Can. J. Math., Vol. XXXIII, No. 5, 1981, pp. 1232–1244

GENERATORS OF $U_n(V)$ OVER A QUASI SEMILOCAL SEMIHEREDITARY RING

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0. Introduction. Let o be a quasi semilocal semihereditary ring, i.e., o is a commutative ring with 1 which has finitely many maximal ideals $\{A_i | i \in I\}$ and the localization o_{A_i} at any maximal ideal A_i is a valuation ring. We assume 2 is a unit in o. Furthermore * denotes an involution on o with the property that there exists a unit θ in o such that $\theta^* = -\theta$. V is an n-ary free module over o with f: $V \times V \rightarrow o$ a λ-Hermitian form. Thus λ is a fixed element of o with $\lambda\lambda^* = 1$ and f is a sesquilinear form satisfying $f(x, y)^* = \lambda f(y, x)$ for all x, y in V. Assume the form is nonsingular; that is, the mapping $M \rightarrow \mathrm{Hom}\ (M, A)$ given by $x \rightarrow f(\ , x)$ is an isomorphism. In this paper we shall write $f(x, y) = xy$ for x, y in V.

Let U be a submodule of V. If there exist n vectors $x_1, \ldots, x_r, \ldots, x_n$ such that $U = ox \oplus \ldots \oplus ox_r$ and $V = ox_1 \oplus \ldots \oplus ox_r \ldots \oplus ox_n$, then we call U a subspace of V and r the dimension of U, r is denoted by $\dim U$.

Let U be a subspace of V. We call U a line if $\dim U = 1$, a plane if $\dim U = 2$, and a hyperplane if $\dim U = n - 1$.

Let $U_n(V)$ or $U(V)$ be the unitary group on V. We call an element σ in $U(V)$ an isometry on V. An isometry τ on V which fixes every vector in a hyperplane V_τ of V is called a quasi-symmetry if V_τ is nonsingular, and a unitary transvection if V_τ is singular: Let S be the set of all those τ, i.e., the set of quasi-symmetries and unitary transvections.

In the present paper, we shall determine the length $l(\sigma)$ of any isometry σ in $U(V)$, i.e., the minimal number of factors that are needed to express σ as a product of elements in S. The result is

$$l(\sigma) = n - d$$

where d is the dimension of a maximal subspace of V which is contained in the module V_σ of σ. In this paper set theoretic difference of A and B will be written $A - B$. $M \oplus N$ is a direct sum of modules M and N.

Clearly, this is a generalization of [**7**].

1. Statement of the theorem. $\{A_i | i \in I\}$ is the set of all maximal ideals of o. For i in I, let π_i or $-$ be the canonical homomorphism from o onto $\bar{o} = o/A_i$. We use the same notation π_i or $-$ to denote the canonical map from V onto $\bar{V} = V/A_iV$. We note that we consider no form on

Received April 28, 1980 and in revised form January 28, 1981.

$\bar{V}$ and we only regard $\bar{V}$ as a module. Further, for σ in $U(V)$ we define $\bar{\sigma}$ in Aut $(\bar{V})$ by $\bar{\sigma}\bar{x} = \overline{\sigma x}$, $x \in V$.

For a subset U of V, $U^\perp = \{x \in V | xU = 0\}$. For submodules U and W of V, $U \perp W$ means $UW = 0$ and $U \cap W = \{0\}$. For $\sigma \in U(V)$ let V_σ be the fix module of σ, i.e.,

$$V_\sigma = \{x \in V | \sigma x = x\} \quad \text{and} \quad d = \min \{\dim \pi_i(V_\sigma) | i \in I\}.$$

We define $l(\sigma) = 0$ for $\sigma = 1$.

Now, with these notations, we state our theorem.

THEOREM. *For any σ in $U_n(V)$ we have $l(\sigma) = n - d$.*

2. Preliminary lemma. We have finitely many maximal ideals $\{A_i | i \in I\}$, I is the index set. For each i in I, ψ_i or $'$ denotes the canonical homomorphism of $\mathfrak{o}$ into $\mathfrak{o}_{A_i}$ which carries an element a of $\mathfrak{o}$ to the class a' of $\mathfrak{o}_{A_i}$ represented by $a/1$.

Therefore for a and b in $\mathfrak{o}$, $a' = b'$ if and only if $ca = cb$ for some c in $\mathfrak{o} - A_i$. We use the same notation ψ_i or $'$ to denote the canonical homomorphisms $V \to \mathfrak{o}_{A_i}V$ or $U(V) \to$ Aut $(\mathfrak{o}_{A_i}V)$. We consider no form on $\mathfrak{o}_{A_i}V$ and only regard it as a module.

