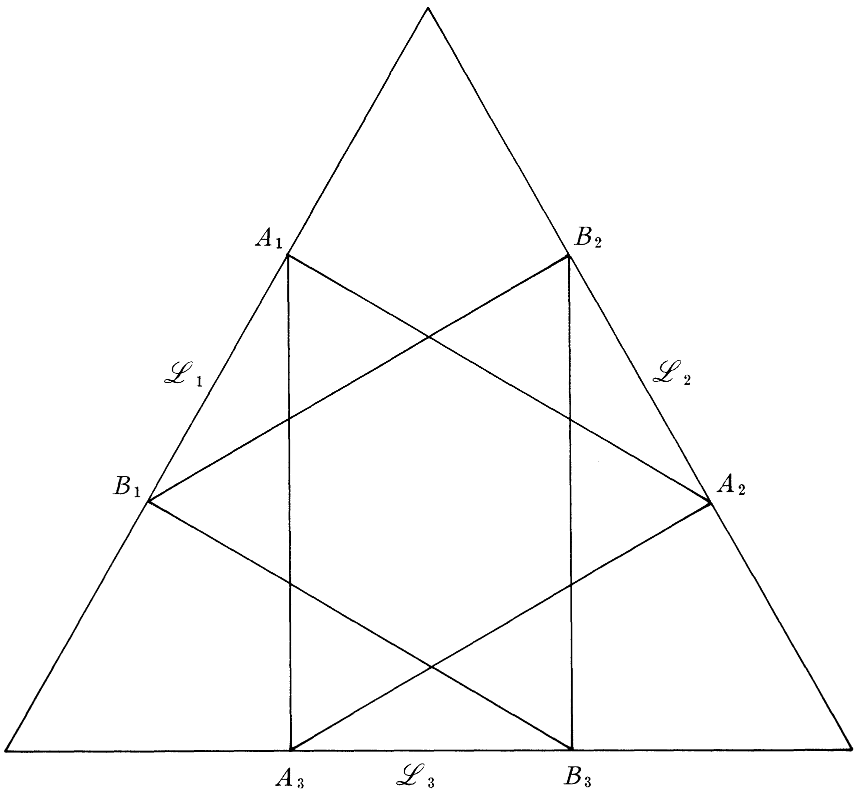

FIGURE 1

Received June 9, 1981. This research was supported in part by NSERC Grant A5137.

https://doi.org/10.4153/CJM-1982-065-x Published online by Cambridge University Press

More precisely, two families F and G of convex sets of equal size are said to have the same intersection pattern if there is a bijection $\psi : F \rightarrow G$ such that $\cap \{\mathbf{F} : \mathbf{F} \in F'\} \neq \emptyset$ if and only if $\cap \{\psi(\mathbf{F}) : \mathbf{F} \in F'\} \neq \emptyset$ for all subfamilies $F' \subset F$.

Given a family of convex sets, we wish to contract each set into a convex polytope in such a way that the total number of vertices of the polytopes is minimal. The family of 4 rectangles in Fig. 2 can be contracted to a line $\mathbf{AB}$, with $\mathbf{S}_1$ becoming the single point $\mathbf{A}$, $\mathbf{S}_4$ the single point $\mathbf{B}$ and $\mathbf{S}_2$ and $\mathbf{S}_3$ the line $\mathbf{AB}$. On the other hand, the family in Fig. 1 cannot be contracted.

