NOTE UPON THE GENERALIZED CAYLEYAN OPERATOR

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1. The following note which deals with the effect of a certain determinantal operator when it acts upon a product of determinants was suggested by the original proof which Dr. Alfred Young gave of the property

$$(NP)^2 = \theta NP$$

subsisting between the positive P and the negative N substitutional operators, θ being a positive integer[1]. This result which establishes the idempotency of the expression $\theta^{-1}NP$ within an appropriate algebra is fundamental in the Quantitative Substitutional Analysis that Young developed. The present note, which is couched in the language of determinants, proves a result which is equivalent to Young's alternative statement $(PN)^2 = \theta PN$.

These operators P and N take their rise in the theory of groups. In fact let

$$p = p_1 + p_2 + \ldots + p_h$$

be a partition of a positive integer p into h non-zero parts which are arranged in descending order: that is

$$p_1 \geqslant p_2 \geqslant \ldots \geqslant p_h .$$

Let p distinct elements be arranged in the following fashion

$$\begin{array}{l} u_1\, u_2 \ldots \ldots \ldots \ldots u_{p_1} \\ v_1\, v_2 \ldots \ldots \ldots v_{p_2} \\ \ldots \ldots \ldots \ldots \\ w_1\, w_2 \ldots w_{p_h} \end{array}$$

so as to form an array of h rows and p_1 columns, each row being filled consecutively from the left and starting at the first column, while each column is filled consecutively downwards and starts at the first row. No row can exceed in length any row which lies above it, and no column can exceed any column which is upon its left. If $p_1 = p_h$ the array is rectangular: but usually $p_1 > p_h$ and the array has a zigzag boundary upon its right. This array is called a tableau.

Let $f(u_1, \ldots, w_{p_h})$ be a function of these p elements, treated as p arguments of the function, and let $p!$ expressions be formed by interchanging the arguments in every possible way. Usually these expressions will be distinct, as for instance the $2!$ expressions $f(x, y)$ and $f(y, x)$ differ, unless f happens to be symmetric in these two arguments. Let δ_i denote the operation of producing the i^{th} of these expressions, namely

$$\delta_i f(u_1, \ldots, w_{p_h}) = f(u'_1, \ldots, u'_p)$$

where $u'_1, \ldots$ denote the corresponding arrangement of the p arguments

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[1] [5] p. 366.

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$u_1, \ldots w_{p_h}$. There are therefore $p!$ such operations δ_i and they characterize the symmetric group of order $p!$ Let those $p_1!$ distinct operations be performed which permute the elements belonging to the first row only of the tableau. Let the sum of the resulting functions be regarded as the effect of a resultant operation, P_1 say, acting upon the original function: namely

$$P_1 f = \Sigma\delta_i f(u_1, \ldots, u_{p_1}, v_1, \ldots) = \Sigma f(u'_1, \ldots, u'_{p_1}, v_1, \ldots)$$

where the first p_1 arguments only are to be permuted, while the remainder are unchanged. This summation has $p_1!$ terms.

Let a corresponding operation be performed for the i^{th} row of the tableau. By taking the rows successively in turn we thus obtain h such operations P_i. Since each of these operations affects a distinct set of arguments, the h operations are independent of one another. We can therefore combine them in any order and form a further resultant operation

$$P = P_1 P_2 \ldots P_h = P_2 P_1 \ldots P_h = \ldots,$$

which consists of $p_1!\ p_2! \ldots p_h!$ terms, obtained by all the possible different permutations of the elements, each within its own row of the tableau. Since these terms are added together, this P is called the *positive* symmetric group associated with the tableau.

