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# ON THE NUMBER OF PARITY SETS IN A GRAPH

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**1. Introduction.** The graphs considered in this paper are finite and have no loops or multiple edges. If $G$ is such a graph, we denote its vertex set by $VG$ and its edge set by $EG$. If $X$ and $Y$ are disjoint subsets of $VG$, we define $\delta(X, Y)$ to be the set of edges of $G$ that join a vertex in $X$ to one in $Y$.

Let $G$ be a graph, and let $A$ and $B$ be disjoint subsets of $VG$. Then in [**1**] a subset $W$ of $VG$ is called a *parity set relative to* $(A, B)$ if for every vertex $v \in A \cup B$, $|\delta(\{v\}, W - \{v\})|$ is even if $v \in A$ and odd if $v \in B$. A necessary and sufficient condition for the existence of a parity set relative to $(A, B)$ is given in [**1**], where this theorem is seen to be a generalisation of a theorem of Kasteleyn. We now enumerate the parity sets relative to $(A, B)$ and illustrate the use of the resulting formula with some applications. In this paper, all congruences are understood to be taken modulo 2.

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**2. The number of parity sets relative to (A, B).** We begin with a definition. Let $G$ be a graph, and let $U = \{u_1 \ldots, u_m\}$ and $X = \{x_1, \ldots, x_n\}$, where $U$, $X \subseteq VG$. For each $i \leqq m$, let $U_i$ be the set of vertices of $X$ adjacent to $u_i$, and associate with $U_i$ the vector $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ where, for all $j \leqq n$, $a_{ij}$ is 1 if $x_j \in U_i$ and 0 otherwise. Then the *rank* of $U$ in $X$ is defined to be the dimension of the vector space (over the field of residue classes modulo 2) spanned by the vectors $A_1, \ldots, A_m$.

In [**1**], the following necessary and sufficient condition for the existence of a parity set is proved.

THEOREM 1. *Let $G$ be a graph and let $A$, $B$, $X$ be subsets of $VG$ such that $A \cap B = \phi$. Then a necessary and sufficient condition for the existence of a subset $W$ of $X$ such that $W$ is a parity set relative to $(A, B)$ is that there does not exist a subset $S$ of $A \cup B$ such that* (i) *$|S \cap B|$ is odd, and* (ii) *$|\delta(\{v\}, S - \{v\})|$ is even for every $v \in X$.*

The proof of Theorem 1 given in [**1**] is based on the following theorem of linear algebra. Given scalars $a_{ij} (i = 1, \ldots, m; \ j = 1, \ldots, n)$ and $z_i (i = 1, \ldots, m)$, the equations

$$(1) \quad \sum_{j=1}^{n} a_{ij} y_j = z_i \quad (i = 1, \ldots, m)$$

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can be solved for $y_1, \ldots, y_n$ if and only if, for every sequence of coefficients $\lambda_1, \ldots, \lambda_m$ such that the vector $\sum_{i=1}^{m} \lambda_i(a_{i1}, a_{i2}, \ldots, a_{in})$ vanishes, we have $\sum_{i=1}^{m} \lambda_i z_i = 0$.

Suppose now that all the scalars above belong to a finite field $F$. If the system of equations (1) has rank $r$ then the homogeneous system corresponding to (1) clearly has exactly $|F|^{n-r}$ distinct solutions. Since each of these solutions added to a fixed particular solution of (1) yields a solution of (1), and all solutions of (1) are of this form, we see that the number of solutions of (1) is $|F|^{n-r}$, provided a solution exists. Interpreting this fact in terms of parity sets by the method indicated in the proof of Theorem 1 given in [**1**], we immediately obtain the following theorem.

THEOREM 2. *Let $G$ be a graph and let $A$, $B$, $X$ be subsets of $VG$ such that $A \cap B = \phi$. If there exists a subset of $X$ that is a parity set relative to $(A, B)$, then there are exactly $2^{|X|-r}$ such sets, where $r$ is the rank of $A \cup B$ in $X$.*

**3. Application.** We now illustrate how Theorem 2 may be used in certain types of enumeration problems.

If $G$ is a graph, a *principal forest* of $G$ is defined to be a spanning subgraph whose components are spanning trees of components of $G$. The *cycle rank* of $G$ is defined to be the number of chords of any principal forest of $G$. If $G$ has $p_0$ components, then a formula for the cycle rank of $G$ is $|EG| - |VG| + p_0$. (See [**2**].) We define an *odd factor* of $G$ to be a subset $S$ of $EG$ such that every vertex of $G$ is incident on an odd number of edges of $S$. As our first example of an application of Theorem 2, we show that if $G$ has an odd factor, then the number of them is the number of subsets of the chords of a fixed principal forest of $G$. This result is a corollary of the next theorem.