Now, we take a base $\{x_\mu | \mu = 1, \ldots, n\}$ for V and fix it.

LEMMA 2.1. *Let $i \in I$.*

(a) *For vectors u and v in V if we have $u' = v'$, then $cu = cv$ for some c in $\mathfrak{o} - A_i$.*

(b) *For any vector v in V we can express $cv = ay$ for some $a \in \mathfrak{o}$, $c \in \mathfrak{o} - A_i$ and $y \in V - A_iV$.*

Proof. First we prove (a). We have the base $\{x_1, \ldots, x_n\}$ for V. Write $u = \sum a_\mu x_\mu$ and $v = \sum b_\mu x_\mu$, $a_\mu, b_\mu \in \mathfrak{o}$. Then $a_\mu' = b_\mu'$ for $\mu = 1, \ldots, n$. Hence for each μ we have $c_\mu a_\mu = c_\mu b_\mu$ for some c_μ in $\mathfrak{o} - A_i$. Putting $c = \prod c_\mu$, we have (a).

Next we prove (b). Write $v = \sum a_\mu x_\mu$, $a_\mu \in \mathfrak{o}$. First let $v' = 0$, i.e., $a_1' = \ldots = a_n' = 0$. This means that for some $c_1, \ldots, c_n$ in $\mathfrak{o} - A_i$ we have $c_1 a_1 = \ldots = c_n a_n = 0$. So, if we put $c = \prod c_\mu$, $a = 0$ and $y =$ any vector in $V - A_iV$, then we have $cv = ay$. Next let $v' \neq 0$. Therefore at least one a_τ', say a_1', is not zero. Since $\mathfrak{o}_{A_i}$ is a valuation ring, we may assume a_1' divides all a_τ' in $\mathfrak{o}_{A_i}$. From this and by (a) we have (b).

3. Proof of the theorem. For i in I, throughout this paper, $-$ denotes π_i, $'$ denotes ψ_i and ϵ_i denotes an element in $\mathfrak{o}$ with $\pi_i\epsilon_i = 1$ and $\pi_j\epsilon_i = 0$ for $j \neq i$; such ϵ_i exists by the Chinese Remainder Theorem.

LEMMA 3.1. *Let $\{E_s | 1 \leqq s \leqq r\}$ be r hyperplanes of V, then*

$$\dim \overline{\bigcap_{s=1}^{r} E_s} \geqq n - r \quad \textit{for any } i \textit{ in } I.$$

https://doi.org/10.4153/CJM-1981-092-1 Published online by Cambridge University Press

Proof. Take any i in I. If $r = 1$, then the lemma is clear. So let $r > 1$. Write

$$D = \bigcap_{s=1}^{r-1} E_s \quad \text{and} \quad E = E_r.$$

We suppose $\dim \bar{D} \geqq n - (r-1)$ and show

$$\dim \overline{D \cap E} \geqq n - r,$$

which gives us the lemma by induction on r.

We write $d = \dim \bar{D}$. Take a base $\bar{x}_1, \ldots, \bar{x}_d$ for $\bar{D}$ where $x_1, \ldots, x_d$ are in D. Since E is a hyperplane, we can write $V = E \oplus ox$, $x \in V$. We may express $x_\mu = u_\mu + a_\mu x$, $u_\mu \in E$ and $a_\mu \in o$ for each $\mu = 1, \ldots, d$.

If $a_\mu' = 0$ for all μ, then we have an element c in $o - A_i$ with $ca_\mu = 0$ for all μ. Hence $cx_\mu = cu_\mu$ is contained in $D \cap E$, and so $\overline{D \cap E} = \bar{D}$. Consequently, $\dim \overline{D \cap E} > n - r$.

Next, we treat the case that at least one $a_\mu' \neq 0$. Since o_{A_i} is a valuation ring, we may assume a_1' divides any a_μ' in o_{A_i}. Put $a_\mu' = (b_\mu'/c'_\mu)a_1'$ for some b_μ in o and c_μ in $0 - A_i$. Then

$$(c_\mu a_\mu)' = c_\mu' a_\mu' = b_\mu' a_1' = (b_\mu a_1)'.$$

Hence $e_\mu c_\mu a_\mu = e_\mu b_\mu a_1$ for some e_μ in $o - A_i$. Put

$$v_\mu = e_\mu c_\mu x_\mu - e_\mu b_\mu x_1.$$

Then v_μ is in $D \cap E$. Since c_μ, e_μ are in $o - A_i$, we have

$$\dim \overline{D \cap E} \geqq d - 1 \geqq n - r.$$

COROLLARY 3.2. $l(\sigma) \geqq n - d$.

