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UNIVERSAL ARROWS TO FORGETFUL FUNCTORS FROM CATEGORIES OF TOPOLOGICAL ALGEBRA

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We survey the present trends in theory of universal arrows to forgetful functors from various categories of topological algebra and functional analysis to categories of topology and topological algebra. Among them are free topological groups, free locally convex spaces, free Banach-Lie algebras, and more. An accent is put on the relationship of those constructions with other areas of mathematics and their possible applications. A number of open problems is discussed; some of them belong to universal arrow theory, and other may become amenable to the methods of this theory.

INTRODUCTION

The concept of a universal arrow was invented by Samuel in 1948 [**149**] in connection with his investigations of free topological groups. The following definition is taken from the book [**80**].

DEFINITION: If $S : D \to C$ is a functor and c an object of C, a universal arrow from c to S is a pair $< r, u >$ consisting of an object r of D and an arrow $u : c \to Sr$ of C, such that to every pair $< d, f >$ with d an object of D and $f : c \to Sd$ an arrow of C, there is a unique arrow $f' : r \to d$ of D with $Sf' \circ u = f$.

In other words, every arrow f to S factors uniquely through the universal arrow u, as in the commutative diagram

$$\begin{array}{ccc} c & \xrightarrow{\;u\;} & Sr \\ \| & & \downarrow{\scriptstyle Sf'} \\ c & \xrightarrow{\;f\;} & Sd \end{array}$$

This notion bears enormous generality and strength, and at present it is an essential ingredient of a metamathematical viewpoint of mathematics [**80, 59**]. Many mathematical constructions can be interpreted as universal arrows of one or another kind. Examples are: quotient structures and substructures, products and coproducts, including algebraic and topological tensor products, universal enveloping algebras, transition

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from a Lie algebra to a simply connected Lie group and *vice versa*, compactifications of all kinds (Stone-Čech, Bohr, and others), completions, prime spectra of rings, and much more.

We are interested in the particular case where S is a *forgetful functor* from some category of topological algebra or functional analysis, D, to another category of topological algebra or functional analysis or a category of topology, C. Historically the first, and studied in most detail, was the construction of the *free topological group*, $F(X)$, over a topological space X, where C is the category of Tychonoff topological spaces and continuous mappings and D is the category of Hausdorff topological groups and continuous homomorphisms. A number of similar constructions have received a comprehensive treatment. Among them are free abelian topological groups, free compact groups, free locally convex spaces. At the same time, in recent years similar constructions have arisen — either explicitly or implicitly — in other areas of mathematics. In some cases no attempt has been made to establish a bridge between those and former types of universal arrows — although seemingly such a connection would facilitate a study of new constructions. Among the disciplines where new types of universal arrows to forgetful functors are likely to play a noticeable role, are infinite-dimensional Lie theory, supermanifold theory, differential geometry, C^*-algebras and "quantised" functional analysis, and quantum groups.

We do not aim at a comprehensive presentation of the subject outlined in the title of this paper, nor do we give detailed proofs of the results: such an elaborate approach would lead to a voluminous treatise. Instead, we discuss a few carefully selected lines of development which, as we see it, dominated the research over more than 50 years. We are focussing on the most interesting unsolved problems. Also, we do our best in predicting the future directions of the theory, paying special attention to recent germs of it in areas of mathematics bordering topological algebra (Lie theory, functional analysis and mathematical physics).

This small survey inevitably tends to the results and ideas coming from the Russian (or, in a more politically correct language, ex-Socialist, to cover Ukrainian, Moldavian, Georgian, Bulgarian and other contributors) school of universal arrow theorists, where the author himself comes from. Most probably and to the author's regret, the contributions from the other two major centres — the Australian and the American schools — are under represented in this article. The author's personal tastes and research work of his own were prevalent in selecting topics for discussion.

Our bibliography, although (intentionally) not complete, is hopefully "everywhere dense" in the subject (a comparison due to Victor Kac [**62**]).

1. Major classical examples

The following are major examples of universal arrows to forgetful functors from categories of topological algebra, which are the subject of a traditional study in this area. We mark with a lozenge ($^\diamond$) those notions which will be considered later in this survey to some extent. By abuse of terminology and notation, we shall sometimes identify a universal arrow with its target object (however, no confusion should result from that).

1.$^\diamond$ C = **Tych** (the category of Tychonoff topological spaces and continuous mappings) and D = **TopGrp** (the category of Hausdorff topological groups and continuous homomorphisms). The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \to C$ is the *(Markov) free topological group over* X, $F(X)$.

This notion was introduced in 1941 by Markov [**85**] who presented his results in most detail somewhat later [**86**]. Among those mathematicians who responded first to the new concept, were Nakayama [**103**], Kakutani [**64**], Samuel [**149**] and Graev [**41**]; the latter work has had an enormous impact on later investigations in the area, and the paper by Samuel, as we have already mentioned, has produced a deep methodological insight.

2. C = **Tych**$_*$ (the category of pointed Tychonoff topological spaces and continuous mappings preserving base points) and D = **TopGrp** (the base point of a topological group being e, the identity). The universal arrow from an object $X \in C$ (a pointed Tychonoff space) to the forgetful functor $S : D \to C$ is the *Graev free topological group over* X, $F_G(X)$.

In fact, Markov and Graev free topological groups are very closely related to each other by means of the following short exact sequence:

$$e \to \mathbb{Z} \to F_M(X) \to F_G(X) \to e.$$

The choice of a basepoint $* \in X$ does not affect the topological group $F_G(X)$. The Markov free group of X is isomorphic to the Graev free group of the disjoint sum $X \oplus \{*\}$. [**41, 42**]. This is why we consider Markov free topological groups only. Anyway, the Graev approach seems more convenient in some other cases such as free Banach spaces and free Banach-Lie algebras over metric spaces.

3. C = **Met**$_*$ (the category of pointed metric spaces) and D = **MetGrp** (the category of groups endowed with a bi-invariant metric). The universal arrow from an object $(X, \rho, *) \in C$ (a pointed metric space) to the forgetful functor $S : D \to C$ is the *free group over* $X \backslash \{*\}$ *endowed with the Graev metric* $\overline{\rho}$.

This concept is due to Graev [**41, 42**]. The metrised group $(F(X),\overline{\rho})$ is of no particular interest by itself; it deserves attention as an auxiliary device for studying the free topological group $F(X)$. An amazing example of such a kind is Arhangel'skiĭ's theorem from [**8**]. If one wants to consider Graev metrics on a Markov free group then one should start with a fixed metric ρ on the set $X \oplus \{e\}$.

4. C = **Tych** and $D = \mathcal{V}$ is a *variety* of Hausdorff topological groups, considered as a subcategory of **TopGrp**. The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \to C$ is the *free topological group over* X *in the variety* $\mathcal{V}$, $F_{\mathcal{V}}(X)$.

Varieties of topological groups can be understood in different senses (see [**91, 92, 95**] and [**138, 139, 140**]). It would not be clear what the "right" notion is, until a non-disputable version of the Birkhoff theorem for topological groups is obtained (see, however, [**158**]). Anyway, all of the most important classes of topological groups fit both definitions. Examples of varieties are: the variety of SIN groups (topological groups with equivalent left and right uniformities) [**99**]; that of topological groups with quasi-invariant basis [**66**] (= $\aleph_0$-balanced groups in [**9**]); of totally bounded, or precompact, groups; of $\aleph_0$-bounded groups [**46, 9**] et cetera. There is a survey on free topological groups in varieties [**95**]. A free topological group, $F_{\mathcal{V}}(X)$, in a variety $\mathcal{V}$ is actually the composition of the universal arrow $F(X)$ and the universal arrow from $F(X)$ to the natural embedding functor $\mathcal{V} \to$ **TopGrp**. The notion of a free topological group relative to classes of topological groups, considered by Comfort and van Mill [**27**], also belongs in this problematics.

The following is the most important particular case.

5. C = **Tych** and D = **AbTopGrp** (the category of abelian topological groups and continuous homomorphisms). The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \to C$ is the *(Markov) free abelian topological group over* X, $A(X)$.

6. $C = \mathbf{Tych}_*$ and D = **AbTopGrp**. The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \to C$ is the *Graev free abelian topological group over* X, $A_G(X)$.

Of course, $A(X)$ (respectively, $A_G(X)$) is just the abelianisation of $F(X)$ (respectively, $F_G(X)$).

7. C = **Tych** and D = **CompGrp** (the category of compact topological groups and continuous homomorphisms). The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \to C$ is the *free compact group over* X, $F_C(X)$.

Remark that the free compact group $F_C(X)$ is nothing but the Bohr compactifi-

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cation, $bF(X)$, of the free topological group, $F(X)$. (The Bohr compactification, bG, of a topological group G [**94**] is the universal arrow from G to the embedding functor **CompGrp** $\to$ **TopGrp**.) We do not touch free compact groups in our survey, and refer the reader to the series of papers by Hofmann and Morris [**50, 51, 52, 53, 54, 55**]. Also free compact groups may be viewed as completions of *free precompact groups* (or, just the same, free totally bounded groups), that is, free topological groups in the corresponding variety. Free precompact groups have been studied recently in connection with some questions of dimension theory [**151**].

Of course, the notion of the free compact abelian group over X also makes sense, and the structure of such groups has been described in detail (*loco citato*).

8.$^\diamond$ $C =$ **Unif** is the category of uniform spaces and $D =$ **TopGrp**. There are at least four "natural" forgetful functors from D to C; our choice as S is the functor assigning to a topological group G the *two-sided* uniform structure on it; we shall denote the resulting uniform space by G_t. The universal arrow from an object $X \in$ **Unif** (a uniform space) to the functor S is the *free topological group over* X, or the *uniform free topological group*, $F(X)$.

This was an invention of Nakayama [**103**]. Free topological groups over uniform spaces later proved to be a most natural framework for analysing some aspects of free topological groups, see [**110**]. Free topological groups over uniform spaces provide a straightforward generalisation of free topological groups over Tychonoff spaces, because for a Tychonoff space X the free topological group over X is canonically isomorphic to the free topological group over the finest uniform space associated to X.

9. By replacing the category **Tych** by **Unif** in items 2,4,6 one comes to the obviously defined concepts of a *(Graev) free (abelian) topological group over a uniform space.*

10. $C =$ **Tych** (respectively, **Unif**) and $D =$ **LCS** (the category of locally convex spaces and continuous linear operators). The universal arrow from an object $X \in C$ (a Tychonoff space; respectively, a uniform space) to the forgetful functor $S : D \to C$ (which in the second case is defined unambiguously, unlike in item 8) is termed the *free locally convex space over a topological (uniform) space* X, and denoted by $L(X)$.