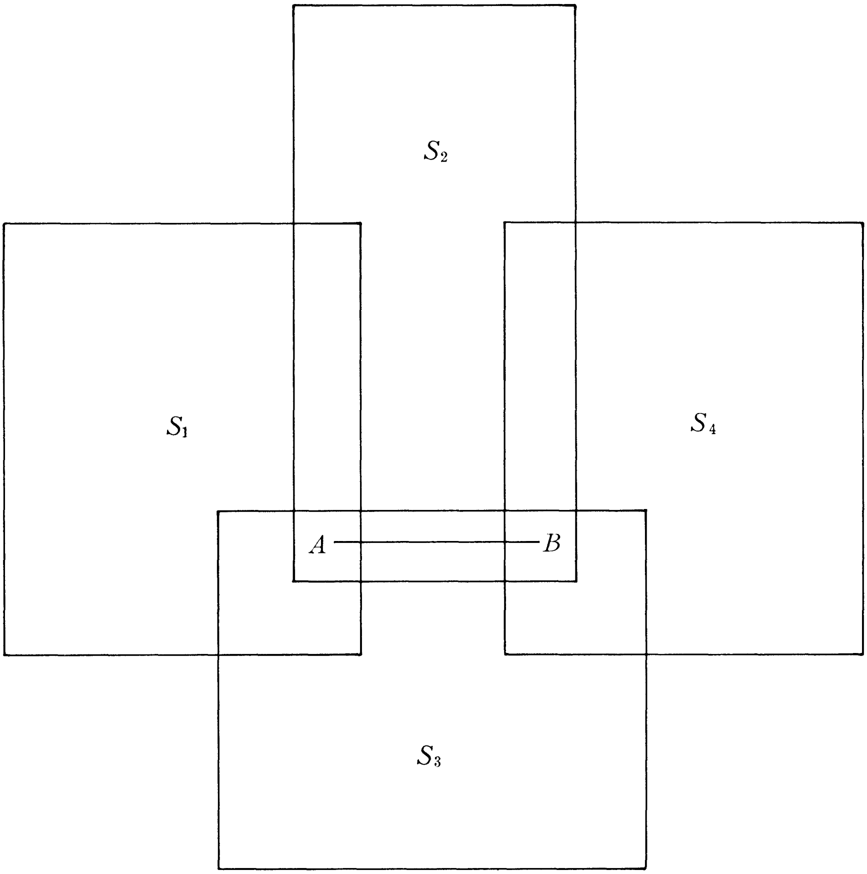

FIGURE 2

This process will be made more precise in the following sections. It will be treated in Section 2 and Section 4. In Section 1 and Section 3, we consider a different way of representing intersection patterns as a preliminary approach to the main problem.

1. Representation by points. Let F be a family of n convex sets in $\mathbf{R}^k$. A finite set $\mathbf{S}$ of points in $\cup F$ is said to be a point representation of F if for every subfamily F' of F with non-empty intersection,

$$\cap \{\mathbf{F} : \mathbf{F} \in F'\} \cap \mathbf{S} \neq \emptyset.$$

In other words, every subfamily of F with non-empty intersection is represented by at least one point in the intersection of the subfamily.

That point representation is always possible is easy to see. Since the number of subfamilies of F is 2^n, no more than 2^n points are required. However, we are interested in representation with minimal size.

Let

$$p(F) = \min \{|\mathbf{S}| : \mathbf{S} \text{ a point representation of } F\} \quad \text{and}$$
$$p(n, k) = \max \{p(F) : F \text{ in } \mathbf{R}^k, |F| = n\}.$$

We first prove a lower bound.

THEOREM 1. $p(n, k) \geqq \binom{n}{k}$.

Proof. In $\mathbf{R}^1$, take n disjoint intervals. This shows that $p(n, 1)$ is at least n. In $\mathbf{R}^2$, take n lines in general position, i.e., every two intersect and no three intersect. This shows that $p(n, 2) \geqq \binom{n}{2}$. The construction for higher dimensions is analogous and the theorem is proved.

To obtain an upper bound for $p(n, k)$, we first prove two lemmas which are variations of classical results. We begin with a definition.

Let $\mathbf{A}$ be any non-empty compact subset of $\mathbf{R}^k$. Define $h(\mathbf{A})$ to be the point $(a_1, a_2, \dots, a_k) \in \mathbf{A}$ where

$$a_1 = \max \{x_1 : (x_1, x_2, \dots, x_k) \in \mathbf{A}\}$$

and for $2 \leqq i \leqq k$,

$$a_i = \max \{x_i : (a_1, \dots, a_{i-1}, x_i, \dots, x_k) \in \mathbf{A}\}.$$

Let $\textcircled{>}$ be the lexicographical order on R^k, i.e.,

$$(a_1, a_2, \dots, a_k) \textcircled{>} (b_1, b_2, \dots, b_k)$$

if $a_i = b_i$ for $1 \leqq i \leqq n$ or if there exists some $t \leqq k$ such that $a_i = b_i$ for $i < t$ and $a_t > b_t$.