In contrast to this a new operator N_1 is defined, with reference to the first column of the tableau, and consisting of $h!$ terms caused by the complete set of permutations among the elements of this column: only in this case each term that belongs to an odd permutation is accompanied by a negative sign, and otherwise by a positive sign: namely

$$N_1 f = \Sigma(-)_j \delta_j f(u_1, \ldots)$$

where the summation has $h!$ terms, δ_j denotes the typical permutation of $u_1, v_1, \ldots w_1$, and $(-)_j$ denotes a positive or negative sign according as the corresponding permutation is even or odd. Let p_1 such operations be defined, one for each column, and combined as before into a resultant operation

$$N = N_1 N_2 \ldots N_{p_1} = N_2 N_1 \ldots N_{p_1} = \ldots$$

which consists of $q_1!\, q_2! \ldots$ terms, where q_j denotes the number of elements in the j^{th} column ($j = 1, 2, \ldots, p_1$). This N is called the *negative* symmetric group associated with the tableau. If a row or a column possesses a single element only, the corresponding factor of P or N may be omitted as it has the effect of the factor unity in the whole product.

When a further operator is made by using P and N in succession the products PN and NP usually differ. They do however satisfy the same quadratic relation $X^2 = \theta X$, where $X = PN$ or NP, as already mentioned. One more preliminary remark should be made, before turning to the application of this theory of Young's Substitutional Operators: namely, that the expression $f(u_1, \ldots)$ upon which the operator takes effect may be construed in a most general sense, provided only that each particular arrangement of the p elements $u_1, \ldots$ defines the expression and that they make sense when they are permuted. For instance f might be a determinant, and the u_i might denote suffixes which indicate the columns of the determinant.

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The connexion between the abstract analysis and the determinantal theory is as follows. The $n \times n$ determinant $\Sigma \pm x_{11}x_{22} \ldots x_{nn}$ may be written $N_1\phi$, where ϕ is $x_{11}x_{22} \ldots x_{nn}$ and N_1 is the operator which permutes the n second suffixes of the x_{ij} in all possible ways and sums the results accompanied by a negative sign for each interchange of a pair of suffixes. A product ϕ of v determinants may consequently be written $N\phi = N_1N_2 \ldots N_v\phi$, where N_j is the operator which generates the j^{th} determinant from its leading term. If the determinants which compose the product $N\phi$ are not all of the same order, the factors are to be arranged in a descending order. The operation P_1 is then that which generates a sum of $v!$ such terms $N\phi$ by permuting the first columns, one from each of the r determinants, in all their different ways and adding together the results: P_2 likewise permutes all the second columns; and so on until all the columns are so treated. Then $P = P_1P_2 \ldots$, and $PN\phi$ is the final expression. This positive substitutional operation is reflected, in what follows, by taking a single product of determinants and making all *first* columns that occur the *same*; and so on. Except for a factor $v!$ the two expressions are substitutionally equivalent. Again, instead of taking a product ϕ of determinants whose orders may differ, all the factors have been brought up to the same order $n \times n$, by the introduction of arbitrary constant borders, in distinction from which those elements x_{ij} that undergo permutation (or, equivalently, differentiation) are called the variables. Young's formula is implicit in (13) below.

2. Let $x_1, x_2, \ldots, x_n$ denote n sets of n independent variables such that x_i denotes the i^{th} set $\{x_{i1}, x_{i2}, \ldots, x_{in}\}$ when it is arranged in a column. Let $\Delta = (x_1x_2 \ldots x_n)$ denote the $n \times n$ determinant of these n columns in this order, so that Δ is a function of n^2 independent variables x_{ij}. Let $\Omega = (\partial/\partial x_1 \ldots \partial/\partial x_n)$ denote the corresponding determinant when each element x_{ij} is replaced, in its own position, by the corresponding differential operator $\partial/\partial x_{ij}$: thus $\partial/\partial x_i$ denotes the column of the operators which correspond to the column x_i.