This concept is also an invention of Markov [**85**]. However, for some reason it received no immedeate attention from the mathematical community until the papers by Arens-Eells [4], Michael [**88**] and Raĭkov [**142**]. The most important of the later developments is due to Flood [**37, 38**] who also proposed a spectacular research program on categorial foundations of analysis (*ibidem*). (We believe that Flood's ideas may become of vital importance in the coming era of noncommutative analysis and geometry.) Some of his results were later rediscovered by Uspenskiĭ [**168**].

A particular case of this construction — the notion of a vector space endowed with the finest locally convex topology — is well known in functional analysis [**150**]; it is actually the free locally convex space over a discrete topological space X.

11. Like in item 4, one can consider universal arrows from an object of **Tych** to the forgetful functor $\mathcal{V} \to$ **Tych** where $\mathcal{V}$ is a *variety* of locally convex spaces in one or another sense. We denote the resulting *free locally convex space over* X *in the variety* $\mathcal{V}$ by $L_{\mathcal{V}}(X)$.

We refer the reader to a very solid paper [**31**] by Diestel, Morris and Saxon, and a survey [**95**] by Morris. Other references include [**18**].

12.$^\diamond$ If $\mathcal{V}$ is the variety of locally convex spaces with weak topology then the resulting *free locally convex space with weak topology* over a Tychonoff space X is denoted by $L_p(X)$.

This concept was apparently well known in functional analysis for decades, because the space $L_p(X)$ is the weak dual of the space $C_p(X)$ of continuous functions on X in the topology of pointwise (simple) convergence. See, for example, [**175**] and references therein.

13. $C =$ **Met**$_*$ and $D =$ **Ban** is the category of *complete normed linear spaces and linear operators of norm* $\leqslant 1$. The universal arrow from an object $X \in$ **Met**$_*$ to the forgetful functor $S : D \to C$ (the origin is a base point) is the *free Banach space over a pointed metric space*, $B(X)$.

This object first appeared in the paper by Arens and Eells [4]; see also [**88, 142, 37, 38, 122**]. However, it was considered by functional analysts independently and from a different point of view: the normed space $B(X)$ is known as the *predual of the space Lip* (X) *of Lipschitz functions* on a pointed metric space X.

14. $C =$ **Tych** and D is the category of universal topological algebras of a given signature Ω. In this case the universal arrow from a space X to the forgetful functor $D \to C$ is the *free universal topological algebra* (of a given signature) over X.

Such algebras were first considered by Mal'cev [**82**] and others [**158, 138, 140**]. We will not touch on them in our survey.

15. $C =$ **Tych** and D is the category of topological associative rings or associative algebras. The resulting *free topological rings* and *free topological algebras* have been also considered by Arnautov, Mikhalev, Ursul and others [**14**].

Later in our survey we shall consider also a number of less traditional examples of universal arrows to forgetful functors. All of them are universal arrows to forgetful functors of one or another kind.

16.$^\diamond$ Let $C = \mathbf{TopGrp} \times \mathbf{TopGrp}$, $D = \mathbf{TopGrp}$, and let S be the diagonal functor $\mathbf{TopGrp} \to \mathbf{TopGrp} \times \mathbf{TopGrp}$. (That is, $S(G) = (G, G)$.) The universal arrow from a pair (G, H) of topological groups to the functor S is called the *free product of G and H* and denoted by $G * H$. In other terms, $G * H$ is just the coproduct of G and H in the category $\mathbf{TopGrp}$.

This notion (belonging to Graev [**43**]) is of the same nature as that of a free topological group. Both constructions share a number of common properties and indeed, can be (if necessary) reshaped as a universal arrow to an appropriate *forgetful* functor. Let $C = \mathbf{TopGrp} \times \mathbf{TopGrp}$ be as above, and let D denote the category of all topological groups with two fixed subgroups. Then $G * H$ can be viewed as the universal arrow from a pair (G, H) to the forgetdul functor from D to C which forgets the first group and sends a triple (F, G, H) to (G, H).

17.$^\diamond$ In an obvious way, the concept of the free product can be defined for arbitrary families of topological groups, $\{G_\alpha : \alpha \in A\}$. This product is denoted by $*_{\alpha \in A} G_\alpha$.

In all the aforementioned cases, similar methods, which are actually of a categorial nature, are used to prove the existence, uniqueness and a number of other properties of universal arrows. We shall summarise those results as follows.

THEOREM 1.1.

(1) *In all cases 1–17 the universal arrow exists and is unique.*
(2) *In all cases apart from 4, the universal arrow is an isomorphic embedding.*
(3) *In case 4, the universal arrow is a homeomorphic embedding if the variety $\mathcal{V}$ contains at least one non totally path-disconnected topological group.*
(4) *In all cases apart from 4 and 7, the image of the universal arrow is topologically closed.*

2. Structure of free topological groups

Among the first, and most vital, questions to be asked about any universal arrow to a forgetful functor from a category of topological algebra or functional analysis, is the question of description of the algebraic-topological structure of the target object of this arrow. In some cases such a description poses no serious problems, but for most (especially noncommutative) examples it is rather challenging. Since this question seems to be best investigated for free topological groups, we find it necessary — and very instructive — to survey the state of affairs in this area.

The topology of a free topological group $F(X)$ is rather complicated, and among the achievements of Graev [**41, 42**] was a description of the topology of $F(X)$ in the

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case where X was a compact space. Later his description was transferred to the so-called k_ω-spaces by Mack, Morris and Ordman [**81**], which result has substantially widened the sphere of applicability of the original description. We shall present it in the strongest form.

Denote by $\widetilde{X} = X \oplus (-X) \oplus \{e\}$, the disjoint sum of a Tychonoff space X, its topological copy $-X = \{-x : x \in X\}$, and a one-point space $\{e\}$. For each $n = 0, 1, 2, \ldots$ there is an obviously defined canonical continuous mapping $i_n : \widetilde{X}^n \to F(X)$. Denote by $F_n(X)$ the subspace of $F(C)$ image of i_n; it is closed. A topological space X is called a *k_ω-space* if it can be represented as a union of countably many compact subsets X_n in such a way that the topology of X is a *weak* topology with respect to the cover $\{X_n : n \in \mathbb{N}\}$, that is, a subset $A \subset X$ is closed if and only if so are all intersections $A \cap X_n$, $n \in \mathbb{N}$. Not only every compact space is a k_ω space; so is every countable CW-complex, every locally compact space with countable base, et cetera.

Theorem 2.1. (Graev–Mack–Morris–Ordman) *Let X be a k_ω-space. Then every mapping i_n is a quotient one, and a subset A of $F(X)$ is closed if and only if so are all intersections $A \cap F_n(X)$. In particular, $F(X)$ is a k_ω-space.*

The above theorem does not admit any noticeable further generalisation, apart from some openly pathological cases, such as the spaces X where every G_δ set is open (the author, unpublished, 1981). In fact, it was shown in [**36**] that the mapping i_3 is not quotient even for $X = \mathbb{Q}$. Answering both questions raised in that paper, the author has proved the following result [**114, 117, 120**].

Theorem 2.2. *Let X be a Tychonoff space. The mapping i_2 is quotient if and only if X is a strongly collectionwise normal space (that is, every neighbourhood of the diagonal in $X \times X$ is an element of the finest uniform structure of X).*

The following property of the mappings i_n proved to be extremely useful.

Theorem 2.3. (Arhangel'skiĭ [**6,7**]) *Let Y be the set of all $y \in \widetilde{X}^n$ such that $i_n^{-1} i_n(y) = \{y\}$. Then $i_n|_Y$ is a homeomorphism.*

A very substantial body of results concerning the structure of free topological groups over k_ω spaces have been deduced (mostly by Australian and American mathematicians) from Theorem 2.1 [**23, 24, 36, 48, 67, 68, 69, 70, 71, 72, 73, 95, 96, 105, 106, 107, 112**].

The following charming theorem of Zarichnyĭ [**179, 180**] connects free topological groups with infinite-dimensional topology. The original result was stated for free Graev topological groups, but it extends to free Markov groups immediately because topologically the group $F(X)$ is a disjoint sum of countably many copies of $F_G(X)$.

Theorem 2.4. (Zarichnyĭ [**179**]) *Let X be a compact absolute neighbourhood retract and $0 < \dim X < \infty$. Then the free topological group $F(X)$ and the free* *abelian topological group $A(X)$ are homeomorphic to an open subset of the locally convex space with finest topology* $\mathbb{R}^\omega = \varinjlim \mathbb{R}^n$.

Returning to general Tychonoff spaces X, one can still describe the topology of $F(X)$ with the help of mappings i_n, but in a fairly non-constructuve way. The following construction was performed by Mal'cev [**82**]. Denote by $\mathfrak{T}_0$ the quotient topology on $F(X)$ with respect to the direct sum of the mappings i_n, $n \in \mathbb{N}$ from the space $\oplus_{n\in\mathbb{N}}\widetilde{X}^n$. It is a Hausdorff, but not necessarily a group, topology. Now construct recursively a transfinite chain of topologies $\mathfrak{T}_\lambda$ on $F(X)$ by defining $\mathfrak{T}_{\lambda+1}$ as the quotient of the topology on $F(X) \times F(X)$ with respect to the mapping $(x,y) \mapsto x^{-1}y$, and $\mathfrak{T}_\tau$ for a limit cardinal τ as the infimum of the chain of topologies $\mathfrak{T}_\lambda$, $\lambda < \tau$. It is clear that for some λ large enough, the topology $\mathfrak{T}_\lambda$ coincides with the topology of $F(X)$. Denote the least λ with this property by $\lambda(X)$. The following question has been open for more than three decades.

PROBLEM. (Mal'cev [**82**]) Which values can $\lambda(X)$ assume?

Seemingly, all one knows is that $\lambda(X) = 0$ for k_ω-spaces, and $\lambda(X) > 0$ for most spaces beyond this class (for instance, for $X = \mathbb{Q}$). In connection with this problem, see also [**135**].

