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A NOTE ON AMALGAMS OF INVERSE SEMIGROUPS

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Abstract

This note gives a necessary condition, in terms of graded actions, for an inverse semigroup to be a full amalgam. Under a mild additional hypothesis, the condition becomes sufficient.

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This note introduces graded actions of inverse semigroups, a notion already implicit in the work of Lawson on ordered representations [2] and the author on semidirect products of inverse semigroups [7]. If one studies inverse semigroups from the inductive groupoid point-of-view (see for instance [3]), then these are exactly the sorts of actions which arise from inductive groupoid actions as per the author's [7]. Subsequent work of the author and Lawson [4] will show that graded partial actions of inverse semigroups lead to a natural proof of the structure theorem for idempotent pure extensions of inverse semigroups [3]. In this paper, we use graded actions to provide a generalization of the 'ping-pong' theorem for amalgamated products of groups [5, Proposition 12.4] to full amalgams of inverse semigroups. Namely, we give a condition for an inverse monoid to be a full amalgam and show that essentially all full amalgams arise in this way. We assume some basic familiarity with inverse semigroups [3].

1. Graded actions

If I is an inverse monoid, we use $E(I)$ for the set of idempotents of I. Viewing $E(I)$ as a partially ordered set, via the natural partial order, we let, for $e \in E(I)$,

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$[e] = \{f \in E(I) | f \leq e\}$. If X is a set, we use $I(X)$ to denote the inverse monoid of all partial bijections of X (acting on the left); that is, the monoid of all bijections between subsets of X with composition of relations. An *action* of an inverse semigroup I on X is then a homomorphism $\varphi : I \to I(X)$. Normally, we write mx for $\varphi(m)(x)$ ($m \in I, x \in X$). If I is an inverse monoid, the action is called *unital* if $\varphi(1) = 1$. We say the action is *graded* if there exists a function $p : X \to E(I)$ such that

(i) $\mathrm{dom}(\varphi(e)) = p^{-1}([e])$;
(ii) if $t^{-1}t = p(x)$, then $tt^{-1} = p(tx)$.

We call p the *grading*. Observe that any (unital) action of a group is graded. If only condition (i) holds, we call the action *weakly* graded. Weakly graded actions will be enough to prove our main theorem, but from the point-of-view of inductive groupoids, graded actions are more natural.

PROPOSITION 1.1. *Let I have a weakly graded action on X via φ with grading p. Then for $t \in I$,* $\mathrm{dom}(\varphi(t)) = p^{-1}([t^{-1}t])$ *and* $\mathrm{ran}(\varphi(t)) = p^{-1}([tt^{-1}])$.

PROOF. By considering inverses, it suffices to deal with the domain of $\varphi(t)$. For $f \in I(X)$, $\mathrm{dom}(f) = \mathrm{dom}(f^{-1}f)$ and so

$$\mathrm{dom}(\varphi(t)) = \mathrm{dom}(\varphi(t^{-1}t)) = p^{-1}([t^{-1}t])$$

as desired. □

To show that graded actions are natural, in fact, prevalent, we point out that the Preston-Wagner representation [3] of an inverse semigroup is graded. The action is given by $\varphi : S \to I(S)$ with $\varphi(s) : s^{-1}sS \to ss^{-1}S$ given by $\varphi(s)(t) = st$. If one defines $p : S \to E(S)$ by $p(s) = ss^{-1}$, then it is not difficult to see that p is a grading for φ.

2. Full amalgams of inverse semigroups

If I is an inverse semigroup and $e \in E(I)$, we will write I_e for the $\mathscr{H}$-class of e. If T is an inverse subsemigroup of I, we say that T is a *full* subsemigroup if $E(T) = E(I)$. If I_1 and I_2 are inverse semigroups with $I_1 \cap I_2 = T$, then the *amalgam* $I_1 *_T I_2$ is the inverse semigroup with the usual universal property. The amalgam is called *full* if T is a full subsemigroup of both I_1 and I_2. The following result is a reformulation of [1, Theorem 2] which is, in turn, a reformulation of the main theorem of [6].