Proof. Remember that quasi-symmetries and unitary transvections fix hyperplanes. Apply the lemma.

By the corollary it suffices to show that $l(\sigma) \leqq n - d$. The proof will proceed by induction on $n - d$.

LEMMA 3.3. *Let U be a submodule of V. If $\bar{V} = \bar{U}$ for all i in I, then $V = U$.*

Proof. We have

$$V = \bigoplus_{\mu=1}^{n} ox_\mu.$$

Take $\{u_{i_\mu}\}$ in U with $\bar{x}_\mu = \bar{u}_{i_\mu}$ for i in I and μ in $\{1, \ldots, n\}$. Put

$$u_\mu = \sum_{i \in I} \epsilon_i u_{i_\mu}.$$

Then u_μ is contained in U and $\bar{x}_\mu = \bar{u}_\mu$ for each i and μ. This means

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$x_\mu - u_\mu$ is in AV, where $A = \bigcap_{i \in I} A_i$. So, we may write

$$x_\mu = u_\mu + \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu, \quad a_{\mu\nu} \in A.$$

Put $M = \{a_{\mu\nu}\}$. Then, we have

$${}^t(u_1, \ldots, u_n) = (E - M)\,{}^t(x_1, \ldots, x_n),$$

E is the identity matrix. Since $E - M$ is invertible, we have $V = U$.

Let $n - d = 0$. Recall we have defined

$$d = \min \{\dim \pi_i(V_\sigma) | i \in I\}.$$

Hence by the lemma we have $V = V_\sigma$. Therefore, $\sigma = 1$ and we have $l(\sigma) = 0 = n - d$, whence there is nothing to do.

So, let $n - d > 0$, i.e., $\sigma \neq 1$. We shall show that there exists τ in S such that

$$\min \{\dim \pi_i(V_{\tau\sigma}) | i \in I\} = d + 1,$$

which will imply $l(\tau\sigma) \leqq n - (d + 1)$ by induction on $n - d$, and so $l(\sigma) \leqq n - d$ as we desire. Thus, all that we have to do is to find such τ in S.

Definition.

$$I_0 = \{i \in I | \sigma' - 1 = 0 \quad \text{for} \quad i\}$$

$$I_1 = \{i \in I | \sigma' - 1 \neq 0 \quad \text{for} \quad i\}.$$

In other words,

$$I_0 = \{i \in I | c(\sigma - 1)V = 0 \text{ for some } c \in \mathfrak{o} - A_i\}$$

and

$$I_1 = \{i \in I | c(\sigma - 1)V \neq 0 \text{ for any } c \in \mathfrak{o} - A_i\}.$$

Clearly, $I = I_0 + I_1$ (direct sum).

LEMMA 3.4. *Let $i \in I$. If a vector y is contained in $V - A_iV$, then there exists a vector x in V with $yx \in \mathfrak{o} - A_i$.*

Proof. Write

$$y = \sum_{\mu=1}^{n} p_\mu x_\mu, \quad p_\mu \in \mathfrak{o}.$$

Since $y \notin A_iV$, at least one p_μ, say p_1, is not in A_i. On the other hand, since V is nonsingular and $\{x_1, \ldots, x_n\}$ is a base for V, there exists a vector x in V with $x_1x = 1$ and $x_\mu x = 0$ for $\mu \neq 1$. So we have $yx = p_1 \notin A_i$.

The following lemma is essential for the proof of the theorem.

LEMMA 3.5. *Let $i \in I_1$. Then there exist a, c in $\mathfrak{o}$ and x, y in V such that*
(C_1) $c(\sigma - 1)x = ay$ *with* $c \notin A_i$,
(C_2) $yx \notin A_i$,
(C_3) $c(\sigma - 1)V \subset aV$.