Another long-standing problem asked by Mal'cev in the same paper [**82**] — that of presenting a constructive description of a neighbourhood system of the identity in a free topological group — was solved by Tkachenko [**163**]. Later simpler versions of Tkachenko's theorem were obtained by the author [**120**] and Sipacheva [**152**]. We shall give one of the possible forms of the result. It is more reasonable to state it for free topological groups $F(X)$ over *uniform spaces* (bearing in mind that for a Tychonoff space X the free topological grop over X is canonically isomorphic to the free topological group over the finest uniform space associated to X). Let $X = (X, \mathcal{U}_X)$ be a uniform space. Denote by j_2 a mapping from X^2 to $F_2(X)$ of the form $(x,y) \mapsto x^{-1}y$, and by $j^*{}_2$ a similar mapping of the form $(x,y) \mapsto xy^{-1}$. If $\Psi \in (\mathcal{U}_X)^{F(X)}$ is a family of entourages of the diagonal indexed by elements of the free group over X, then we put

$$\mathcal{V}_\Psi =_{def} \bigcup \{x \cdot [j_2(\Psi(x)) \cup j^*{}_2(\Psi(x))] \cdot x^{-1} : x \in F(X)\}$$

If B_n is a sequence of subsets of some group then, following [**146**], we set

$$[(B_n)] =_{def} \bigcup_{n\in\mathbb{N}} \cup_{\pi\in S_n} B_{\pi(1)} \cdot B_{\pi(2)} \cdot \ldots \cdot B_{\pi(n)},$$

where S_n is a symmetric group.

THEOREM 2.5. (Pestov [**120**]) *Let $(X, \mathcal{U}_X)$ be a uniform space. A base of neighbourhoods of the identity in the free topological group $F(X)$ is formed by all sets of the* *form* $[(\mathcal{V}_{\Psi_n})]$, *where* $\{\Psi_n\}$ *runs over the family of all countable sequences of elements of* $(\mathcal{U}_X)^{F(X)}$.

Among other results on the algebraic-topological structure of free topological groups let us mention a nice theorem of Tkachenko [**160, 161**] stating that the free topological group over a compact space has the countable chain condition (together with its subsequent generalisation due to Uspenskiĭ [**167**]), and a characterisation of Tychonoff spaces X such that the free topological group $F(X)$ embeds into a direct product of a family of separable metrisable groups [**46**].

There exists a very convenient and simple description of the topology of the free abelian topological group $A(X)$ which has no analog in the non-commutative case. One can define Graev metrics on $A(X)$ in the same way as for $F(X)$, and it turns out that they describe the topology of $A(X)$. It follows from this observation that the canonical morphism from $A(X)$ to the free locally convex space $L(X)$ is an embedding of $A(X)$ as a closed topological subgroup (see, for example, [**162**]).

The embedding $A(X) \hookrightarrow L(X)$ enables one to describe the topology of $A(X)$ as the topology of uniform convergence on all equicontinuous families of characters of $A(X)$, and this way Pontryagin-van Kampen duality comes into play. The first time the Pontryagin-van Kampen duality for free abelian topological groups was studied by was Nickolas [**106**] who showed, answering a question by Noble [**108**], that the topological group $A[0,1]$ is non-reflexive (that is, does not verify the statement of the Pontryagin duality theorem). Later the author obtained the following result.

THEOREM 2.6. (Pestov [**121**]) *Let* X *be a Dieudonné complete* k*-space with* $\dim X = 0$. *Then the free abelian topological group* $A(X)$ *is reflexive.*

3. DIMENSION OF FREE TOPOLOGICAL BASES

In 1945 Markov in one of his important papers [**86**] asked whether any two Tychonoff topological spaces, X and Y, with isomorphic free topological groups $F(X)$ and $F(Y)$, are necessarily homeomorphic. Soon Graev in his no less important papers [**41, 42**] answered in the negative by constructing a whole series of pairs X, Y of spaces with $F(X) \cong F(Y)$, and thus the resulting relation of equivalence between Tychonoff spaces turned out to be substantial. Graev called such spaces X and Y *F-equivalent*; we follow the terminology due to Arhangel'skiĭ [**7, 9, 10, 11, 12**] and call such spaces *Markov equivalent* or *M-equivalent*. Graev paid special attention to the pairs of spaces X, Y with Graev free topological groups isomorphic, $F_G(X) \cong F_G(Y)$; however, the distinction between the two relations of equivalence is — from the viewpoint of their topological properties — inessential. With the help of Arhangel'skiĭ's terminology, one of the central results of the Graev's paper [**41**] can be formulated like this.

THEOREM 3.1. (1948, Graev) *If X and Y are M-equivalent compact metrisable spaces then* $\dim X = \dim Y$.

(Here $\dim X$ stands for the Lebesgue covering dimension of a space X.)

This result — as well as the technique of the proof — has received a lot of attention. The generalisations of the result came in two directions: firstly, the equivalence relation was replaced by more and more loose ones, and secondly, the topological restrictions on the spaces X, Y were weakened.

In 1976 Joiner [**60**] noticed that the conclusion $\dim X = \dim Y$ remains true if X and Y are both *locally compact metrisable* spaces such that the free *abelian* topological groups, $A(X)$ and $A(Y)$, are isomorphic. (Following Arhangel'skiĭ, we call such spaces X, Y *A-equivalent.*) Of course, A-equivalence of two topological spaces follows from their M-equivalence, because the universal arrow $A(X)$ is a composition of the universal arrow $F(X)$ and the functor of abelianisation **TopGrp** $\to$ **AbTopGrp**.

Consider the universal arrow from the free abelian topological group $A(X)$ to the forgetful functor from the category of locally convex spaces with weak topology to **AbTopGrp**. The composition of two universal arrows is obviously the free locally convex space in weak topology, $L_p(X)$. Therefore, we come to a still looser relation of equivalence between two spaces: X and Y are *l-equivalent* if $L_p(X) \cong L_p(Y)$. In 1980 Pavlovskiĭ [**113**] showed that $\dim X = \dim Y$ if X and Y are *l-equivalent* spaces which are *either locally compact metrisable* or *separable complete metrisable.*

So far all proofs relied on a suitable refinement of the original Graev techniques. A basically new method — that of inverse spectra — was invoked and applied to this problem by Arhangel'skiĭ [**7, 10**] who deduced from Pavlovskiĭ's theorem the following landmark result.

THEOREM 3.2. (Arhangel'skiĭ 1980) *Let X and Y be l-equivalent compact spaces. Then* $\dim X = \dim Y$.

Independently a weaker version was obtained by Zambakhidze [**177**]: the covering dimension of any two *M-equivalent compact* spaces is the same. Later this result was generalised by him to the class of *Čech complete, scaly, normal, totally paracompact* spaces [**178**] (it remained not quite clear how wide this class actually was). About the same time the result was independently somewhat generalised by Valov and Pasynkov [**173**].

Further efforts were boosted by a question asked by Arhangel'skiĭ [**9**] on whether it was true that for Tychonoff M-equivalent spaces X and Y one has $\dim X = \dim Y$?

The answer "yes" came from the author, who proved in late 1981 [**116**] the following result by combining and adjusting both Graev's lemma and the spectral technique of Arhangel'skiĭ:

Theorem 3.3. (Pestov, 1981) *If X and Y are l-equivalent Tychonoff spaces then* $\dim X = \dim Y$.

As a matter of fact, the aforementioned Graev's lemma, which forms the core of the proofs, is not a single result but rather a *scheme* of results, improved and adjusted from one situation to another. We present it as it appears in [**118**], not in the most general form possible, but in a quite elegant one.

Graev's Lemma 3.4. *If X and Y are M-equivalent Tychonoff spaces then X is a union of countably many closed subspaces each of which is homeomorphic to a subspace of Y.*

Then one uses addition theorems for covering dimension valid for spaces with countable base; to proceed from such spaces to a general situation, the Tychonoff spaces X and Y are decomposed in inverse spectra of spaces with countable base and the same dimension as $\dim X$ and $\dim Y$; the property of l-equivalence of the two limit spaces is partly delegated to the spectrum spaces, in a form strong enough to ensure a version of Graev's lemma.

It was shown by Burov [**25**] that both the result and the scheme of the proof remain true also for cohomological dimension $\dim_G$ where the group of coefficients G is a finitely generated abelian group (for instance, $\dim_{\mathbb{Z}} X \equiv \dim X$).

The weak dual space to $L_p(X)$ is the space of continuous functions on X with the topology of simple (pointwise) convergence, $C_p(X)$. (It follows actually from a version of the Yoneda lemma). The theory of the linear topological structure of the LCS $C_p(X)$ has grown out of Banach space theory, after the following observation proved useful [**28**]: any Banach space E in weak topology is a subspace of $C_p(X)$ where X is the closed unit ball of the dual to E with weak* topology. This theory is developing now on its own, and a good survey is [**13**]. A bridge between the theory of spaces $C_p(X)$ and universal arrow theory is erected by means of the following observation: since the two LCS's in weak topology, $L_p(X)$ and $C_p(X)$, are in duality, then two topological spaces X and Y are l-equivalent if and only if $C_p(X)$ and $C_p(Y)$ are isomorphic.

Arhangel'skiĭ was the first to suggest an even weaker relation of equivalence between two Tychonoff topological spaces, X and Y: two such spaces are called *u-equivalent* if the locally convex spaces $C_p(X)$ and $C_p(Y)$ are isomorphic as uniform spaces (with the natural additive uniformity). Surprisingly, it was possible to make one more step in extending the original Graev result.

Theorem 3.5. (Gul'ko, [**44**]) *If X and Y are u-equivalent Tychonoff spaces then* $\dim X = \dim Y$.

The proof of Gul'ko's result [**44**] develops along the same lines as the author's earlier theorem, but technically it is considerably more sophisticated, and — to the best of author's knowledge — no "soft" version of the proof exists at the moment.

One can consider an even weaker relation of equivalence: two topological spaces, X and Y, are said to be *t-equivalent* [45] if the locally convex spaces $C_p(X)$ and $C_p(Y)$ are *homeomorphic* as topological spaces. It is not known whether the dimension is preserved under the relation of t-equivalence. It is worth mentioning that all the aforementioned equivalence relations (those of M-, l-, u-, t-equivalence) have been distinguished from each other.

What remains still unclear, is the existence of a reasonable straightforward characterisation of dimension of a Tychonoff space X in terms of the additive uniformity of the LCS $C_p(X)$, or the linear topological structure of the space $L_p(X)$, or — at the very least — the algebraic-topological strucuture of $F(X)$. The existing proofs are in a sense obscure and do not reveal the real machinery keeping dimension preserved by the equivalence relations.

It is an opinion of the author that emerged from discussions with Gul'ko in April 1991 that a complete understanding of the phenomenon of preservation of dimension is to be sought on the following way.

CONJECTURE. The Lebesgue dimension of X can be expressed in terms of a certain (co)homology theory associated with the LCS in weak topology $L_p(X)$.

It is not clear if one can use any of the already existing (co)homology theories for locally convex spaces, because the desired theory should make a sharp distinction between week and normable topologies. For instance, the space $C(X)$ endowed with the topology of *uniform convergence on compacta* instead of the pointwise topology carries essentially no information about the dimension of X, according to the celebrated Milyutin isomorphic classification theorem [90].