LEXICOGRAPHICAL LEMMA. *Let B be a family of n compact convex sets in $\mathbf{R}^k$ with non-empty intersection, where $n \geqq k$. Then*

$$h(\cap B) = \min \{h(\cap A) : A \subset B, |A| = k\}.$$

Proof. Let $a = \min \{h(\cap A) : A \subset B, |A| = k\}$. Since $\cap B \subset \cap A$ for all $A \subset B$, $h(\cap B) \textcircled{>} a$. Suppose $h(\cap B) \neq a$. Define

$$\mathbf{D} = \{x \in \mathbf{R}^k : x \textcircled{>} a\}.$$

https://doi.org/10.4153/CJM-1982-065-x Published online by Cambridge University Press

Clearly $\mathbf{D}$ is convex. Now $D \cap (\bigcap A) \neq \emptyset$ for all $A \subset B$, $|A| = k$ while $D \cap (\bigcap B) = \emptyset$. This contradicts Helly's Theorem [**1**]. Hence $h(\bigcap B) = a$.

We point out that the Lexicographical Lemma, which had appeared in a slightly different form in an earlier paper [**2**], also implies Helly's Theorem. The next result is a variation of Sperner's Theorem [**3**].

ANTICHAIN LEMMA. *Let C be a collection of subfamilies of a family F of n sets such that $A \not\subset B$ for any $A, B \in C$ and $|A| \leqq k$ for all $A \in C$, where $k \leqq (n+1)/2$. Then* $|C| \leqq \binom{n}{k}$.

Proof. Let C be of maximal size. Let t be the minimal size of a subfamily in C. Assume that $t < k$. Let

$$C_1 = \{A \in C : |A| = t\} \quad \text{and}$$
$$C_2 = \{B \in F : |B| = t+1,\ B \supset A \text{ for some } A \in C_1\}.$$

Now every $A \in C_1$ belongs to exactly $n - t$ B's while every $B \in C_2$ contains at most $t + 1$ A's. Hence

$$|C_2| \geqq \frac{n-t}{t+1}|C_1| \geqq |C_1|.$$

However, $(C - C_1) \cup C_2$ still satisfies the conditions of the problem. Repeating this procedure will raise t to k. It follows that $|C| \leqq \binom{n}{k}$.

Combining with Theorem 1, the following result yields the exact value $p(n, k) = \binom{n}{k}$.

THEOREM 2. $p(n, k) \leqq \binom{n}{k}$, $n \geqq 2k - 1$.

Proof. Let F be any family of n convex sets in $\mathbf{R}^k$. Let C be the collection of all subfamilies of F of size at most k with non-empty intersection, except for those which are contained in a larger member of C. By the Antichain Lemma, $|C| \leqq \binom{n}{k}$.

Let $\mathbf{S} = \{h(\bigcap A) : A \in C\}$. Now subfamilies of size at most k with non-empty intersection are clearly represented by $\mathbf{S}$. That subfamilies of size exceeding k with non-empty intersection are represented by $\mathbf{S}$ follows from the Lexicographical Lemma. This completes the proof of the theorem.

2. Representation by convex polytopes. Consider once again the construction in $\mathbf{R}^2$ used in the proof of Theorem 1. We have n lines in general position. For point representation, we need $\binom{n}{2}$ points. This seems somewhat extravagant as two points for each line are adequate representation. Thus we are led to consider a different representation of intersecting convex sets.

Let F be a family of n convex sets in $\mathbf{R}^k$. A finite set $\mathbf{T}$ of points in $\cup F$ is said to be a *vertex representation* of F if for every subfamily F' of F with non-empty intersection, $\cap \{\mathbf{F}^* : \mathbf{F} \in F'\} \neq \emptyset$, where $\mathbf{F}^*$ is the convex hull of $\mathbf{T} \cap \mathbf{F}$. In other words, $\{\mathbf{F}^* : \mathbf{F} \in F\}$ is a contraction of F into a family of convex polytopes, including those degenerated to lower dimensions, which has the same intersection pattern as F.

We point out that every point representation is a vertex representation but the converse is false. It is easy to show that for n lines in $\mathbf{R}^2$ in general position, $2n - 3$ points are sufficient for vertex representation, far less than the $\binom{n}{2}$ points needed for point representation.

Let

$$v(F) = \min \{|\mathbf{T}| : \mathbf{T} \text{ a vertex representation of } F\} \quad \text{and}$$

$$v(n, k) = \max \{v(F) : F \text{ in } \mathbf{R}^k, |F| = n\}.$$

We have $v(n, k) \leqq p(n, k)$ but unfortunately no improvement on this, even for $k = 2$. However, we do have a lower bound.