Proof. We have a direct sum $V = \bigoplus_{\mu=1}^{n} \mathfrak{o}x_\mu$. For each x_μ, by (b) of Lemma 2.1 we can express

$$c_\mu(\sigma - 1)x_\mu = a_\mu y_\mu$$

where a_μ is in $\mathfrak{o}$, c_μ in $\mathfrak{o} - A_i$ and y_μ in $V - A_iV$. Since $i \in I_1$, we have $\sigma' \neq 1$. This implies that at least one a_μ', say a_1', is not zero. Since $\mathfrak{o}_{A_i}$ is a valuation ring, we may assume a_1' divides all a_μ', say, let $a_\mu' = a_1'b_\mu'$ for b_μ in $\mathfrak{o}$. Hence for each μ by Lemma 2.1 there exists d_μ in $\mathfrak{o} - A_i$ such that $a_\mu d_\mu = a_1 b_\mu d_\mu$. We may take $b_1 = d_1 = 1$. Therefore, putting $a = a_1$ and $c = \prod_{\mu=1}^{n} c_\mu d_\mu$, we have

(1) $\quad c(\sigma - 1)x_\mu = ae_\mu y_\mu$

where $e_\mu \in \mathfrak{o}$, $e_1 \in \mathfrak{o} - A_i$ and $c \in \mathfrak{o} - A_i$. From this (C_3) $c(\sigma - 1)V \subset aV$ is now clear.

Next, since $y_1 \in V - A_iV$, by Lemma 3.4 for some x_μ we have

(2) $\quad y_1 x_\mu \notin A_i$.

Let p, q be variables in $\mathfrak{o}$. We put

$$x = px_1 + qx_\mu, \quad y = pe_1y_1 + qe_\mu y_\mu.$$

Then by (1) we have (C_1) $c(\sigma - 1)x = ay$ and the equation

(3) $\quad yx = pp^*e_1y_1x_1 + pq^*e_1y_1x_\mu + p^*qe_\mu y_\mu x_1 + qq^*e_\mu y_\mu x_\mu$

holds. Hence it suffices to show that we can choose p, q in $\mathfrak{o}$ with $yx \notin A_i$, which completes our proof. We recall that we have the unit θ in $\mathfrak{o}$ with $\theta^* = -\theta$. Therefore the answer is given by the following table, where $-$ denotes π_i (note $\bar{e}_1 \neq 0$ by (1) and $\overline{y_1x_\mu} \neq 0$ by (2)).

Cases	$\overline{y_1x_1}$	$\overline{e_\mu}$	$\overline{y_\mu x_\mu}$	p	q
1	$\neq 0$			1	0
2	0	0		1	1
3	0	$\neq 0$	$\neq 0$	0	1
4	0	$\neq 0$	0	1	1 or θ

In case 4 above, we take $q \in \{1, \theta\}$ with

$$\overline{q^*e_1y_1x_\mu + qe_\mu y_\mu x_1} \neq 0;$$

in fact such q exists, since $\overline{e_1y_1x_\mu} \neq 0$. Thus we have shown (C_2) $yx \notin A_i$.

Let us call the above four elements $a, c \in \mathfrak{o}$ and $x, y \in V$ in the lemma "a good foursome for i" if they satisfy (C_1), (C_2) and $C(_3)$, and denote it by (a, y, c, x). Further, when there exits a good foursome (a, y, c, x) for i, we say "x is good for i". With this definition we can say that if $i \in I_1$ then there exists a vector x in V which is good for i.

Now, since the involution * on $\mathfrak{o}$ induces a permutation on the set of maximal ideals $\{A_i | i \in I\}$ of $\mathfrak{o}$, we can define a permutation on the index set I by defining $i^* = j$ if and only if $A_i^* = A_j$.

LEMMA 3.6. *If $i, i^* \in I_1$, then there exists a vector u_i in V which is good for both i and i^*.*

Proof. If $i = i^*$ then there is nothing to do (apply Lemma 3.5). So let $i \neq i^*$. To simplify the notation we write $j = i^*$. Now by Lemma 3.5 we have good foursomes (a_i, y_i, c_i, x_i) for i and (a_j, y_j, c_j, x_j) for j. By condition (C_1), (C_2) we have respectively,

(1) $\quad c_i(\sigma - 1)x_i = a_i y_i \quad$ and $\quad c_j(\sigma - 1)x_j = a_j y_j$

with $c_i \in \mathfrak{o} - A_i$ and $c_j \in \mathfrak{o} - A_j$, and

(2) $\quad \pi_i(y_i x_i) \neq 0 \quad$ and $\quad \pi_j(y_j x_j) \neq 0.$

By condition (C_3) we can express

(3) $\quad c_i(\sigma - 1)x_j = a_i w_j \quad$ and $\quad c_j(\sigma - 1)x_i = a_j w_i$

for some $w_i, w_j \in V$.