THEOREM 3. *Let $G$ be a graph, and let $A$ and $B$ be disjoint subsets of $VG$. Then a necessary and sufficient condition for the existence of a subgraph $W$ of $G$ such that $A \cup B \subseteq VW$ and every vertex of $A$ has even valency in $W$ while every vertex of $B$ has odd valency in $W$ is that for every component $C$ of $G$ such that $|VC \cap B| \equiv 1$, some vertex of $C$ belongs to $VG - (A \cup B)$. Furthermore, if $W$ exists, then there exist exactly $2^{|EG|-|A \cup B|+m}$ choices for $W$, where $m$ is the number of components $C$ of $G$ for which $VC \subseteq A \cup B$.*

*Proof.* The first statement of the theorem is Corollary 1 of Theorem 5 in [**1**].

To establish the second statement, we define a bipartite graph $H$ and a subset $X$ of $VH$ as in the proof of Corollary 1 in [**1**], and use the fact that if $W$ exists the number of possible choices for $W$ is exactly $2^{|EG|-r}$ where $r$ is the rank of $A \cup B$ in $X$. This fact is deduced from Theorem 2 by noticing that $EW$ is a parity set of $H$ relative to $(A, B)$. It remains only to calculate $r$. We note that a necessary and sufficient condition for a subset $S$ of $VG$ to have the property that every vertex of $X$ is adjacent in $H$ to an even number of vertices of $S$ is that $S$ be the union of the vertex sets of some collection of components of $G$.

If follows that in the graph $H$ the rank of $A \cup B$ in $X$ is $|A \cup B| - m$. Hence the number of possible choices for $W$ is $2^{|EG|-|A \cup B|+m}$.

COROLLARY. *$G$ has an odd factor if and only if every component of $G$ has an even number of vertices. Furthermore, if $G$ has an odd factor, then the number of odd factors of $G$ is exactly $2^{p_1}$ where $p_1$ is the cycle rank of $G$.*

*Proof.* Apply Theorem 3 with $A = \phi$ and $B = VG$, using the fact that $p_1 = |EG| - |VG| + p_0$ where $p_0$ is the number of components of $G$.

This corollary suggests a 1 : 1 correspondence between odd factors of $G$, if they exist, and subsets of the set of chords of a given principal forest of $G$. We now establish such a correspondence directly, thereby providing an alternative proof of this corollary.

Let $T$ be a principal forest of a graph $G$, and let $C$ be a set of chords of $T$. We will show that there exists at most one odd factor $F$ of $G$ such that $F \cap (\mathrm{EG} - ET) = C$.

Letting $|VG| = n$, we define a sequence $T_0, T_1, \ldots, T_{n-p_0}$ of forests of $G$ and a sequence $C_0, C_1, \ldots, C_{n-p_0}$ of subsets of $EG$ as follows. Set $T_0 = T$ and $C_0 = C$. Now let $i < n - p_0$, and suppose that $T_i$ and $C_i$ have been defined, that $C_i \cap ET_i = \phi$ and that if $F$ is any odd factor of $G$ for which $F \cap (EG - ET) = C$, then $F \cap (EG - ET_i) = C_i$. Let $e$ be an edge of $T_i$ incident on a vertex $v$ of valency 1 in $T_i$. Let $T_{i+1} = T_i - \{v\}$. Let $C_{i+1} = C_i$ or $C_i \cup \{e\}$ according to whether the number of edges of $C_i$ incident on $v$ is odd or even. Then the number of edges of $C_{i+1}$ incident on $v$ is odd, and clearly if $F$ is an odd factor of $G$ for which $F \cap (EG - ET_i) = C_i$, then $F \cap (EG - ET_{i+1}) = C_{i+1}$.

Since $|ET| = n - p_0$, we have $ET_{n-p_0} = \phi$. Furthermore, $T_{n-p_0}$ has exactly $p_0$ vertices, one in each component of $G$. Let $u$ be a vertex of $T_{n-p_0}$, and let $J$ be the component of $G$ containing $u$. By construction, every vertex of $VJ - \{u\}$ is incident on an odd number of edges of $C_{n-p_0}$. Since every edge of $C_{n-p_0} \cap EJ$ is incident on exactly two vertices of $J$, we conclude that $u$ is incident on an odd number of edges of $C_{n-p_0}$ if and only if $|VJ - \{u\}| \equiv 1$. Therefore $C_{n-p_0} \cap EJ$ is an odd factor of $J$ if and only if $|VJ| \equiv 0$. Thus $C_{n-p_0}$ is an odd factor of $G$ if and only if every component of $G$ has an even number of vertices.

Furthermore, by construction, if $F$ is an odd factor of $G$ for which $F \cap (EG - ET) = C$, then $F \cap (EG - ET_{n-p_0}) = C_{n-p_0}$. In other words, $F = C_{n-p_0}$. Hence $G$ has at most one odd factor $F$ satisfying $F \cap (EG - ET) = C$. By the previous paragraph, such an $F$ exists if and only if every component of $G$ has an even number of vertices. Furthermore, if $G$ satisfies this condition, then for each subset $C$ of $EG - ET$, there exists a unique odd factor $F$ of $G$ satisfying $F \cap (EG - ET) = C$. Hence the number of odd factors of $G$ is $2^{p_1}$, as asserted in the corollary.