THEOREM 3. *$v(n, k) \geqq (n/k)^k$ for $k|n$.*

Proof. In $\mathbf{R}^1$, take n disjoint intervals. This shows that $v(n, 1)$ is at least n. We now describe a construction in $\mathbf{R}^2$. Let $\mathbf{P}$ be a convex $\frac{1}{2}n$-gon and B be the collection of all edges of $\mathbf{P}$. Let A be a family of $\frac{1}{2}n$ inscribed $\frac{1}{2}n$-gons in $\mathbf{P}$ such that each $\mathbf{A} \in A$ intersects each $\mathbf{B} \in B$ at a distinct interior point of $\mathbf{B}$. In order to preserve the intersection pattern of $F = A \cup B$, all of these intersection points must be included in the representation. Hence $v(n, 2) \geqq (\frac{1}{2}n)^2$.

We next describe a construction in $\mathbf{R}^3$. Let $\mathbf{P}$ be a convex polyhedron with $n/3$ faces and C be the collections of all faces of $\mathbf{P}$. In the interior of each $\mathbf{C} \in C$, construct a configuration $A^{\mathbf{C}} \cup B^{\mathbf{C}}$ as described above, where $A^{\mathbf{C}} = \{\mathbf{A}_i^{\mathbf{C}} : 1 \leqq i \leqq n/3\}$ are convex $n/3$-gons and

$$B^{\mathbf{C}} = \{\mathbf{B}_i^{\mathbf{C}} : 1 \leqq i \leqq n/3\}$$

are segments. For $1 \leqq i \leqq n/3$, let $\mathbf{A}_i$ be the convex hull of $\cup \{\mathbf{A}_i^{\mathbf{C}} : \mathbf{C} \in C\}$ and $\mathbf{B}_i$ be the convex hull of $\cup \{\mathbf{B}_i^{\mathbf{C}} : \mathbf{C} \in C\}$. Finally let $A = \{\mathbf{A}_i : 1 \leqq i \leqq n/3\}$ and $B = \{\mathbf{B}_i : 1 \leqq i \leqq n/3\}$. Now for $\mathbf{A} \in A$, $\mathbf{B} \in B$ and $\mathbf{C} \in C$, $\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}$ consists of a single point. In order to preserve the intersection pattern of $F = A \cup B \cup C$, all of these intersection points must be included in the representation. Hence $v(n, 3) \geqq (n/3)^3$.

The inductive construction for higher dimensions is analogous and we omit the descriptions.

https://doi.org/10.4153/CJM-1982-065-x Published online by Cambridge University Press

The divisibility condition $k|n$ is for convenience and not essential. We point out that the correct order of magnitude of $v(n, k)$ is n^k and the upper and lower bounds differ by a constant dependent only on k. We conjecture that our lower bound gives the correct value.

3. Related problems. Consider once again the construction in $\mathbf{R}^2$ used in the proof of Theorem 3. The family F includes $n/2$ convex polygons all of which have a common point. Clearly, the larger the size of a subfamily with non-empty intersection, the smaller the number of points required for either point representation or vertex representation. In the extreme case where the entire family has non-empty intersection, a single point is sufficient.

We introduce the parameter $i(F)$ which denotes the maximum number of sets in F that contain a common point. Define

$$p(n, k, l) = \max \{p(F) : F \text{ in } \mathbf{R}^k, |F| = n \text{ and } i(F) = l\} \quad \text{and}$$
$$v(n, k, l) = \max \{v(F) : F \text{ in } \mathbf{R}^k, |F| = n \text{ and } i(F) = l\}.$$

As before, we have $v(n, k, l) \leqq p(n, k, l)$. Also,

$$p(n, k) = \max_l p(n, k, l) \quad \text{and} \quad v(n, k) = \max_l v(n, k, l).$$

For $p(n, k, l)$, we have the exact value for most values of l.

THEOREM 4. *$p(n, k, l) \geqq \binom{n}{k} - \binom{l}{k} + 1$ for $l \geqq k$.*

Proof. In $\mathbf{R}^1$, take l copies of one interval and $n - l$ intervals that are disjoint from one another and from the l intervals. We need $n - l + 1$ points for representation. In $\mathbf{R}^2$, take l concurrent lines and add $n - l$ lines in general position with respect to one another and to the l lines. Thus we need $\binom{n-l}{2} + l(n - l) + 1 = \binom{n}{2} - \binom{l}{2} + 1$ points. The construction for higher dimensions is analogous and the theorem is proved.