THEOREM 2.1. *Let S be an inverse semigroup generated by full inverse subsemigroups S_1 and S_2 and let $T = S_1 \cap S_2$. Then $S \cong S_1 *_T S_2$ if and only if given a product $s = a_1 \cdots a_n t$ with the a_j alternately in $S_1 \setminus T$ and $S_2 \setminus T$, $a_j^{-1}a_j = a_{j+1}a_{j+1}^{-1}$, $a_n^{-1}a_n = tt^{-1}$, $t \in T$, and with $t \notin E(S)$ if $n = 0$, one has $s \notin E(S)$.*

We now give a condition for an inverse semigroup to be an amalgam in terms of weakly graded actions. We use $\subset$ to denote strict containment.

THEOREM 2.2. *Let S be an inverse semigroup generated by full inverse subsemigroups S_1 and S_2 with $S_1 \cap S_2 = T$ such that there exists $i \in \{1, 2\}$ with $[(S_i)_e : T_e] > 2$ for all $e \in E(S)$. Let S have a weakly graded action on a set X with grading p and let X_1 and X_2 be disjoint non-empty subsets of X. Suppose further:*

(i) $(S_1 \setminus T)X_1 \subseteq X_2$, $(S_2 \setminus T)X_2 \subseteq X_1$;
(ii) $TX_1 \subseteq X_1$, $TX_2 \subseteq X_2$;
(iii) *for $e \in E(S)$, $p^{-1}([e]) \cap X_1$ and $p^{-1}([e]) \cap X_2$ are both non-empty.*

Then $S = S_1 *_T S_2$.

PROOF. We imitate the proof of [5, Proposition 12.4]. Suppose $s = a_1 \cdots a_n t \in E(S)$ with the a_j alternately in $S_1 \setminus T$ and $S_2 \setminus T$, $n > 0$, $t \in T$, $a_j^{-1}a_j = a_{j+1}a_{j+1}^{-1}$, and $a_n^{-1}a_n = tt^{-1}$; we obtain a contradiction. Without loss of generality, we may assume that $a_n \in S_1$. For $e \in E(S)$, let $X_{i,e} = X_i \cap p^{-1}([e])$, $i = 1, 2$. By assumption, for all $e \in E(S)$, $X_{i,e} \neq \emptyset$, $i = 1, 2$. Also, by Proposition 1.1 and from conditions (i) and (ii), it follows that if $r \in S_1 \setminus T$, then $rX_{1,r^{-1}r} \subseteq X_{2,rr^{-1}}$ (and dually for $r \in S_2 \setminus T$) and if $r \in T$, then $rX_{i,r^{-1}r} \subseteq X_{i,rr^{-1}}$, $i = 1, 2$. Observe that $s = s^{-1}s = t^{-1}t$ and $s = ss^{-1} = a_1a_1^{-1}$. Thus, since $s \in E(S)$ must act as a partial identity and

$$sX_{1,s} = a_1 \cdots a_n t X_{1,t^{-1}t},$$

we see that $n = 2k$ for some $k > 0$.

We claim, for $i = 1, \dots, n-1$,

$$a_i a_{i+1} X_{1,a_{i+1}^{-1}a_{i+1}} \subset X_{1,a_i a_i^{-1}}$$

where $a_{i+1} \in S_1 \setminus T$ (and hence $a_i \in S_2 \setminus T$). Our above observations show that $a_i a_{i+1} X_{1,a_{i+1}^{-1}a_{i+1}} \subseteq X_{1,a_i a_i^{-1}}$. We must now show that this containment is strict. First suppose that, for all $e \in E(S)$, $[(S_1)_e : T_e] > 2$. Let $e = a_{i+1}^{-1}a_{i+1}$ and choose $r, u \in (S_1)_e \setminus T_e$ such that $ru^{-1} \notin T_e$. Then it follows that at least one of $a_{i+1}r$ and $a_{i+1}u$ is not in T; say $h = a_{i+1}r \notin T$. Note that $h^{-1}a_{i+1} = r^{-1}e = r^{-1} \in S_1 \setminus T$, $h^{-1}h = r^{-1}er = e$, and $hh^{-1} = a_{i+1}rr^{-1}a_{i+1}^{-1} = a_{i+1}a_{i+1}^{-1}$. So $h^{-1}a_{i+1}X_{1,e} \subseteq X_{2,e}$ whence $a_{i+1}X_{1,e} \subseteq hX_{2,e}$. Thus