Let
$$\sum_{i=1}^{n} \alpha_i \partial/\partial x_{ni} = \left(\alpha \,\middle|\, \frac{\partial}{\partial x_n}\right)$$
denote the polar operator which substitutes a set of n arbitrary constants α_i for the set of variables x_n. Since Δ is a linear form in the n components of x_n it follows at once by differentiation that
$$(\alpha|\partial/\partial x_n)\,\Delta = (x_1x_2 \ldots x_{n-1}\alpha)$$
which we abbreviate to $(X_{n-1}\alpha)$. More generally, and by further such polarizations of the x_i, let r of these sets, say the last r, be replaced by r columns of arbitrary constants, namely

(1) $$\Delta_r = (x_1x_2 \ldots x_{n-r}\beta_1\beta_2 \ldots \beta_r)$$

which we write as $A_Z\Delta = \Delta_r = (X_{n-r}A_r)$, where A_Z denotes the operator which substitutes the block A_r of r columns for Z the block of the last r columns of Δ. (The above single column operator is therefore written as α_z, with $z = x_n$.)

If this is done for the first $n - 1$ values of r we obtain altogether n different

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determinants, each of which involves the first column x_1 of the variables, all but one involve the second column x_2, and so on, until Δ alone involves the last column x_n only. Let a power product

(2) $$\phi = \Delta^{p_0}\Delta_1^{p_1}\dots\Delta_{n-1}^{p_{n-1}} = \prod_{r=0}^{n-1}(X_{n-r}A_r)^{p_r}$$

of these determinants be constructed, where the exponents p_r are zero or positive integers, and where all the blocks of constants A_r are arbitrary.

For example $\phi = (xyz)^p(xy\alpha)^q(x\beta\gamma)^r$ is such a product of three rowed determinants.

It is well known, and indeed it is a fundamental result in the theory of projective invariants, that the effect of the Cayleyan[2] operator $\Omega = |\partial/\partial x_{ji}|$, already mentioned, acting upon a perfect p^{th} power of Δ, is analogous to ordinary differentiation with regard to Δ and yields the identity

(3) $$\Omega\Delta^p = p(p+1)\dots(p+n-1)\Delta^{p-1}.$$

The object of the present note is to extend this property to the more general power product ϕ, and to shew that

(4) $$\Omega\phi = p_0(p_0+p_1+1)\dots(p_0+p_1+\dots+p_{n-1}+n-1)\phi_1,$$

where $\phi_1\Delta = \phi$, that is to reduce the index p_0 by unity while leaving the remaining indices unchanged. Naturally if $p_0 = 0$, $\Omega\phi$ vanishes.

To prove this we shall first establish a more general theorem. In fact let a set of positive integers λ_r be introduced where

$$\lambda_r = p_0(p_0+p_1+1)\dots(p_0+p_1+\dots+p_{r-1}+r-1),$$

with $r = 1, 2, \dots, n$. From an $n \times n$ determinant of arbitrary constants let the last r columns be chosen and called B. Furthermore let

(5) $$B_Z = (b_1b_2\dots b_r \mid \partial/\partial z_1\,\partial/\partial z_2\dots\partial/\partial z_r)$$

denote the bideterminantal (or compound inner product) operator obtained by combining the r columns of B with the last r columns of Ω. Here for convenience the $(n-r+1)^{\text{th}}$ set x has been renamed z_1, and so on until the last x_n is the same as z_r. With this understanding the following result holds:

(6) THEOREM. $B_Z\phi = \lambda_r(X_{n-r}B)\phi_1$.

Proof. We proceed by induction upon r. For if $r = 1$, and b denotes a single column and z denotes x_n, then, by differentiation,

$$b_z\Delta^{p_0} = p_0\Delta^{p_0-1}\,b_z\Delta.$$

But since $\Delta = (X_{n-1}z)$, $\quad b_z\Delta = (X_{n-1}b)$.

Hence $$b_z\Delta^{p_0} = p_0(X_{n-1}b)\,\Delta^{p_0-1},$$

that is $b_z\phi = \lambda_1(X_{n-1}b)\phi_1$ since z is absent from all the remaining factors belonging to ϕ: which proves the result when $r = 1$. By assuming it true for r we shall prove it true for $r+1$. To do this, write

$$X_{n-r} = Xy, \quad y = x_{n-r},$$

so that y denotes the last of the $n-r$ columns x, and X denotes all the earlier columns. The original set of n columns is now exhibited by

$$\Delta = (XyZ).$$

[2] [2].