Our next application of Theorem 2 is concerned with the circuits of a graph. If $C$ is a circuit of a graph $G$, we designate one of the two senses of $C$ as *clock-*

*wise*. If $G$ is a directed graph, then $C$ is said to be *clockwise odd* if the number of edges of $C$ that are directed in agreement with the clockwise sense is odd. Otherwise $C$ is said to be *clockwise even*.

THEOREM 4. *Let $G$ be a directed graph. Let $S_0$ be the set of clockwise even circuits of $G$, and let $S_1$ be the set of clockwise odd circuits of $G$. Then there exist exactly $2^{|VG|-p_0}$ orientations of $G$ in which every circuit of $S_0$ is clockwise even and every circuit of $S_1$ is clockwise odd, where $p_0$ is the number of components of $G$.*

*Proof*. Let $S$ be the set of circuits of $G$. Let $H$ be the bipartite graph defined as follows. Let $VH = S \cup EG$, and let vertices $v, w \in VH$ be adjacent if and only if $v \in S$, $w \in EG$ and $w \in Ev$ in $G$. Let $X = EG$.

Let $Q$ be the given orientation of $G$. We must show that there are exactly $2^{|VG|-p_0}$ orientations $R$ of $G$ with the property that, for every circuit $C$ of $G$, there are an even number of edges of $EC$ whose orientations under $Q$ and $R$ differ. For any orientation $R$ of $G$ with this property, let $W_R$ be the set of edges whose orientations under $Q$ and $R$ differ. Then $W_R$ is clearly a parity set of $H$ relative to $(S, \phi)$. Thus we require the number of parity sets of $H$ relative to $(S, \phi)$ that are subsets of $EG$. Such parity sets exist since $\phi$ is one of them. Hence by Theorem 2, the number of them is $2^{|EG|-r}$ where $r$ is the rank of $S$. But it is well known that $r = |EG| - |VG| + p_0$ (see [**2**]). Hence the required number of orientations is $2^{|EG|-(|EG|-|VG|+p_0)} = 2^{|VG|-p_0}$.

We apply this theorem to some work of P. W. Kasteleyn. Let $G$ be a planar graph, and $M$ a representation of $G$ in the plane. For each circuit $C$ of $M$, we define the clockwise sense of $C$ in the usual manner. Then Kasteleyn shows in [**3**] that the edges of $M$ may be oriented so that for any circuit $C$, the number of edges of $C$ that are oriented in the clockwise sense has opposite parity to the number of vertices enclosed by $C$. We shall call an orientation of $M$ with this property a *Kasteleyn orientation*. Kasteleyn orientations are used in the enumeration of the 1-factors of a planar graph, as explained in [**3**].

The following corollary of Theorem 4 is now clear.

COROLLARY 1. *If $G$ has a Kasteleyn orientation, then it has exactly $2^{|VG|-p_0}$ of them.*

If $G$ is a graph and $X \subseteq VG$, then $\delta(X, VG - X)$ is called the *coboundary* of $G$ determined by $X$ (or by $VG - X$). We now have the following additional application of Theorem 4.

COROLLARY 2. *A subset $W$ of $EG$ satisfies $|W \cap EC| \equiv 0$ for every circuit $C$ of $G$ if and only if $W$ is a coboundary.*

*Remark*. It follows that if every circuit of $G$ has even length, then $EG$ is a coboundary. This corollary therefore generalises the theorem that a graph is bipartite if and only if every circuit has even length.

*Proof*. It is clear that every coboundary has the required property, as the edge set of any circuit of $G$ must intersect every coboundary in an even number of edges. By the proof of Theorem 4, there are only $2^{|VG|-p_0}$ sets of edges having the required property; hence it remains only to show that the number of distinct coboundaries of $G$ is $2^{|VG|-p_0}$. Let $H_1, \ldots, H_{p_0}$ be the components of $G$. Let $S \subseteq VG$, and let $\delta$ be the coboundary of $G$ determined by $S$. For all $i$ such that $1 \leqq i \leqq p_0$, let $S_i = S \cap VH_i$. Then for any set $T$ of components of $G$, the set obtained from $S$ by replacing $S_i$ by $VH_i - S_i$ for every $H_i \in T$ determines the same coboundary $\delta$. Since it is clear that all sets which determine $\delta$ are of this form, there are exactly $2^{p_0}$ sets which determine $\delta$. Since the number of sets of vertices is $2^{|VG|}$, there exist exactly $2^{|VG|-p_0}$ coboundaries.The proof is complete.

## References


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