THEOREM 5. *$p(n, k, l) \leqq \binom{n}{k} - \binom{l}{k} + 1$ for $l \geqq 2k - 2$.*

Proof. We shall use induction on n. For $n = l$, $\binom{n}{k} - \binom{l}{k} + 1 = 1$ point is all we need, and we can choose this point to be $h(\bigcap F)$. Suppose now that F' is a family of $n - 1$ convex sets in $\mathbf{R}^k$ with $i(F') = l$. Let C' be the collection of subfamilies of F' of size at most k. Eliminate from C' members that are contained in larger members of C'. As in Theorem 2, the set $\mathbf{S}' = \{h(\bigcap H) : H \in C'\}$ is a point representation of F'. Our induction hypothesis is that

$$|\mathbf{S}'| \leqq \binom{n-1}{k} - \binom{l}{k} + 1.$$

https://doi.org/10.4153/CJM-1982-065-x Published online by Cambridge University Press

Now let F be a family of n convex sets in $\mathbf{R}^k$ with $i(F) = l$ where $n > l$. There is a set $\mathbf{F} \in F$ such that $i(F - \{\mathbf{F}\}) = l$. Let C be the collection of subfamilies of F of size at most k. Eliminate from C members that are contained in larger members of C. As in Theorem 2, the set $\mathbf{S} = \{h(\bigcap H) : H \in C\}$ is a point representation of F. It remains to show that

$$|\mathbf{S}| \leqq \binom{n}{k} - \binom{l}{k} + 1.$$

Let $\mathbf{S} = \mathbf{S}' \cup \mathbf{S}''$ where $\mathbf{S}' = \{h(\bigcap H) \in S : \mathbf{F} \notin H\}$ and $\mathbf{S}'' = \mathbf{S} - \mathbf{S}'$. By induction hypothesis,

$$|\mathbf{S}'| \leqq \binom{n-1}{k} - \binom{l}{k} + 1.$$

Now $\{H - \{\mathbf{F}\} : h(\bigcap H) \in \mathbf{S}, \mathbf{F} \in H\}$ is a collection of subfamilies of $F - \{\mathbf{F}\}$ of size at most $k - 1$ such that no one contains another. By the Antichain Lemma,

$$|\mathbf{S}''| \leqq \binom{n-1}{k-1}.$$

Hence

$$\mathbf{S}| = |\mathbf{S}'| + |\mathbf{S}''| \leqq \binom{n-1}{k} - \binom{l}{k} + 1 + \binom{n-1}{k-1} = \binom{n}{k} - \binom{l}{k} + 1.$$

For $l \leqq k$, clearly $p(n, k, l) = \binom{n}{l}$. For $k < l < 2k - 2$, the induction argument in Theorem 5 breaks down at the early stages, when $n < 2k - 1$. However, we believe that $\binom{n}{k} - \binom{l}{k} + 1$ is still the correct value for $p(n, k, l)$ in that range.

4. A case study. For $v(n, k, l)$, we have

$$\tfrac{1}{4}n^2 \leqq v(n, 2, \tfrac{1}{2}n) \leqq 3n^2/8$$

from Theorems 3 and 5. We shall consider only one other case. The example of n lines in general position in $\mathbf{R}^2$ shows that

$$2n - 3 \leqq v(n, 2, 2)$$

while

$$v(n, 2, 2) \leqq p(n, 2, 2) = \binom{n}{2}.$$

We shall improve on these bounds, giving an instance where the function p is strictly greater than the function v.

THEOREM 6. *$v(n, 2, 2) > cn$ for any $c > 0$ provided that n is sufficiently large.*

Proof. We shall construct a sequence of families as follows. Consider an $r \times r$ lattice where r is sufficiently large. Describe with the lattice points as centre circles of equal radius which is sufficiently small. For each line passing through a subset of the lattice points, we define its cover as the two exterior common tangents to the circles with centres on the line.