Let $\Delta_0 = (XyB)$. The assumed identity is therefore

(7) $$B_Z\phi = \lambda_r \Delta_0 \Delta^{p_0-1} \Delta_1^{p_1} \ldots \Delta_r^{p_r} N$$

where N denotes all those factors into which the column y does not enter, since in (2) y does not enter Δ_s whenever $s > r$. Now each of the $r + 2$ (unrepeated and repeated) factors $\Delta_0, \Delta, \ldots, \Delta_r$ is of the form

$$(XyT)$$

where the block of r columns T differs but X and y are always present in each factor. Operate with c_y, that is $\Sigma c_i \,\partial/\partial y_i$, upon both sides of the equation (7). On the right-hand side we obtain a sum of $r + 2$ terms, one for each different Δ. Thus, the affected parts in the various terms are

$$c_y\Delta_0 = (XcB), \quad c_y(XyT)^{p_s} = p_s(XcT)(XyT)^{p_s-1}.$$

Now perform[3] the determinantal permutation $\{c, B\}'$ which consists of $r + 1$ terms interchanging c with each of the r columns of B in turn, accompanied by a change of sign, and adding the term (the *static* term, let us say) in which c remains unmoved. The result of this upon the left member $c_yB_Z\phi$ of our equation produces the corresponding operator of order $r + 1$, namely

$$\{c, B\}' c_yB_Z \equiv \left(cB \left| \frac{\partial}{\partial y} \frac{\partial}{\partial Z}\right.\right) \equiv \left(cb_1b_2 \ldots b_r \left| \frac{\partial}{\partial y} \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \cdots \frac{\partial}{\partial z_r}\right.\right),$$

as is seen at once on expanding this last determinantal expression by its first column.

On the right there are $r + 2$ terms, as already seen. In the first term Δ_0 has been altered to (XcB), and in any other term a single Δ_s, say, has been altered to (XcT) multiplied by p_s. The effect of the new operation $\{c, B\}'$ on the first term produces

$$(r + 1)\ (XcB)$$

from $C_y\Delta_0$, merely by deranging the $r + 1$ columns of cB within this determinant. In each of the other r terms the new operation convolves the columns c, B which occur entirely within Δ_s and Δ_0 respectively. But by the fundamental identity[4]

$$(c, B)'(XyB)(XcT) = (XcB)(XyT)$$

that is, the operation interchanges the c, wherever it occurs with the y, which occurs in Δ_0 the first determinantal factor. This restores the full exponent p_s to Δ_s for $s = 1, 2, \ldots, r$, and in the case of Δ itself restores $p_0 - 1$ which had dropped to $p_0 - 2$ through the operation c_y. Gathering these results together we infer that

(8) $$\left(cB \left| \frac{\partial}{\partial y} \frac{\partial}{\partial Z}\right.\right) \phi = \lambda_r[(r + 1) + (p_0 - 1) + p_1 + p_2 + \ldots + p_r](XcB)\phi_1$$
$$= \lambda_{r+1}(XcB)\ \phi_1,$$

which is of the same form as the assumed identity but with $r + 1$ replacing r. Since the identity is true when $r = 1$ this proves it by induction for $r = 1, 2, \ldots, n$. In the last stage when $r = n - 1$ in (8) the operator factorizes into

[3][4] p. 27. [4][4] p. 44.

$(cB)\Omega$, and all the columns of X have disappeared. On taking the arbitrary $n \times n$ determinant (cB) to be the unit determinant $|\delta_{ij}|$ the original identity

$$\Omega\phi = \lambda_n \phi_1$$

emerges.

COROLLARY 1. The same identity is true if each A_r that occurs is replaced by p_r arbitrary blocks A'_r, A''_r, etc. This follows since, in the above proof, no use is made of the value of T, but only of its extent.