$$a_{i+1}X_{1,e} \cap hX_{1,e} \subseteq hX_{2,e} \cap hX_{1,e} = \emptyset.$$

Since $X_{1,e} \neq \emptyset$, $h^{-1}h = e = a_{i+1}^{-1}a_{i+1}$, and $hh^{-1} = a_{i+1}a_{i+1}^{-1}$, it follows, from condition (iii) and Proposition 1.1, that

$$\emptyset \neq hX_{1,e} \subseteq X_{2,a_{i+1}a_{i+1}^{-1}}$$

and so

$$a_{i+1}X_{1,a_{i+1}^{-1}a_{i+1}} \subset X_{2,a_{i+1}a_{i+1}^{-1}}.$$

It now follows, since $a_i^{-1}a_i = a_{i+1}a_{i+1}^{-1}$, that $a_i a_{i+1} X_{1,a_{i+1}^{-1}a_{i+1}} \subset X_{1,a_i a_i^{-1}}$. A similar argument shows that if, for all $e \in E(S)$, $[(S_2)_e : T_e] > 2$, then $a_i X_{2,a_i^{-1}a_i} \subset X_{1,a_i a_i^{-1}}$ and the claim follows.

From the above claim and since $tX_{1,s} \subseteq X_{1,s}$, it follows that $sX_{1,s} \subset X_{1,s}$ contradicting s being an idempotent. □

We now prove a strong converse to the above theorem. Let T be a full inverse subsemigroup of an inverse semigroup S. For $s, s' \in S$ with $s \mathcal{R} s'$, one says that $s \sim_L s'$ if $s^{-1}s' \in T$. It is not hard to see that this is an equivalence relation and that if $a^{-1}a = ss^{-1}$, then $s \sim_L s'$ implies $as \sim_L as'$. By a *complete set of left coset representatives of T in S*, we mean a complete set of representatives for $\sim_L$. Note that, for $t, t' \in T$ with $t \mathcal{R} t'$, $t \sim_L t'$.

THEOREM 2.3. *Suppose S_1, S_2 are inverse semigroups with $T = S_1 \cap S_2$ a full subsemigroup of S_1 and S_2. Then there exists a graded action of $S_1 *_T S_2$ on a set X with grading p and disjoint non-empty subsets X_1 and X_2 such that*

(i) $(S_1 \setminus T)X_1 \subseteq X_2$, $(S_2 \setminus T)X_2 \subseteq X_1$;
(ii) $TX_1 \subseteq X_1$, $TX_2 \subseteq X_2$;
(iii) *for $e \in E(S)$, $p^{-1}([e]) \cap X_1$ and $p^{-1}([e]) \cap X_2$ are both non-empty.*

PROOF. Let $S = S_1 *_T S_2$ and consider the Preston-Wagner representation of S; we saw earlier that this action is graded. Choose a complete set of left coset representatives of T in S. Then it is shown in [1], using the results of [6], that each element $s \in S$ has a unique factorization of the form $s = a_1 \cdots a_n t$ with the a_j left coset representatives, alternately in $S_1 \setminus T$ and $S_2 \setminus T$; $t \in T$; $a_j^{-1}a_j = a_{j+1}a_{j+1}^{-1}$; and $a_n^{-1}a_n = tt^{-1}$. Let X_1 be the collection of elements of S whose factorizations begin with an element of $S_2 \setminus T$ and X_2 the collection of elements whose factorizations begin with an element of $S_1 \setminus T$. It is straightforward to verify that, for the Preston-Wagner representation of S, X_1 and X_2 have the desired properties. □

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