There are r horizontal lines each with r lattice points on it, and similarly there are r such vertical lines. Replacing these lines by their covers generates a family of $4r$ lines defining r^2 squares circumscribing the circles. Let F be a family consisting of the $4r$ lines and r^2 squares inscribed in the squares mentioned above, one in each. All 4 vertices of each of the inscribed squares are required for a vertex representation of F. Hence we have

$$v(r^2 + 4r, 2, 2) \geqq 4r^2.$$

Consider now in addition the lines of slope 1 or -1 that pass through the lattice points. There are $4r - 2$ of them. Replacing these lines by their covers yields octagons circumscribing the circles. Consideration of the family of the $4r + 2(4r - 2)$ lines and r^2 inscribed octagons yields

$$v(r^2 + 12r - 4, 2, 2) \geqq 8r^2.$$

A portion of this configuration is illustrated in Fig. 3.

The general construction consists of adding in stages covers of families of lines through the lattice points. The most efficient way is to use lines of slope p/q where p and q are relatively prime integers and $\max \{p, q\}$ is as small as possible.

It is necessary to ensure that $i(F) = 2$ for the family F we construct. The inscribed polygons are clearly disjoint and intersect the covers at distinct points. The only thing that needs to be checked is that no three covers intersect. Since the circles used in the construction are sufficiently small, this will not happen if the lines which they cover do not intersect. When the lines do intersect, they describe a circle of the same size with the point of intersection as centre. The covers will then be tangent to this circle and it is impossible to draw three distinct tangents to a circle from the same point.

The number of inscribed polygons, which increase in their number of sides, remains r^2 while the number of lines will be a linear function of r. Hence $v(n, 2, 2) > cn$ for every $c > 0$ provided that n is sufficiently large.

THEOREM 7. $v(n, 2, 2) \leqq 3n^2/8$.

Proof. Let F be a family of n convex sets. We may assume that no one contains another or is disjoint from all others. For $\mathbf{F}_i \cap \mathbf{F}_j \neq \emptyset$, choose a point h_{ij} on their common boundary and let $\mathbf{T}$ be the collection of these

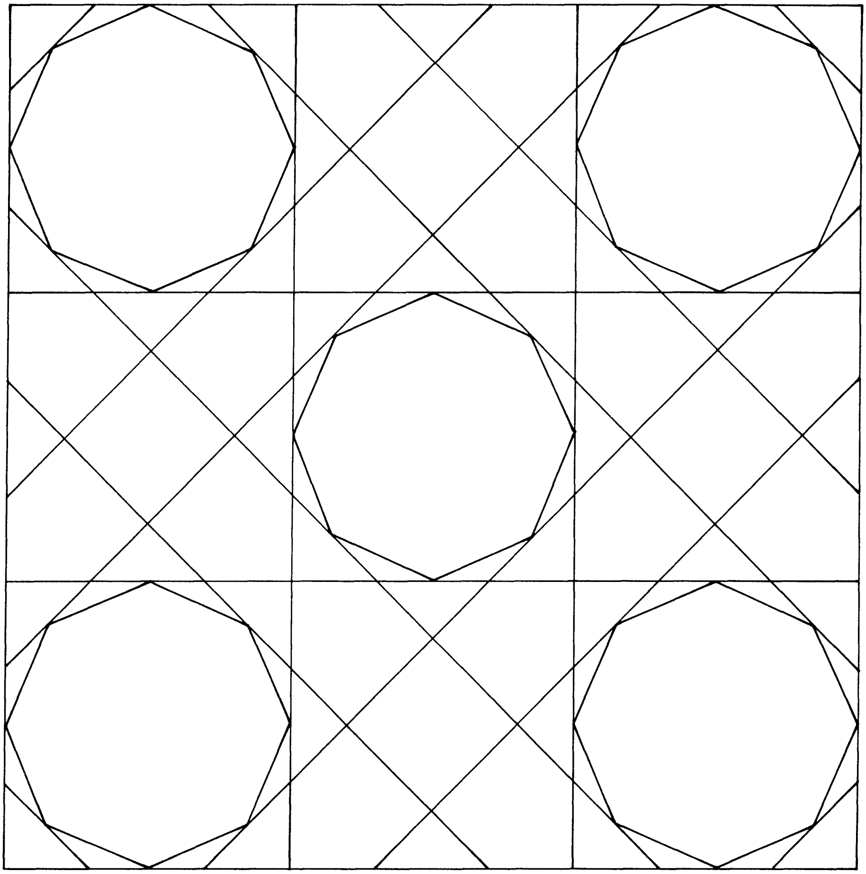

FIGURE 3

points. Clearly $\mathbf{T}$ is a vertex representation, in fact a point representation, of F.