3. On writing

$$\begin{aligned} p_0 + p_1 + \ldots + p_{n-1} &= q_1 \\ p_0 + p_1 + \ldots + p_{n-2} &= q_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ p_0 &= q_n \end{aligned} \tag{9}$$

where $q_i = p_0 + p_1 + \ldots + p_{n-i}$, we obtain the numbers of times which each x_i occurs in the product ϕ, x_i occurring exactly q_i times, for $i = 1, 2, \ldots, n$. In particular $z = x_n$, and the last column of Δ, appears q_n times. Accordingly, if we operate q_n times in succession with Ω and apply the theorem, we obtain

$$\Omega^{q_n}\phi = \mu_0 \Delta_1^{p_1} \Delta_2^{p_2} \ldots \Delta_{n-1}^{p_{n-1}} = \mu\psi_1, \text{ say,} \tag{10}$$

where μ_0 is a product of positive integers λ_r, and from which the last column x_n has disappeared. From (4) we obtain

$$\begin{aligned} \mu_0 &= p_0! \frac{(p_0 + p_1 + 1)!}{(p_1 + 1)!} \cdots \frac{(p_0 + p_1 + \ldots + p_{n-1} + n - 1)!}{(p_1 + \ldots + p_{n-1} + n - 1)!} \\ &= q_n! \frac{(q_{n-1} + 1)!}{(p_1 + 1)!} \cdots\cdots\cdots\cdots \frac{(q_1 + n - 1)!}{(q_1 + n - 1 - p_0)!}. \end{aligned} \tag{11}$$

Now let Ω_1 denote the $(n - 1)$-fold column of operators (each component being a determinant of order $n - 1$):

$$\{ \, | \partial/\partial x_1 \, \partial/\partial x_2 \ldots . \partial/\partial x_{n-1} | \, \}$$

and let $C_X = (C \,|\, \Omega_1)$ be such as the operator (5) but with $r = n - 1$. Then, by the theorem, $C_X \psi_1$ reduces the exponent p_1 of Δ_1 by unity, and C_X applied p_1 times replaces the X in this factor by C, and introduces a positive integral factor

$$\mu_1 = p_1! \frac{(p_1 + p_2 + 1)!}{(p_2 + 1)!} \cdots \frac{(p_1 + \ldots + p_{n-1} + n - 2)!}{(p_2 + \ldots + p_{n-1} + n - 2)!}.$$

Now write C_{n-1} for this block C. We may proceed in this way with further operators $(C_{n-r} \,|\, \Omega_r)$ of this type, where $r = n - 2$, $n - 3$ and so on, in succession: for the theorem is directly applicable at each such stage, and replaces all the X_{n-r} in $\Delta_r^{p_r}$ by an equal number of C_{n-r} while attaching a further positive integral factor μ_r. If preferred all the C_{n-r} which are p_r in number can be distinct, for they are arbitrary. The whole operation can now be written

$$\Omega \prod_{s=1}^{n-1} (C_{n-s} \,|\, \Omega_s)$$

and, since it is composed entirely of differential operators $\partial/\partial x_{ji}$ and constants, the order of its factors is immaterial. On retaining the original $n \times n$ constant

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determinant (now called $|C_n|$) along with Ω we may drop the factor Ω and let s run from 0 to $n-1$ in the product; for $(C_n|\,\Omega_0)$ factorizes to $|\,C_n|\,\Omega$ since $|\Omega_0| = \Omega$. It is then convenient to express the whole operator in terms of the integers q_i as follows:

$$(12)\qquad \prod_{s=0}^{n-1} (C_{n-s}\,|\,\Omega_s) \equiv (C_{q_1q_2\,\ldots\,q_n}\,|\,\Omega_{q_1q_2\,\ldots\,q_n}) \equiv (C_Q\,|\,\Omega_Q) \equiv C_\Omega\,.$$