The following remarkable theorem by Pavlovskiĭ may be also sugggestive; to our knowledge, no attempt has been made yet to generalise it to arbitrary CW-complexes.

THEOREM 3.6. (Pavlovskiĭ [113]) *Two polyhedra (simplicial complexes) X and Y are l-equivalent if and only if* $\dim X = \dim Y$.

In addition to Gel'fand-Naĭmark duality, general interest in the problem of preservation of properties of topological spaces by different functors from the category **Tych** to the categories of topological algebra has been heated for a long time by the following result of Nagata [102].

THEOREM 3.7. (1949, Nagata) *Two Tychonoff spaces X and Y are homeomorphic if and only if the topological rings $C_p(X)$ and $C_p(Y)$ are isomorphic. In other terms, the functor $C_p(\cdot)$ from* **Tych** *to* **TopRings** *is a (contravariant) inclusion functor.*

By considering for every Tychonoff space X the universal arrow from X to a

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forgetful functor from the category **TopGrp** to **Tych** sending a topological group to a topological subspace consisting of all elements of order 2, one comes to the following result [**124**].

THEOREM 3.8. *There exists a (covariant) inclusion functor* **Tych** $\to$ **TopGrp**.

There is no *full* inclusion functor of such kind [**124**].

The following question seems very natural in connection with our problematics, and it was asked independently by many (for example, by Zarichnyĭ at the International Topological Conference, Baku-1987):

QUESTION. Is it true that K-groups of M-equivalent Tychonoff topological spaces are isomorphic?

An obvious idea, to obtain the affirmative answer with the help of universal classifying groups, fails, because if G is a non-abelian topological group and X and Y are M-equivalent, then it follows (from Yoneda's lemma, actually) that $K(X)$ and $K(Y)$ are isomorphic as *sets*, not *groups*: contrary to what is asserted in [**173**], the set $Hom_c(F(X), G)$ does not carry a natural group structure because of non-commutativity of G — and the universal classifying groups in K-theory are noncommutitive.

The general classification of topological spaces up to an M-equivalence (as well as l-equivelence and other relations mentioned in this section) seems a totally hopeless problem. For numerous results on preservation and non-preservation of particular properties of set-theoretic topology by M-equivalence, l-equivalence, et cetera, see [**7, 9, 11, 12, 38, 39, 111, 165, 166**].

4. TOWARDS A TOPOLOGICAL VERSION OF THE NIELSEN-SCHREIER THEOREM

The celebrated Nielsen-Schreier theorem states that every subgroup of a free group is free, and it is equally well known that every subgroup of a free abelian group is free abelian. The analogous result is no longer true for free (abelian) topological groups [**41, 23, 56**]. However, there exist certain sufficient conditions for a subgroup of a free topological group to be topologically free [**24**]. Namely, the following result is true.

THEOREM 4.1. (Brown and Hardy [**24**]) *Let X be a k_ω-space and let H be a closed topological subgroup of $F(X)$. Suppose there exists a continuous Schreier transversal $s : F(X)/H \to F(X)$ for the subgroup H. Then the canonical Nielsen-Schreier basis consisting of all elements*

$$s(Hg)xs(Hgx)^{-1},\ g \in F(X),\ x \in X,$$

is a free topological basis for H if endowed with the induced topology.

In particular, an open subgroup of a free topological group over a k_ω space is topologically free (*ibidem*). For a detailed account of corollaries of the Brown-Hardy techniques and related developments, see [**105**].

It seems that there exists no similar result for subgroups of free abelian topological groups. A recent development is the following:

THEOREM 4.2. (Morris and Pestov [98]) *An open subgroup of a free abelian topological group over a completely regular space X is a free abelian topological group over a free topological basis of the same covering dimension as* dim X.

The following natural question arises as a part of the general topological version of the Nielsen-Schreier subgroup theorem. Let X and Y be some particular topological spaces; in which cases can the free (abelian) topological group over X be embedded (not necessarily in some "canonical" way) as a topological subgroup into the free (abelian) topological group over Y? This problem was treated for a long time, and the main device was the above Theorem 2.1. Main achievements belong in the realm of k_ω spaces. We shall mention three astonishing results in this direction.

THEOREM 4.3. [74] *If X is a closed topological subspace of the free topological group $F(I)$ then the free topological group $F(X)$ is a closed topological subgroup of $F(I)$.*

COROLLARY 4.4. [104, 105] *If X is a finite-dimensional metrisable compact space then $F(X)$ is a closed topological subgroup of $F(I)$.*

THEOREM 4.5. (Katz and Morris [69]) *If X is a countable CW-complex of dimension n, then the free abelian topological group on X is a closed subgroup of the free abelian topological group on the closed ball B^n.*

Now let us return for a while to Theorem 3.3. This result bears a striking similarity to the well-known property of free bases of a discrete free (abelian) group: every two of them have the same cardinality, called the *rank* of a free (abelian) group. For this reason, it seems natural to refer to the dimension of any free topological basis of a free (abelian) topological group as the *topological rank* of it. The rank of a subgroup of a free group can exceed the rank of the group itself (the free group with two generators contains as a subgroup the free group over infinitely many generators), and the same is true for topological rank, according to the above Theorem 4.4. At the same time, the rank of a subgroup of a free abelian group cannot exceed the rank of the group itself, and the following conjecture was put forward in [71, 96]: *the topological rank of a free topological abelian subgroup of a free abelian topological group $A_G(X)$ cannot exceed* dim X.

In spite of strong evidence in support of the conjecture (for example, Theorem 4.2 above implies that the topological rank of an open subgroup of a free abelian topological group is the same as the topological rank of the group itself), the problem was recently solved in the negative for $X = [0,1]$. A surprising point about the aforementioned conjecture is that it turned out to be nearly equivalent to Hilbert's Problem 13 —

so no wonder it remained open for long! (Fortunately enough, Hilbert's problem 13 had been solved by Kolmogorov, and the proof of the following is based on his famous Superposition Theorem.)

THEOREM 4.6. (Leĭderman, Morris and Pestov [**78**]) *For a completely regular space X the following are equivalent.*

(i) *The free abelian topological group $A(X)$ embeds into $A(I)$ as a topological subgroup.*
(ii) *The free topological group $F(X)$ embeds into $F(I)$ as a topological subgroup.*
(iii) *X is homeomorphic to a closed topological subspace of $A(I)$.*
(iv) *X is homeomorphic to a closed topological subspace of $F(I)$.*
(v) *X is homeomorphic to a closed topological subspace of $\mathbb{R}^\infty$.*
(vi) *X is a k_ω space such that every compact subspace of X is metrisable and finite-dimensional.*
(vii) *X is a submetrisable k_ω space such that every compact subspace of X is finite-dimensional.*

The following particular case of the topological version of the Nielsen-Schreier theorem has received a complete solution. If X is a subset of a set Y, then the free group over the set of generators X is a subgroup of the free group over Y. Now let X be a topological subspace of a Tychonoff space Y; there is still a canonical continuous group monomorphism $F(X) \hookrightarrow F(Y)$, but it need not be a topological embedding. Graev has shown [**41, 42**] that if Y is compact and X is closed in Y then $F(X) \hookrightarrow F(Y)$ is a isomorphic embedding of topological groups. This result was transferred to k_ω-spaces. It it known that a necessary condition for the monomorphism $F(X) \hookrightarrow F(Y)$ to be topological is that the restriction $\mathcal{U}_Y|_X$ of the universal uniformity $\mathcal{U}_Y$ from Y to X coincides with the universal uniformity $\mathcal{U}_X$ of X. (It is just an immedeate consequence of the fact that both left and two-sided uniformities on $F(X)$ induce on X its universal uniform structure — which in its turn follows from the existence of Graev's pseudometrics on $F(X)$ and was essentially known to Graev; perhaps Flood [**37**] was the first to state the condition in an explicit form.) In [**114, 115, 117**] and [**110**] it was shown independently, as an answer to a question by Hardy, Morris and Thompson [**48**], that the above condition $\mathcal{U}_Y|_X = \mathcal{U}_X$ is sufficient in the case where X is dense in Y. A final positive answer was obtained by Sipacheva [**153**] after a series of results of intermediate strength [**169, 171**].

THEOREM 4.7. (Sipacheva [**153**]) *Let X be a topological subspace of a Tychonoff space Y. Then the monomorphism $F(X) \hookrightarrow F(Y)$ is a topological embedding if and only if $\mathcal{U}_Y|_X = \mathcal{U}_X$.*

A similar result for free abelian topological groups and free locally convex spaces was earlier obtained by Flood [**37**]. The noncommutative case treated by Sipacheva proved to be much more difficult.

OPEN PROBLEM. Describe those subgroups of a free (abelian) topological group $F(X)$ over a completely regular space X which are themselves free (respectively, free abelian) topological groups.

5. Completeness

Our next topic can be also traced back to Graev's papers [**41, 42**]. Graev deduced from his description of the topology of the free group over a compact space that any such free topological group is Weil complete (that is, complete with respect to the left uniform structure). The result remains true for free topological groups over k_ω-spaces.

Examples of topological groups which are complete in their two-sided uniformity but not Weil complete (and therefore admit no Weil completion at all) have been known for decades, but seemingly it remains still unclear *whether or not free topological groups admit Weil completion*. This question was asked by Hunt and Morris [**56**].

An obvious necessary condition for a free topological group to be Weil-complete is the Dieudonné completeness of X, that is, completeness of X with respect to the finest uniformity $\mathcal{U}_X$. There are only a series of partial results stating the Weil completeness of free topological groups over particular spaces [**169**].

However, it seems in a sense more natural to examine free topological groups for another form of completeness — completeness with respect to the two-sided uniformity (sometimes also called Raĭkov completeness), [**141, 109**]. There exists a fascinating comprehensive result for completeness of this kind, and the question about the validity of such a result was first asked independently by Nummela [**110**] and the author (in oral form, talk at Arhangel'skiĭ's seminar on topological algebra at Moscow University, February 1981).