We shall reduce the size of $\mathbf{T}$ as much as possible. In the following, the set $\mathbf{T}$ may change, but we shall use the symbol $\mathbf{T}$ throughout to denote the vertex representation.

Let $\mathbf{F}_i^*$ be the convex hull of $\mathbf{T} \cap \mathbf{F}_i$. We point out that as $\mathbf{T}$ changes, $\mathbf{F}_i^*$ may change too. All of the points in $\mathbf{T} \cap \mathbf{F}_i$ are on the boundary of $\mathbf{F}_i^*$ and not in the interior of any other $\mathbf{F}_j^*$. However, some of these points may not be vertices of $\mathbf{F}_i^*$. In particular, if $\mathbf{F}_i^*$ is a line, only the two end points are considered to be vertices.

For $\mathbf{F}_i^* \cap \mathbf{F}_j^* \neq \emptyset$, we shall use the following steps to either eliminate the point h_{ij} from $\mathbf{T}$ or ensure that $\mathbf{F}_i^* \cap \mathbf{F}_j^* = \{h_{ij}\}$.

(1) If h_{ij} is neither a vertex of $\mathbf{F}_i^*$ nor of $\mathbf{F}_j^*$, we can remove it from $\mathbf{T}$ without affecting the intersection pattern.

https://doi.org/10.4153/CJM-1982-065-x Published online by Cambridge University Press

(2) If h_{ij} is a vertex of $\mathbf{F}_i^*$ but not of $\mathbf{F}_j^*$, let a_i and b_i be the two points still in $\mathbf{T}$ that are adjacent to h_{ij} along the boundary of $\mathbf{F}_i^*$. Note that if $\mathbf{F}_i^*$ is a line, then a_i and b_i are identical. Let a_j and b_j be defined in analogous manner. We consider three subcases:

(a) Let a_ih_{ij} and b_ih_{ij} be on opposite sides of a_jb_j. Now a_ib_i must intersect a_jb_j as otherwise one of a_j and b_j will be inside $\mathbf{F}_i^*$. Hence we may remove h_{ij} from $\mathbf{T}$, thereby reducing $\mathbf{F}_i^*$ by the triangle $a_ib_ih_{ij}$. We point out that the line a_ib_i is still in $\mathbf{F}_i^*$. The situation is illustrated in Fig. 4.

It remains to show that the intersection pattern is unchanged. The set $\mathbf{F}_j^*$ remains the same. It is therefore only necessary to show that if some $\mathbf{F}_t^*$ intersect $\mathbf{F}_i^*$ only in the triangle $a_ib_ih_{ij}$, then $\mathbf{F}_t^*$ intersects a_ib_i. Note that $\mathbf{F}_t^*$ must intersect two edges of the triangle $a_ib_ih_{ij}$ or