Here the capital suffix Q denotes the multiple suffix $q_1q_2\ldots$: and in the latter, which is a set of positive integers written in descending order since their first differences are the p_i which are $\geqslant 0$, it is unnecessary to include any zero suffixes. This Q therefore denotes a partition $\{q_1q_2\ldots\}$ of Σq_i, written in the usual way. Reference to (2) shews that the operator effects the substitution of the C_Q's for the X's as follows:

$$(13)\qquad \prod_{s=0}^{n-1} (C_{n-s}\,|\,\Omega_s)\phi = \theta_Q \prod_{r=0}^{n-1} (C_{n-r}A)_r{}^{p_r}\,,$$

where $\theta_Q = \mu_0\mu_1\ldots\mu_r\ldots\mu_{n-1}$, a numerical constant which is a positive integer. The more general case when all p_r of the C_{n-r} are distinct, for each value of r, can be written down without serious difficulty (only it is rather prolix !). It has the same numerical factor θ_Q.

Two further corollaries follow at once:

COROLLARY 2. Take all the A_r which occur in the product ϕ to be non-zero portions of the unit matrix $[\delta_{ij}]$, so that $(X_{n-r}A_r)$ is then an $(n-r)$ rowed minor of the determinant $|x_{ij}|$. Thus ϕ is a power product of such minors of all orders (every minor of a lower order being a minor within the columns but not necessarily the rows occupied by a minor of a higher order, owing to the original condition imposed upon the columns x_i). Take each C_{n-r} to be the complementary portion of the unit matrix so that $(C_{n-r}A_r) = 1$. Then the corresponding operator C_Ω reduces ϕ to the positive integer θ_Q.

COROLLARY 3. Replace each C_{n-s} by the corresponding matrix X_{n-s}. Then $(X_{n-s}|\,\Omega_s)$ is the well-known *Capelli operator*.[5] The generalized operator X_Ω will produce two sorts of terms when it operates on any function of the x_{ij}—(i) intrinsic terms due to differentiating those parts X_t of the operator which stand in factors to the right of the partial operator $\partial/\partial x_{ij}$, and (ii) extrinsic terms due to direct operation on the operand. Since the right-hand side member of (13) reverts to ϕ itself on substituting the X for the C, it follows that

$$\text{extr}\; X_\Omega\phi = \theta_Q\phi$$

where the notation indicates the extrinsic terms only.

What happens to the intrinsic terms? Is there a result comparable in beauty to the original formula of Capelli? This formula expresses the operator

$$\sum_I (x_1x_2\ldots x_s)_I(\partial/\partial x_1\;\partial/\partial x_2\ldots\partial/\partial x_s)_I$$

(for $I = i_1i_2\ldots i_s$, any set of s different integers $1, 2, \ldots, n$) as a determinant

$$|\,(x_i|\,\partial/\partial x_j) + (n-i)\delta_{ij}\,|\,, \qquad i, j = 1, 2, \ldots n,$$

[5][1].

where the first $n - 1$ integers appear in descending order, finishing with zero, as additions to the elements upon the leading diagonal. These additions are caused by the intrinsic terms, and the expansion of the whole determinant must be taken in the strict order of its columns.[6]

As an example of the complete operator acting upon

$$\phi = (xyz)(xy\alpha)^2(x\beta\gamma)$$

take $C_\Omega \equiv (\partial/\partial x\ \partial/\partial y\ \partial/\partial z)\,.\,(\delta\epsilon \mid \partial/\partial x\ \partial/\partial y)^2(\zeta \mid \partial/\partial x)$ where δ, ϵ, ζ are arbitrary columns, and all the columns consist of three elements each. Then Q, as in (12), denotes the suffix row 4, 3, 1 which indicate the numbers of appearances of x, y, z respectively in ϕ. Then

$$C_\Omega = \theta_{431}(\delta\epsilon\alpha)^2(\zeta\beta\gamma) = 576(\delta\epsilon\alpha)^2(\zeta\beta\gamma).$$

Again, if $\alpha\beta\gamma\delta\epsilon\zeta$ denote the columns of the unit matrix, the result is zero unless $\delta\epsilon\alpha$ include the three different columns, as well as $\zeta\beta\gamma$. For instance

$$(\partial/\partial x\ \partial/\partial y\ \partial/\partial z)(\partial/\partial x\ \partial/\partial y)^2_{13}(\partial/\partial x)_1\ \phi = 576$$

when $\phi = (xyz)(xy)^2_{13}\,x_1$.