THEOREM 5.1. (Sipacheva, [**153**]) *The free topological group $F(X)$ over a Tychonoff space X is complete if and only if X is Dieudonné complete.*

The idea of the proof is based on the notion of a special universal arrow, $F_\rho(X)$, introduced by Tkachenko [**163, 164**]. Say that a subspace Y of a topological group G is *Tkachenko thin* if for every neighbourhood of the identity, U, the set $\bigcap\{yUy^{-1} : y \in Y\}$ is a neighbourhood of the identity. Consider the category of pairs (G,Y) where G is a Hausdorff topological group and Y is a Tkachenko thin subset of G, and obvious morphisms between them, and let S be the functor from this category to **Tych** of the form $(G,Y) \mapsto Y$. Now by $F_\rho(X)$ one denotes the universal arrow from a Tychonoff space X to the functor S. There is a canonical continuous algebraic isomorphism

$F(X) \to F_\rho(X)$, and it can be shown without serious difficulties that the topological group $F_\rho(X)$ is complete if and only if X is Dieudonné complete [**163, 164**]. In the case where X is compact, just $F_\rho(X) \cong F(X)$ [**163, 164**]. Sipacheva proved that the free topological group $F(X)$ has a base of neighbourhoods of the identity that are closed in the topology of the topological group $F_\rho(X)$.

The completeness of the free abelian topological group $A(X)$ over Dieudonné complete spaces X was established in [**162**] (and, as a matter of fact, much earlier — in [**37**]).

Let X be a set, and let $\mathcal{V}$ and $\mathcal{W}$ be any two uniformities on X generating the same Tychonoff topology. (Such a triple $(X, \mathcal{V}, \mathcal{W})$ is termed sometimes a *bi-uniform space.*)

QUESTION. Does there exist a topological group $F(X, \mathcal{V}, \mathcal{W})$ algebraically generated by X (free over X) such that $\mathcal{V}$ is the restriction to X of the *left* uniform structure of G, and $\mathcal{W}$ is the restriction to X of the *right* uniform structure?

This question can be obviously reformulated in terms of universal arrows to forgetful functors. This concept may help in understanding how the completeness works.

6. Applications to general topological groups

In this section we consider some applications of free topological groups to the general theory of topological groups. We remark that perhaps one owes the very existence of the concept of free topological group to a stimulating applied problem of such a kind: in his historical note [**85**] Markov was openly guided by the idea to answer a question asked by Pontryagin and to construct the first ever example of a Hausdorff topological group whose underlying space was not normal. (The free topological group over any Tychonoff non-normal space X is such.)

Free topological groups provide flexible "building blocks" for erecting more sophisticated constructions. Also, the following theorem is of crucial importance.

THEOREM 6.1. (Arhangel'skiĭ [**5**]) *Let f be a quotient mapping from a topological space X onto a topological group G. Then the continuous homomorphism $\widehat{f}: F(X) \to G$ extending f is open and therefore G is a topological quotient group of $F(X)$.*

Seemingly, analogs of this theorem exist for other types of universal arrows as well, and one wonders whether this result can be given a universal categorical shaping. Such results are invaluable for examining questions of existence of couniversal objects of one kind or another.

1. NSS PROPERTY. A topological group G *has no small subgroups* if there is a neighbourhood U of the identity element e such that the only subgroup in U is $\{e\}$. This

is abbreviated to *NSS*. The crucial role of the NSS property in finite-dimensional Lie theory (especially in connection with Hilbert's Fifth Problem) is well known.

In 1971 Kaplansky wrote [**65**, p.89]: *"The following appears to be open: if G is NSS and H is a closed normal subgroup of G, is G/H NSS? This is true if in addition G is locally compact (...) (Of course it is an old result for Lie groups.)"*

Very soon Morris [**93**] answered in the negative by constructing a counter-example. Moreover, he proved the following result.

THEOREM 6.2. *Let X be a submetrisable Tychonoff topological space (that is, a Tychonoff space admitting a continuous metric). Then the Graev free abelian topological group $A_G(X)$ over X is an NSS group.*

The proof of this result is so transparent that it deserves a special discussion. The free (Graev) abelian topological group $A_G(X)$ admits a continuous one-to-one group homomorphism into the additive group of the free Banach space over the metric space (X,ρ). (Here ρ stands for a continuous metric on X). Since the additive group of a Banach space is obviously an NSS group (the unit ball contains no nontrivial additive subgroups) then so is $A_G(X)$. We shall return to this proof later.

The above result was generalised to the noncommutative case by considerable successive efforts of Morris and Thompson [**100**], Thompson [**159**], and Sipacheva and Uspenskiĭ [**154**]:

THEOREM 6.3. *Let X be a submetrisable Tychonoff topological space. Then the Markov free topological group $F(X)$ over X is an NSS group.*

It was asked in [**100**] whether the following result is true.

THEOREM 6.4. *Each topological group is a quotient group of an NSS group.*

The author [**114, 117**] has deduced Theorem 6.4 from Theorems 6.3 and 6.1, and later it turned out that such a deduction follows at once from the above Theorems 6.3 and 6.1 in conjunction with [**61**], see [**8, 9**].

An elaborate proof of Theorem 6.3 can be found in [**154**]. The proof is definitely "hard" — it relies on combinatorial techniques of words and their cancellations in free groups. The concept of free Banach-Lie algebra enables us to provide a purely Lie-theoretic (and certainly "soft") proof of Theorem 6.3. (See Section 8 below.)

2. ZERO-DIMENSIONALITY. Our next story is about quotient groups of zero-dimensional topological groups, and it is strikingly similar to the preceding development. In 1938 Weil (see the note [**8**] for this and the next references) claimed that open continuous homomorphisms of topological groups do not increase dimension. This statement was later refuted by Kaplan by means of a counterexample. Arhangel'skiĭ [**6**] shows that every topological group with a countable base is a quotient group of a zero-

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dimensional group. (Zero-dimensionality here and in the sequel is understood in the sense of Lebesgue covering dimension dim.) Possible ways to represent *any* topological group as a quotient group of a zero-dimensional one were discussed by Arhangel'skiĭ in [5], but until late 1980 the above conjecture remained open.

THEOREM 6.5. (Arhangel'skiĭ [8, 9]) *Any topological group is a topological quotient group of a group G with* $\dim G = 0$.

Subtle topological considerations involving Graev metrics on free groups played a crucial role in the proof of the main auxiliary result: *if a submetrisable topological space X is a disjoint union of a family of spaces each of which has a unique non-isolated point then* $\dim F(X) = 0$. Then the fact that every Tychonoff space is a quotient of a space with the above property is used, together with Arhangel'skiĭ's Theorem 6.1.

This result brought to life a variety of satellite theorems and examples refining the statement. Of them by far the most important one is, from the author's viewpoint, the following.

THEOREM 6.6. (Sipacheva [153]) *If X is a Tychonoff space and* $\dim X = 0$ *then* $\dim F(X) = 0$.

3. TOPOLOGISING A GROUP. As the last example, we discuss a problem of Markov [86] remaining open for 40 years. A subset X of a group G is called *unconditionally closed* in G if X is closed with respect to every Hausdorff group topology on G. Markov asked [86] *whether a group G admits a connected group topology if and only if every unconditionally closed subgroup of G has index* $\geqslant \mathfrak{c}$. (Obviously, this condition is necessary.)

The first counterexample was constructed by the author in [125]. Denote by $L^\flat(X)$ the universal arrow from a uniform space X to the forgetful functor from the category of pairs (E, Y), E a LCS and Y a bounded subset of E (with obviously defined morphisms), to **Unif**, of the form $(E, Y) \to Y$ where Y inherits the additive uniformily from E. If G is a topological group and H a closed subgroup, then the left action of G on the quotient space G/H with a natural quotient uniform structure [146] lifts to a continuous action of G on $L^\flat(G/H)$. The double semidirect product

$$G^\dagger = \Big(G \ltimes L^\flat(G)\Big) \ltimes L^\flat(X),$$

where X is the disjoint sum of a family of copies of a quotient space of $G \ltimes L^\flat(G)$, serves as a counterexample to the Markov question in the case where G is an uncountable totally disconnected topological group.

Later it was observed by Remus [143] that the infinite symmetric group $S(X)$ with pointwise topology provides another — much more transparent — counterexample to Markov's conjecture.

The author's techniques were also used by him to construct an example of a group admitting a nontrivial Hausdorff group topology but admitting no non-trivial Hausdorff metrisable topology [**123**].

Another problem of Markov still remains open. A subset X of a topological group G is called *absolutely closed* if it is closed in the coarsest topology on G making all mappings of the form

$$x \mapsto w(x)$$

continuous as soon as $w(x)$ is a word in the alphabet formed by all elements of G and a single variable x. This topology is an analog of the Zariski topology in affine spaces; we think it is natural to call it the *Markov topology* on a group.

PROBLEM. (Markov [**86**]) Prove of refute the conjecture: every unconditionally closed subset of a group is absolutely closed.

Denote by $\mathfrak{T}_M(G)$ the Markov topology on a group G, and by $\mathfrak{T}_\wedge(G)$ the topology intersection of all Hausdorff group topologies on G. It is clear that $\mathfrak{T}_M(G) \subset \mathfrak{T}_\wedge(G)$. Markov's problem can be be now put in other terms: *is it true that for an arbitrary group G one has $\mathfrak{T}_M(G) = \mathfrak{T}_\wedge(G)$?*

7. Free products of topological groups

Graev [**43**] presented a constructive description of the topology of the free product $G * H$ of two compact groups. Later his result was generalised to topological groups whose underlying spaces are k_ω [**97**], and a version of the Kurosh subgroup theorem was established [**104, 107**].

One can ask about the free products of topological groups almost the same natural questions as for free topological groups: to give a reasonable description of the topology in the general case, to prove (or refute) that the free product of two (an arbitrary family of) complete topological groups is a complete group; to prove (or refute) that if H_α is a topological subgroup of G_α for every $\alpha \in A$ then $*_{\alpha \in A} H_\alpha$ is a topological subgroup of $*_{\alpha \in A} G_\alpha$. However, here is a question deserving, from our viewpoint, special attention — and not only because of its respectable age.