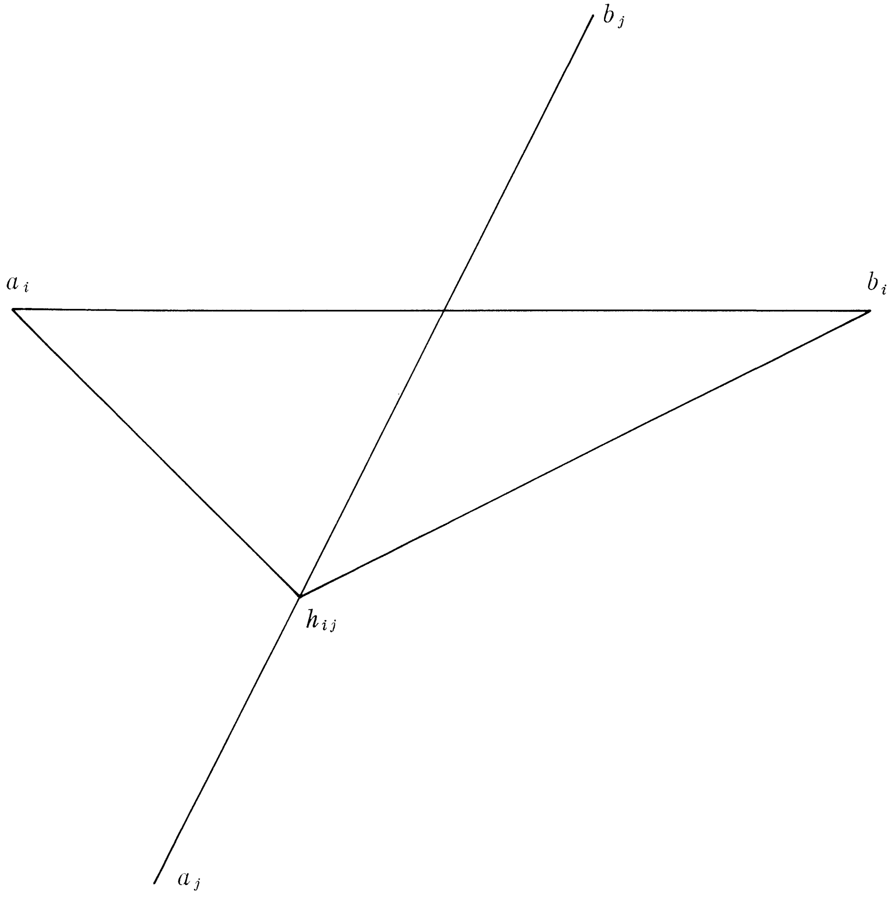

FIGURE 4

one of its vertices will be inside $\mathbf{F}_i^*$. However, if it intersects both a_ih_{ij} and b_ih_{ij}, then $\mathbf{F}_i^* \cap \mathbf{F}_j^* \cap \mathbf{F}_t^* \neq \emptyset$, a contradiction. Hence intersection pattern is indeed preserved.

(b) Let a_ih_{ij} and b_ih_{ij} be on the same side of a_jb_j interior to $\mathbf{F}_j^*$. Now both a_i and b_i are outside $\mathbf{F}_j^*$. If a_ib_i intersects $\mathbf{F}_j^*$, h_{ij} can be eliminated. If not, let x be the point in $\mathbf{F}_j^*$ on the line a_ih_{ij} closest to a_i. We may replace h_{ij} by x in $\mathbf{T}$ and rename x as h_{ij}. $\mathbf{F}_i^*$ will be reduced but it is easy to verify that the intersection pattern is unchanged.

(c) Let a_ih_{ij} and b_ih_{ij} be on the same side of a_jb_j exterior to $\mathbf{F}_j^*$. No reduction is possible or necessary here.

(3) If h_{ij} is a vertex of both $\mathbf{F}_i^*$ and $\mathbf{F}_j^*$, there are three subcases which are analogous to those under case (2).

The above process will terminate since there are finitely many pairs F_i^* and F_j^* with non-empty intersection. At the end of this process, for each such pair, either h_{ij} is eliminated or $F_i^* \cap F_j^*$ is reduced to the single point h_{ij}.

Now define a graph G on n vertices as follows. The vertex v_i corresponds to the reduced polygon $\mathbf{F}_i^*$. Join v_i to v_j if and only if h_{ij} is still in $\mathbf{T}$. While G in general is not a planar graph, we claim that it contains no complete subgraph on 5 vertices.

Suppose there is a complete subgraph on v_1, v_2, v_3, v_4 and v_5. For $1 \leqq i \leqq 5$, assume that $\mathbf{F}_i^*$ is at least a triangle. We may consider the point v_i to be chosen in the interior of $\mathbf{F}_i^*$ and the edges v_iv_j a continuous path from v_i through h_{ij} to v_j so that no two paths intersect. This contradicts the well-known fact that a complete graph on 5 vertices is non-planar.

The cases where some F_i^* is a single point or a line can be handled in similar fashion. It follows from Turán's Theorem on Extremal Graphs [**4**] that G has no more than $3n^2/8$ edges. Since the number of edges in G is equal to the final size of T, $v(n, 2, 2) \leqq 3n^2/8$.

References

1. E. Helly, *Über Mengen Konverser Kurper mit gemeinschaftlichen Punkten*, J. Deutsch. Math. Verein. *32* (1923), 175–176.

2. M. Katchalski and A. Liu, *A problem of geometry in* $\mathbf{R}^n$, Proc. Amer. Math. Soc. *75* (1979), 284–288.

3. E. Sperner, *Ein Satz über Untermengen einer endlichen Menge*, Math. *27* (1928), 544–548.

4. P. Turán, *On an extremal problem in graph theory*, Mat. Fiz. Lapok *48* (1941), 436–452.

Technion,
Haifa, Israel;
University of Alberta,
Edmonton, Alberta