The numerical coefficient

$$\theta_Q \equiv \theta_{q_1q_2\ldots q_n}$$

may be found from the above product $\mu_0\mu_1\ldots\mu_{n-1}$ where

$$\mu_i = p_i!\,\frac{(p_i + p_{i+1} + 1)!}{(p_{i+1} + 1)!}\cdots\frac{(p_i + \ldots + p_{n-1} + n - i - 1)!}{(p_{i+1} + \ldots + p_{n-1} + n - i - 1)!},$$

for these actual values of the μ_i follow directly by repeated use of the identity (6). On substituting for the μ_i in terms of the q_i we obtain[7]

$$(14) \qquad \theta_Q = \Pi\mu_i = \frac{\Pi\,(q_r + r - 1)!}{\prod\limits_{r<s}(q_r - q_s - r + s)}, \qquad \left.\begin{matrix} r \\ s \end{matrix}\right\} = 1, 2, \ldots, n\,.$$

This is a positive integer, since each μ_i is, and it is the well-known cofactor of the number f_Q for $n!$, namely

$$\theta_Q f_Q = n!$$

where f_Q is the group characteristic $\chi_0{}^Q$, or the Q^{th} component of the character χ_0, which was given by Frobenius for the symmetric group of order $N = \Sigma q_i$.

This number θ_Q can also be defined[8] by the determinant

$$\Delta_Q = \left| d_{ij} \right| = \left| \frac{1}{(q_i - i + j)!} \right| = \frac{1}{\theta_Q},$$

where $d_{ij} = 0$ whenever $q_i < i - j$ and $d_{ij} = 1$ whenever $q_i = i - j$. The number of rows and columns in the determinant is taken to be the number of non-zero suffixes in the set $Q = q_1q_2\ldots q_n$ ($q_1 \geqslant q_2 \geqslant$ etc.).

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[6]Cf. [1], and [4] p. 117.

[7]Cf. [5] p. 366.

[8][4] p. 359. A misprint is here corrected from j to $i-j$.

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For example

$$\frac{1}{\theta_{431}} = \begin{vmatrix} \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} \\ \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} \\ 0 & 1 & \frac{1}{1!} \end{vmatrix} = \frac{1}{576}.$$

The proof that the determinantal and the product formulae for θ_Q are equivalent follows at once on evaluating the determinant by Dodgson's method.[9] If by compact minor we mean a minor chosen in any manner from any r consecutive rows and any r consecutive columns of the original determinant, then the method depends upon the systematic condensation of Δ_Q by the use of compact minors. Here for instance, if u_Q denotes $1/\theta_Q$, we have the condensation

$$u_{431} \equiv \begin{vmatrix} u_4 & u_5 & u_6 \\ u_2 & u_3 & u_4 \\ 0 & 1 & u_1 \end{vmatrix}, \quad \begin{vmatrix} u_{43} & u_{54} \\ u_{21} & u_{31} \end{vmatrix}, \quad \begin{vmatrix} u_{431} \end{vmatrix} = u_{431}.$$

In this and all such sequences of condensing determinants those elements which stand *within* the whole outer border of elements are called pivotal elements (u_3 alone is such a pivot in this example). If v is such a pivot and V is the 3×3 minor determinant of which v is the central element then V appears as an element in the next but two member of the sequence. If v happens to be zero, then by definition of Δ_Q, the three consecutive elements which stand in the row immediately below that of v,

$$\begin{matrix} & 0 & & \dots \\ 0 & 0 & 0 & \dots \end{matrix}$$

symmetrically, must also be zero. Thus V also vanishes. When $v \neq 0$ the usual pivotal process is available (for example $u_{43}u_{31} - u_{54}u_{21} = u_{431}u_3$, where u_3 can be cancelled since it does not vanish). In either case the process is definite, and leads to the required result.

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[1] A. Capelli, *Math. Ann.*, vol. 29 (1887), 331-338.

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