The construction of the free product of groups is a generalisation of the construction of a free group: indeed, the free group $F(X)$ over the set X of free generators is just the free product $*_{x \in X} \mathbb{Z}_x$ of $|X|$ copies of the infinite cyclic group $\mathbb{Z}$. This is obviously not the case with free *topological* groups and free products of *topological* groups — unless X is discrete. In 1950 Graev mentioned this and remarked that *"the question of existence of a natural construction which would embrace both free topological groups and free products of topological groups still remains open."*

This problem has been solved for topological groups whose underlying spaces are k_ω [**24**]. Let $\pi : \mathcal{G} \to X$ be a quotient mapping between two completely regular spaces,

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and let every fiber of the map, $\mathcal{G}_x = f^{-1}(x)$, $x \in X$, be endowed with a structure of a group in such a way that a) with the topology inherited from $\mathcal{G}$, the group $\mathcal{G}_x$ is topological, and b) the restriction of π to the subspace $E_{\mathcal{G}} = \{e_{\mathcal{G}_x} : x \in X\}$ is a homeomorphism. Then the topological quotient space $\mathcal{G}/E_{\mathcal{G}}$, obtained by identifying all elements $e_{\mathcal{G}_x}$, $x \in X$ with each other, becomes a so-called *topological groupoid* [**47**]. There exists a universal arrow from any topological groupoid G to the forgetful functor from **TopGrp** to the category of topological gropoids and relevant morphisms; it is called the *universal topological group of* G and denoted by $U(G)$ [**47**]. It is easy to see that if X is discrete then $U(\mathcal{G}/E_{\mathcal{G}})$ is just the free product of topological groups $*_{x\in X}\mathcal{G}_x$, and if all the groups $\mathcal{G}_x$ are isomoprhic to the discrete group $\mathbb{Z}$ then $U(\mathcal{G}/E_{\mathcal{G}})$ is the free topological group $F(X)$. The problem is:

OPEN QUESTION. Is the universal arrow from a topological groupoid of the form $\mathcal{G}/E_{\mathcal{G}}$ an embedding of $\mathcal{G}/E_{\mathcal{G}}$ into $U(\mathcal{G}/E_{\mathcal{G}})$ as a topological subgroupoid?

It was shown [**24, 104**] that it is the the case if $\mathcal{G}$ is a k_ω space. If the answer to the above question is "yes" in the completely regular case as well, then the above construction can be accepted as a fully satisfactory solution of the 1950 Graev problem.

The Graev problem can be connected with deformation theory and quantum groups. In quantum physics, one considers *deformations* of algebraic-topological objects (such as Lie groups) as families of objects, $A_\hbar$, depending on a continuous parameter $\hbar$, which is assumed to be a "very small" real number approaching zero. Physically, $\hbar$ is Planck's constant, and the case $\hbar = 0$ corresponds to the (quasi) classical limit of a theory; what is deformed, is the object A_0. The absence of nontrivial deformations for classical simple Lie groups and algebras was a reason for introducing new kind of objects — the quantum groups [**33, 84, 144, 147, 176**].

While there exists a rich mathematically sound deformation theory for *Lie algebras*, deformations of Lie groups are often treated at a heuristic level. The conjectural Graev construction would enable one to consider the family $G_\hbar$, $\hbar \geqslant 0$ of Lie groups as a veritable continuous path in the topological space $\mathcal{L}(G)$ of all closed subgroups of the topological group $G = U(\mathcal{G}/E_{\mathcal{G}})$, endowed with an appropriate topology. Such spaces have been thoroughly studied [**137**] in connection with extending the Mal'cev Local Theorems to the case of locally compact groups. It is known that there exist numerous "natural" topologies on the set $\mathcal{L}(G)$, including the Vietoris, Chabeauty, and other topologies (*loco citato*).

8. FREE BANACH-LIE ALGEBRAS AND THEIR LIE GROUPS

The *free Banach-Lie* algebra, $\mathfrak{lie}(E)$, over a normed space E is the universal arrow from E to the forgetful functor S from the category **BLA** of complete Lie algebras endowed with submultiplicative norm to the category **Norm** of normed linear spaces.

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THEOREM 8.1. (Pestov [130]) *The free Banach-Lie algebra exists for every normed space E, and $E \hookrightarrow \mathfrak{lie}(E)$ is an isometric embedding. The Lie algebra $\mathfrak{lie}(E)$ is centreless and infinite-dimensional if* dim $E > 0$.

One can also define the free Banach-Lie algebra over an arbitrary pointed metric space X (we shall denote it $\mathfrak{lie}_X$) as the universal arrow from X to the forgetful functor from **BLA** to **Met**$_*$ (zero goes to the basepoint). Obviously, it is just the composition of the free Banach space and free Banach-Lie algebra arrows.

A Banach-Lie algebra $\mathfrak{g}$ is called *enlargable* if it comes from a Banach-Lie group. Every free Banach-Lie algebra is enlargable, and we shall denote the corresponding simply connected Banach-Lie group by $\mathfrak{LG}(E)$ (respectively, $\mathfrak{LG}_X$). Since every Banach-Lie algebra $\mathfrak{g}$ is a quotient Banach-Lie algebra of the free Banach-Lie algebra over the underlying Banach space of $\mathfrak{g}$, then we come to an independent proof of a result due to van Est and Świerczkowski [157]: *every Banach-Lie algebra is a quotient of an enlargable Banach-Lie algebra.*

This result can be strengthened. The couniversality of the Banach space l_1 among all separable Banach spaces is well-known [79]. (Actually, it is due to the fact that l_1 is the free Banach space over a discrete metric space). Therefore, $\mathfrak{lie}(l_1)$ is a couniversal separable Banach-Lie algebra, and the universality property is transfered to the Lie group $\mathfrak{LG}(l_1)$.

THEOREM 8.2. *There exists a couniversal connected separable Banach-Lie group.*

Of course, the same is true for groups containing a dense subset of cardinality $\leqslant \tau$.

One can show using results of Mycielski [101] and an idea of Gelbaum [39] that for any metric space X, the exponential image of $X\setminus\{0\}$ in the Lie group $\mathfrak{GL}_X$ generates an algebraically free subgroup. Now let Y be a submetrisable pointed space admitting a one-to-one continuous mapping to X. The composition of this mapping and the exponential mapping $\exp_{\mathfrak{GL}_X}$ determines a continuous monomorphism $F_G(Y) \to \mathfrak{GL}_X$, and since any Banach-Lie group has the NSS property then it is shared by $F_G(Y)$. This is the promised "soft" proof of the Morris-Thompson-Sipacheva-Uspenskiĭ theorem. In fact, it is just an extension of Morris's original argument (see Theorem 6.2 above) to the non-commutative case: indeed, Banach spaces are exactly simply connected commutative Banach-Lie groups!

In view of the existence of a couniversal separable Banach-Lie group, the following question seems most natural.

QUESTION. Does there exist a universal separable Banach-Lie group?

One should compare it with the following fascinating result of Uspenskiĭ [170].

THEOREM 8.3. (Uspenskiĭ) *The group of isometries of the Banach space $C(I^{\aleph_0})$ endowed with the strong operator topology is a universal topological group with a*

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countable base.

In contrast, the general linear group $GL(E)$ of any Banach space E, endowed with the uniform operator topology, cannot serve as a universal Banach-Lie group because there exist separable enlargable Banach-Lie algebras $\mathfrak{g}$ which do not admit a faithful linear representation in a Banach space [**172**].

The universal arrow from a Lie algebra, $\mathfrak{g}$, to the forgetful functor from the category of associative algebras to the category of Lie algebras is well-known; this is the universal enveloping algebra, $U(\mathfrak{g})$, of $\mathfrak{g}$ [**32**].

It seems that little is known about a topologised version of this, that is, the universal arrow from a locally convex Lie algebra, $\mathfrak{g}$, to the forgetful functor from the category of locally convex associative algebras to the category of locally convex Lie algebras. Let us denote this arrow by $i_{\mathfrak{g}} : \mathfrak{g} \to U_{\mathcal{T}}(\mathfrak{g})$. Is $i_{\mathfrak{g}}$ an embedding of topological algebras? (That is, does a topological version of the Poincaré-Birkhoff-Witt theorem hold?) Is $U_{\mathcal{T}}(\mathfrak{g})$ algebraically isomorphic to $U(\mathfrak{g})$? What about the convergence of the exponential mapping for $U_{\mathcal{T}}(\mathfrak{g})$?

The only result in this connection I am aware of is the following.

THEOREM 8.4. [**22**] *The universal enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ can be made into a normed algebra if and only if $\mathfrak{g}$ is nilpotent.*

This means that, firstly, a metric version of the universal arrow makes no sense and, secondly, in general the algebra $U_{\mathcal{T}}(\mathfrak{g})$ is non-normable even if $\mathfrak{g}$ is finite-dimensional.

A detailed analysis of the structure of the locally convex associative algebra $U_{\mathcal{T}}(\mathfrak{g})$ would be helpful in connection with enlargability problems for $\mathfrak{g}$.

9. The Lie-Cartan theorem

The Lie-Cartan theorem says that finite-dimensional Lie algebras are enlargable, and it seems that the question of the existence of a "direct" proof of the Lie-Cartan theorem, which would be independent of both known proofs (the cohomological one by Cartan [**26**] and the representation-theoretic one by Ado [**1**]), is still open. For a detailed discussion, see the book [**136**], where it is claimed that the above question for a long time received attention from both French and Moscow schools of Lie theorists.

In this Section we discuss the idea of a conjectural proof based entirely on universal arrows type constructions (free topological groups and free Banach-Lie algebras).

It is well known how by means of the Hausdorff series $H(x, y)$ one can associate in the most natural and straightforward way a local Lie group (or, rather, a Lie group germ in the sense of [**145**]) to any Banach-Lie algebra $\mathfrak{g}$ [**22**]. This is why, according to a result of Świerczkowski [**155**], the problem of enlarging a given Banach-Lie algebra $\mathfrak{g}$

is completely reduced to the problem of embedding a local Banach-Lie group U into a topological group G as a local topological subgroup.

Let $\mathfrak{g}$ be a Banach-Lie algebra. Fix a neighbourhood of zero, U, such that the Hausdorff series $H(x,y)$ converges for every $x, y \in U$. (For example, set U equal to a closed ball of radius less than $(1/3) log\ (3/2)$ [**22**].) Denote by $\mathcal{N}_{\mathfrak{g}}$ a normal subgroup generated by all elements of the form $x^{-1}[x.(-y)]y$, $x, y \in U$. Clearly enough, the subgroup $\mathcal{N}_{\mathfrak{g}}$ does not depend on the particular choice of U. Denote by $G_{\mathfrak{g}}$ the topological group quotient of $F(\mathfrak{g})$ by $\mathcal{N}_{\mathfrak{g}}$, and by $\phi_{\mathfrak{g}} : \mathfrak{g} \to G_{\mathfrak{g}}$ the restriction of the quotient homomorphism $\pi_g : F(\mathfrak{g}) \to G_{\mathfrak{g}}$ to $\mathfrak{g}$. One can prove that π_g is a universal arrow of a certain type.