Let p, q be variables in $\mathfrak{o}$ and put $u_i = px_i + qx_j$. Then by (1) and (3) we have

(4) $\quad c_i(\sigma - 1)u_i = a_i(py_i + qw_j) \quad$ and $\quad c_j(\sigma - 1)u_i = a_j(pw_i + qy_j).$

Therefore, two foursomes $(a_i, py_i + qw_j, c_i, u_i)$ and $(a_j, pw_i + qy_j, c_j, u_i)$ satisfy the two conditions (C_1) and (C_3) for i and j respectively. So it suffices to show (C_2) for those two foursomes respectively. Namely, we must show that we can choose p, q in $\mathfrak{o}$ so that

$$\pi_i((py_i + qw_j)u_i) \neq 0 \quad \text{and}$$
$$\pi_j((pw_i + qy_j)u_i) \neq 0.$$

As usual, the Chinese Remainder Theorem will play a central role. To simplify the notation we write

$$f = (py_i + qw_j)u_i \quad \text{and} \quad g = (pw_i + qy_j)u_i.$$

Therefore

(5) $\quad f = pp^*y_i x_i + qp^*w_j x_i + pq^*y_i x_j + qq^*w_j x_j$

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and

(6) $g = pp^*w_ix_i + qp^*y_jx_i + pq^*w_ix_j + qq^*y_jx_j$.

By the Chinese Remainder Theorem we can take p, q in o as in the following table.

Cases	$\pi_i(w_jx_j)$	$\pi_j(w_ix_i)$	$\pi_i(w_jx_i)$	$\pi_j(w_ix_j)$	$\pi_i(y_ix_j)$	$\pi_j(y_jx_i)$	p	q
1	$\neq 0$						0	1
2		$\neq 0$					1	0
3	0	0	$\neq 0$				α	1
4	0	0		$\neq 0$			1	β
5	0	0			$\neq 0$		γ	1
6	0	0				$\neq 0$	1	δ

In the above, α is any element in o with

$$\pi_i(\alpha) = 0, \pi_j(\alpha) \in \{\pm 1\} \quad \text{and}$$
$$\pi_j(\alpha w_ix_j + y_jx_j) \neq 0;$$

γ is any element in o with

$$\pi_i(\gamma^*) = 0, \quad \pi_j(\gamma^*) \in \{\pm 1\} \quad \text{and}$$
$$\pi_j(\gamma^* y_jx_i + y_jx_j) \neq 0.$$

As for β and δ they are chosen symmetrically to α and γ respectively.

We now check that p, q satisfy $\pi_i(f) \neq 0$ and $\pi_j(g) \neq 0$. We treat Cases 1, 3, 5. In Case 1, $p = 0$ and $q = 1$, so $\pi_i(f) = \pi_i(w_jx_j) \neq 0$. Further, by (2), $\pi_j(g) = \pi_j(y_jx_j) \neq 0$. Next we treat Case 3.

Let $-$ denote π_i. Since $\overline{w_jx_j} = 0$, $\bar{p} = \bar{\alpha} = 0$ and $q = 1$, we have $\bar{f} = \overline{\alpha^* w_jx_i}$. Further $\pi_j(\alpha) \in \{\pm 1\}$ implies $\alpha \notin A_j$. Hence $\alpha^* \notin A_j^*$ (note $A_j^* = A_i$), i.e., $\overline{\alpha^*} \neq 0$. Thus we have $\bar{f} \neq 0$. Let $-$ denote π_j. Since $\pi_i(\alpha) = 0$, by the same way as above we have $\pi_j(\alpha^*) = 0$. Hence $\overline{p^*} = \overline{\alpha^*} = 0$. Further, since we have

$$q = 1 \quad \text{and} \quad \overline{\alpha w_ix_j + y_jx_j} \neq 0,$$

we have $\bar{g} \neq 0$. We consider Case 5. Let $-$ denote π_i. By $\overline{w_jx_j} = 0$, $\overline{p^*} = \overline{\gamma^*} = 0$ and $q = 1$, we have $\bar{f} = \overline{\gamma y_ix_j}$. Since $\pi_j(\gamma^*) \neq 0$, we have $\pi_i(\gamma) \neq 0$ and so $\bar{f} \neq 0$. Let $-$ denote π_j. Since $\pi_i(\gamma^*) = 0$, we have $\pi_j(\gamma) = 0$. Further, since $q = 1$, we have

$$\bar{g} = \overline{\gamma^* y_jx_i + y_jx_j}.$$

Hence $\bar{g} \neq 0$. The cases 2, 4, and 6 are symmetric to the cases 1, 3, and 5, respectively and we omit them.