It is well known (in different terms, though — [**156**]) that the enlargability of $\mathfrak{g}$ is equivalent to any of the following conditions:

(a) the intersection $\mathcal{N}_{\mathfrak{g}} \cap \mathfrak{g}$ is discrete in $\mathfrak{g}$;

(b) the restriction of $\phi_{\mathfrak{g}}$ to a neighbourhood of zero in $\mathfrak{g}$ is one-to-one;

(c) the topological group $G_{\mathfrak{g}}$ can be given the structure of an analytical Banach-Lie group in such a way that $\phi_{\mathfrak{g}}$ is a local analytical diffeomorphism; in this case *Lie* $(G_{\mathfrak{g}}) \cong \mathfrak{g}$, $\phi_{\mathfrak{g}} = \exp_{G_{\mathfrak{g}}}$, and $G_{\mathfrak{g}}$ is simply connected.

Although one can show that the closedness of $\mathcal{N}_{\mathfrak{g}}$ in general is not sufficent for any of these conditions to be fulfilled, it *is* so in the following particular case.

THEOREM **9.1.** [131] *A Banach-Lie algebra* $\mathfrak{g}$ *with finite-dimensional centre is enlargable if and only if the subgroup* $\mathcal{N}_{\mathfrak{g}}$ *is closed in* $F(\mathfrak{g})$. *In this case the quotient topological group* $G_{\mathfrak{g}}$ *carries the natural structure of a Banach-Lie group associated to* $\mathfrak{g}$.

The proof of this result goes as follows: firstly, it is reduced to separable Banach-Lie algebras with the help of a local theorem [**126**], and then certain perfectly direct and functorial constructions are used, including the free Banach-Lie algebra over the underlying Banach space of $\mathfrak{g}$, the Banach-Lie group associated to it, and their quotients. Now only one obstacle remains between us and a direct proof of the Lie-Cartan theorem.

CONJECTURE. The closedness of the subgroup $\mathcal{N}_{\mathfrak{g}}$ in the free topological group $F(\mathfrak{g})$ over the underlying topological space of a finite-dimensional Lie algebra $\mathfrak{g}$ can be proved relying solely on the description of the topology of free topological groups over finite-dimensional Euclidean spaces.

In fact, we conjecture that the subgroup $\mathcal{N}_{\mathfrak{g}}$ is topologically free over a k_ω space, and thence complete and closed in any larger topological group.

We already *know* that $\mathcal{N}_{\mathfrak{g}}$ *is always closed in* $F(\mathfrak{g})$ for $\mathfrak{g}$ finite-dimensional (it follows from the Lie-Cartan theorem), and it is tempting to think that the genuine reason why the statement of Lie-Cartan theorem is always true for finite dimensional Lie algebras, is not (co)homological but entirely in the realm of general topology, namely: finite dimensional Lie algebras are k_ω spaces, while infinite dimensional ones are not.

10. Locally convex Lie algebras and groups

Infinite-dimensional groups play a major role in contemporary pure and applied mathematics [**62, 63**]. Many of them cannot be given the structure of a Banach-Lie group (for example, groups of diffeomorphisms of manifolds, some of their subgroups preserving a certain differential-geometric structure, Kac-Moody groups). At the same time, in all particular examples of an infinite-dimensional group there is associated in some natural way an infinite-dimensional Lie algebra, and therefore it is appealing, to try to develop a version of Lie theory with all its attributes general enough to embrace all particular examples of infinite-dimensional groups.

Such attempts have led to the theory of Lie groups modeled over locally convex spaces (bornological and sequentially complete [**89**]), especially Lie groups modeled over Fréchet spaces [**76**]. We shall call a *Fréchet-Lie group* a group object in the category of smooth Fréchet manifolds, that is — in this case — just a smooth manifold modeled over a Fréchet space which carries a group structure such that the group operations are Fréchet C^∞.

There is a striking difference between the Banach and Fréchet versions of Lie theory. For example, although there is a well-defined notion of the Lie algebra, *Lie* (G), of a Fréchet-Lie group G (which is a Fréchet-Lie algebra), the exponential mapping $\exp_G$: *Lie* $(G) \to G$ need not be C^∞ nor a local diffeomorphism; therefore there is in general no canonical atlas on a Fréchet-Lie group. Moreover, the following question seems to be still open:

Question. [**89, 76**] Does the exponential map $\exp_G$: *Lie* $(G) \to G$ always exist for a Fréchet-Lie group G?

Because of such misbehaviour of Fréchet-Lie theory, some mathematicians question its ability to serve as a basis for infinite-dimensional group theory. Among them is Kirillov who once (Novosibirsk, January 1988) even expressed the opinion that obtaining an answer to the above question either in the positive or in the negative sense would be disadvantageous all the same!

Nevertheless, we believe that this question should be answered in order to understand the proper place of Fréchet-Lie theory, and now we want to present a new, universal arrow type construction of locally convex Lie algebras, which may give a clue.

It is convenient to present the results in the spirit of the Δ-normed spaces and algebras of Antonovskiĭ, Boltyanskiĭ and Sarymsakov [**2**].

Let Δ be a directed partially ordered set. A vector space E is said to be Δ-normed if there is fixed a family of seminorms $p = \{p_\delta : \delta \in \Delta\}$ with the property $p_\delta \leqslant p_\gamma \Leftrightarrow \delta \leqslant \gamma$. (The family p is called a Δ-*norm* because it can be treated as a single map $E \times E \to \mathbb{R}^\Delta$ where $\mathbb{R}^\Delta$ is the so-called *topological semifield*, and it satisfies close analogs of all three axioms of a usual norm.)

Let A be an algebra. We shall say that a Δ-norm $p = \{p_\delta : \delta \in \Delta\}$ on A is *submultiplicative* if

(i) *for every* $\delta, \gamma \in \Delta$ *such that* $\delta < \gamma$ *and for every* $x, y \in A$ *one has* $p_\delta(x * y) \leqslant p_\gamma(x) \cdot p_\gamma(y)$, *where* $*$ *denotes the binary algebra operation*;

(ii) *for every* $\delta \in \Delta$ *there is a* γ *such that for every* $x, y \in A$ *one has* $p_\delta(x * y) \leqslant p_\gamma(x) \cdot p_\gamma(y)$.

One can show that the topology of every locally convex topological algebra is given by an appropriate submultiplicative Δ-norm. For example, the locally multiplicatively convex topological algebras introduced by Arens and Michael [**3, 87**] are characterised by the existence of a Δ-norm with the property $p_\delta(x * y) \leqslant p_\delta(x) \cdot p_\delta(y)$ for all $x, y \in A$ and every $\delta \in \Delta$.

For a fixed directed set Δ the class of all complete Δ-normed Lie algebras forms a category with contracting Lie algebra homomorphisms as morphisms. We shall denote this category $\Delta\mathbf{LA}$.

THEOREM 10.1. *For every Δ-normed vector space (E, p) there exists a universal arrow from this space to the forgetful functor from $\Delta\mathbf{LA}$ to the category of Δ-normed spaces. It is an isometric embedding of (E, p) into a Δ-submultiplicateively normed Lie algebra $\mathfrak{lie}(E)$.*

In the particular case where Δ is a one-point set, the above construction coincides with the construction of a free Banach-Lie algebra over a normed space considered earlier.

If Δ has countable cofinality type (in particular, is countable) then the Lie algebra $\mathfrak{lie}(E)$ is a Fréchet-Lie algebra.

The algebra $\mathfrak{lie}(E)$ is centreless and infinite-dimensional (unless $\dim E = 1$). It is completely unclear whether such Fréchet-Lie algebras are enlargable (that is, come from Fréchet-Lie groups). The property of being centreless gives hope that the answer is "yes," at least in some cases. However, if $\Delta = \mathbb{N}$ and the corresponding sequence of seminorms, p, grows "fast enough," there is a good evidence that $\mathfrak{lie}(E, p)$ can have no exponential map.

THEOREM 10.2. *Let $(E, \|\cdot\|)$ be a normed space. Define a Δ-norm p, where $\Delta = \mathbb{N}$, by letting $p_n = n!\|\cdot\|$, $n \in \mathbb{N}$. Suppose there exists a Fréchet-Lie group, G, associated to the Lie algebra $\mathfrak{lie}(E)$. Then there is no exponential map $\mathfrak{lie}(E) \to G$.*

One can also study *free locally convex Lie algebras* over locally convex spaces, that is, universal arrows from an LCS E to the forgetful functor from the category of locally convex topological Lie algebras and continuous Lie algebra homomorphisms to the category of locally convex spaces. We shall denote the free locally convex Lie algebra over E by $\mathfrak{LC}\mathfrak{lie}(E)$. If X is a Tychonoff space, then one can consider the *free locally convex Lie algebra* over X, defined either as the composition of the free locally convex space $L(X)$ and the free locally convex Lie algebra, or directly as the universal arrow from X to the forgetful functor from the category of locally convex topological Lie algebras and continuous Lie algebra homomorphisms to the category **Tych**. We denote this Lie algebra by $\mathfrak{LC}\mathfrak{lie}_X$.

P. de la Harpe has kindly drawn my attention to the following problem.

PROBLEM. (Bourbaki [**22**]) Is it true that every extension of a Lie algebra $\mathfrak{g}$ by means of a $\mathfrak{g}$-module M is trivial (in other terms, $H^2(\mathfrak{g}, M) = (0)$ for every $\mathfrak{g}$-module M) if and only if $\mathfrak{g}$ is a free Lie algebra?

The property $H^2(\mathfrak{g}, M) = (0)$ is readily verifiable for a free Lie algebra $\mathfrak{g}$, but the validity of the inverse implication is not known.

It is not clear yet whether free locally convex Lie algebras can help in answering the above question (supposedly in the negative), but at the very least, they enjoy a similar property for *continuous* second cohomology.

THEOREM 10.3. *Let X be a separable metrisable topological space, and let M be a complete normable locally convex $\mathfrak{LC}\mathfrak{lie}_X$-module. Then every locally convex extension of the Lie algebra $\mathfrak{LC}\mathfrak{lie}_X$ by means of M is trivial. In particular, $H^2_c(\mathfrak{LC}\mathfrak{lie}_X, M) = (0)$.*

The proof follows the argument for free Lie algebras, but the Michael Selection Theorem (Theorem 1.4.9 in [**174**]) is involved.

In some cases one managed to establish the triviality of *algebraic* second cohomology for locally convex (and even Banach) Lie algebras [**29**].

11. SUPERMATHEMATICS

The (unhappy but hardly avoidable) term "supermathematics" is used to designate the mathematical background of dynamical theories with nontrivial fermionic sector in the quasi-classical limit $\hbar \to 0$. The "supermathematics" includes superalgebra, superanalysis, supergeometry et cetera, all of these being obtained from their "ordinary" counterparts by incorporating odd (anticommuting) quantities [**15, 16, 17, 30, 83**].

In one of those approaches an important role is played by the so-called ground algebras, or algebras of supernumbers; in another approach, algebras of this type come into being as algebras of superfunctions over purely odd supermanifolds. As a matter of fact, those algebras turn out to be universal arrows of a special kind, and they also find an independent application in infinite-dimensional differential geometry.

We shall give the necessary definitions. The term *"graded"* in this paper means *"$\mathbb{Z}_2$-graded"*. A graded algebra Λ is an associative algebra over the basic field $\mathbb{K}$ together with a fixed vector space decomposition $\Lambda \cong \Lambda^0 \oplus \Lambda^1$, where Λ^0 is called the *even* and Λ^1 the *odd part (sector)* of Λ, in such a way that the *parity* $\widetilde{x}$ of any element $x \in \Lambda^0 \cup \Lambda^1$ defined by letting $x \in \Lambda^{\widetilde{x}}, \widetilde{x} \in \{0,1\} = \mathbb{Z}_2$, meets the following restriction:

$$\widetilde{xy} = \widetilde{x} + \widetilde{y},\ x, y \in \Lambda^0 \cup \Lambda^1$$

If in addition one has

$$xy = (-1)^{\widetilde{xy}} yx,\ x, y \in \Lambda^0 \cup \Lambda^1$$

then Λ is called *graded commutative.*

THEOREM 11.1. [128, 129] *Let E be a normed space. There exists a universal arrow $\wedge_B E$ from E to the forgetful functor from the category of complete submultiplicatively normed graded commutative algebras to the category of normed spaces. It contains B as a normed subspace of the odd part $(\wedge_B E)^1$ in such a way that $E \cap \{\dot{1}\}$ topologically generates $\wedge_B E$ and every linear operator f from E to the odd part Λ^1 of a complete normed associative unital graded commutative algebra Λ with a norm $\|f\|_{op} \leqslant 1$ extends to an even homomorphism $\widehat{f}\colon \wedge_B E \to \Lambda$ with a norm $\left\|\widehat{f}\right\|_{op} \leqslant 1$.*

Algebraically, $\wedge_B E$ is just the exterior algebra over the space E, endowed with a relevant norm and completed after that. It enjoys one more property. A *Banach-Grassmann algebra* [57] is a complete normed associative unital graded commutative algebra Λ satisfying the following two conditions.

BG_1 *(Jadczyk-Pilch self-duality).* For any $r, s \in \mathbb{Z}_2 = \{0,1\}$ and any bounded Λ^0-linear operator $T\colon \Lambda^r \to \Lambda^s$ there exists a unique element $a \in \Lambda^{r+s}$ such that $Tx = ax$ whenever $x \in \Lambda^r$. In addition, $\|a\|$ equals the operator norm $\|T\|_{op}$ of T.

BG_2. The algebra Λ decomposes into an l_1 type sum $\Lambda \simeq \mathbb{K} \oplus J^0_\Lambda \oplus \Lambda^1$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and J^0_Λ is the even part of the closed ideal J_Λ topologically generated by the odd part Λ^1. In other words, for an arbitrary $x \in \Lambda$ there exist elements $x_B \in \mathbb{K}$, $x^0_S \in J^0_\Lambda$, and $x^1 \in \Lambda^1$ such that $x = x_B + x^0_S + x^1$ and $\|x\| = \|x_B\| + \|x^0_S\| + \|x^1\|$.

THEOREM 11.2. [129] *Let E be a normed space. The following conditions are equivalent:*

(i) dim $E = \infty$;

(ii) $\wedge_B E$ *is a Banach-Grassmann algebra.*

The algebra $\wedge_B l_1$ (denoted by B_∞) was widely used in superanalysis [57].

Algebras of the type $\wedge_B E$ appear in infinite-dimensional differential geometry: in [**75**], Klimek and Lesniewski used them for constructing Pfaffian systems over infinite-dimensional Banach spaces after it became clear that the previously considered Pfaffians over Hilbert spaces are insufficient for applications in mathematical physics.

If one wishes to study algebras of superfunctions on purely odd (that is, including fermionic degrees of freedom only) infinite dimensional supermanifolds modeled over *locally convex* spaces, then another universal arrow comes into being. A *locally convex* graded algebra Λ carries two structures - that of a graded algebra and of locally convex space — in such a way that multiplication is continuous and both even and odd sectors are closed subspaces of Λ. A topological algebra A is called *locally multiplicatively convex*, or just *locally m-convex*, if its topology can be described by a family of submultiplicative continuous seminorms. (Equivalently: A can be embedded into the direct product of family of normable topological algebras.) [**3, 87**]. An *Arens-Michael algebra* [**49**] is a complete locally m-convex algebra.

THEOREM 11.3. [**127, 128**] *Let E be a locally convex space. Then there exists a universal arrow $\wedge_{AM}E$ from E to the forgetful functor from the category of graded commutative Arens-Michael algebras to the category of locally convex spaces.*

Two particular cases are well-known: $\wedge_{AM}\mathbb{R}^{\aleph_0}$ is the DeWitt supernumber algebra [**30**], and $\wedge_{AM}\mathbb{R}^{\omega}$ is the nuclear (LB) algebra considered in [**77**]. (Here $\mathbb{R}^{\aleph_0}$ stands for the direct product of countably many copies of $\mathbb{R}$, and $\mathbb{R}^{\omega}$ denotes the direct limit $\varinjlim \mathbb{R}^n$.) In addition, in the finite-dimensional case, $\wedge_{AM}\mathbb{R}^q$ is just the Grassmann algebra with q odd generators.

Perhaps, the same sort of construction would serve as a base for the study of Pfaffians on infinite dimensional locally convex spaces.

At present one of the most appealing unsolved problems in "supermathematics" is to give a unified treatment of all existing approaches to the notion of a supermanifold by viewing supermanifolds over non-trivial ground algebras Λ as superbundles over *Spec* Λ.

Denote by $\mathcal{G}$ the category of finite-dimensional Grassmann algebras and unital algebra homomorphisms preserving the grading. Let $\mathbf{LCS}^{\mathcal{G}^{op}}$ denote the category of all contravariant functors from $\mathcal{G}$ to the category $\mathbf{LCS}$ of locally convex spaces and continuous linear operators; the category $\mathbf{LCS}^{\mathcal{G}^{op}}$ is called the category of *virtual locally convex superspaces*. Every graded locally convex space $E = E^0 \oplus E^1$ canonically becomes an object of $\mathbf{LCS}^{\mathcal{G}^{op}}$, because it determines a functor of the form $\wedge(q) \mapsto [\wedge(q) \otimes E]^0$; we shall identify this functor with E. The simplest nontrivial example of a virtual graded locally convex space is $\mathbb{R}^{1,1} = \mathbb{R}^1 \oplus \mathbb{R}^1$. The category $\mathbf{LCS}^{\mathcal{G}^{op}}$ is a subcategory of the category $\mathbf{DiffLCS}^{\mathcal{G}^{op}}$ of all contravariant functors from $\mathcal{G}$ to the category $\mathbf{DiffLCS}$ of locally convex spaces and infinitely smooth mappings between them.

CONJECTURE. The set of all morphisms in the category $\mathbf{DiffLCS}^{\mathcal{G}^{op}}$ from a purely odd graded locally convex space E to $\mathbb{R}^{1,1}$ carries a natural structure of a graded locally convex algebra canonically isomorphic to the free graded commutative Arens-Michael algebra, $\wedge_{AM}E'_\beta$, on the strong dual space E'_β.

For more applications of universal arrows in superanalysis, see [**132**].

12. $C^\star$ ALGEBRAS AND NONCOMMUTATIVE MATHEMATICS

Every normed space E admits a universal arrow to the forgetful functor from the category of (commutative) $C^\star$-algebras and their morphisms to the category of normed spaces and contracting linear operators; we shall denote it by $C^\star(E)$ ($C^\star_{com}(E)$, in the commutative case), and refer to it as *the free (commutative) $C^\star$-algebra over a normed space.* The arrows in both cases are isomorphic embeddings. This is simply due to the two facts: firstly, every normed space E embeds into the $C^\star$-algebra of continuous functions on the closed unit ball of the dual space E' with the weak* topology, and secondly, the class of (commutative) $C^\star$ algebras is closed under the l_∞-type sum.

This construction is a particular case of Blackadar's construction of a $C^\star$-algebra defined by generators and relations [**19**]. For example, the *free $C^\star$-algebra* over a set Γ of free generators [**40**] is just the free $C^\star$-algebra in our sense over the Banach space $l_1(\Gamma)$. In non-commutative topology [**21**] the $C^\star$-algebras $C^\star(l_1(\Gamma))$ (treated as objects of the opposite category) are viewed as noncommutative versions of Tychonoff cubes I^τ, because they are couniversal objects (universal — in the opposite category).

It is known that every free $C^\star$-algebra is *residually finite-dimensional (RFD)*, that is, admits a family of $C^\star$-algebra homomorphisms to finite-dimensional $C^\star$-algebras separating points [**40**]. The same is true for our more general objects.

THEOREM 12.1. *For every normed space E the $C^\star$-algebra $C^\star(E)$ is residually finite-dimensional.*

This result seems interesting because there are few known classes of RFD $C^\star$-algebras [**35**].

Both embeddings have been considered earlier [**20, 148**], where the so-called matrix norms on E defined by those embeddings are denoted by MAX and MIN. This construction is especially important for the so-called quantised functional analysis [**34**], of which the idea is that all the main functional-analytic properties and results concerning Banach spaces can be expressed in terms of the universal arrow $C^\star_{com}(E)$, so their non-commutative versions stated for $C^\star(E)$ constitute the object of *quantised* (that is, noncommutative) functional analysis.

In this connection, it may be useful to consider two equivalence relations on Banach spaces, two such spaces, E and F, being equivalent if and only if $C^\star(E) \cong C^\star(F)$ (respectively, $C^\star_{com}(E) \cong C^\star_{com}(F)$).

If one wishes to study "quantised" theory of LCS's then one should turn to the similar universal arrows from a given LCS E to the forgetful functor from the category of the so-called *pro-$C^\star$-algebras* in the sense of N.C. Phillips [**134**] (just inverse limits of $C^\star$-algebras) and their morphisms to the forgetful functor to the category of LCS's; there are both commutative and non-commutative versions of those universal arrows.

Free $C^\star$ algebras and other closely related universal arrows seem to be useful in exploring the co-called compact forms of quantum groups (see [**84, 133**]).

Finally, we expect that a whole new class of examples of the so-called *quantum algebras* in the sense of Jaffe and collaborators [**58**] can be obtained by considering universal arrows from a set of data including graded normed spaces to the relevant forgetful functor.

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