LEMMA 3.7. *If $i \in I_1$, then there exists a vector u_i in V which is good for i and $d <$ dim $\overline{V_\sigma + ou_i}$ for i.*

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Proof. We write $-$ for π_i. Using Lemma 3.5, we have a good foursome (a_i, y_i, c_i, x_i) for i. If it holds that

$$d < \dim \overline{V_\sigma + ox_i},$$

the lemma holds. So we assume this is not the case, i.e.,

$$d = \dim \overline{V_\sigma + ox_i}.$$

Since $d < n$, there exists a vector z in V with

$$d < \dim \overline{V_\sigma + oz}.$$

By condition (C_3) we may write $c_i(\sigma - 1)z = a_i w$ for some w in V. Now for $p \in o$ and $q \in o - A_i$ we put

$$u_i = px_i + qz \quad \text{and} \quad v_i = py_i + qw.$$

Then (a_i, v_i, c_i, u_i) satisfies (C_1), (C_3) and

$$d < \dim \overline{V_\sigma + ou_i}.$$

To show (C_2) compute

$$v_i u_i = pp^* y_i x_i + qp^* wx_i + pq^* y_i z + qq^* wz.$$

Since $\overline{y_i x_i} \neq 0$ and 2 is a unit, we can take $\alpha \in \{\pm 1\}$ with

$$\overline{y_i x_i + \alpha wx_i + \alpha^* y_i z} \neq 0.$$

Therefore, our p, q are given by the following table.

Cases	$\overline{wz}$	p	q
1	$\neq 0$	0	1
2	0	1	α

LEMMA 3.8. *Let $i \in I$. Then there exists a vector u_i in V with $d <$ $\dim \overline{V_\sigma + ou_i}$ for both i and i^*.*

Proof. Write $j = i^*$. Since $d < n$, we can take z_i, z_j in V with

$$d < \dim \overline{V_\sigma + oz_i} \quad \text{for } i \quad \text{and}$$

$$d < \dim \overline{V_\sigma + oz_j} \quad \text{for } j.$$

Let p be any element in o with $\pi_i(p) = 1$ and $\pi_j(p) = 0$, q in o with $\pi_i(q) = 0$ and $\pi_j(q) = 1$. Then $u_i = px_i + qx_j$ is the desired vector.

Definition. Let $i \in I$. We say a vector u is *admissible for* i if u satisfies the following two conditions, where $v = (\sigma - 1)u$ and $- = \pi_i$.

(a) $\bar{V} = \overline{v^\perp + ou}$

(b) $d < \dim \overline{V_\sigma + ou}$.

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We say u is *admissible* if u is admissible for all i in I.

A key point of the proof is to find an admissible vector u in V.

Definition.

$$I_{11} = \{i \in I_1 | i^* \in I_1\}$$
$$I_{10} = \{i \in I_1 | i^* \in I_0\}$$
$$I_{01} = \{i \in I_0 | i^* \in I_1\}$$
$$I_{00} = \{i \in I_0 | i^* \in I_0\}.$$

Therefore we have $I = I_{11} + I_{10} + I_{01} + I_{00}$ (direct sum), and $I_{11}{}^* = I_{11}$, $I_{10}{}^* = I_{01}$, $I_{01}{}^* = I_{10}$, $I_{00}{}^* = I_{00}$.

Definition. * defines a classification of I in which each class consists of $\{i, i^*\}$. Let K be the set of representatives of this classification with $I_{10} \subset K$.

For each k in K, applying Lemmas 3.6, 3.7 and 3.8, we can take a vector u_k in V with the following properties (P_1), (P_2) and (P_3):

(P_1) If $k \in I_{11}$, then u_k is good for both k and k^*.

(P_2) If $k \in I_{10}$, then u_k is good for k and $d < \dim \overline{V_\sigma + ou_k}$ for k.

(P_3) If $k \in I_{00}$, then $d < \dim \overline{V_\sigma + ou_k}$ for both k and k^*.

Further, for each k in K, we take an element α_k in o with $\bar{\alpha}_k = 1$ for k and k^* and $\bar{\alpha}_k = 0$ for $k \in I - \{k, k^*\}$. Put $u = \sum_{k \in K} \alpha_k u_k$ and $v = (\sigma - 1)u$. With these notations our next task is to show that u is an admissible vector.

LEMMA 3.9. (a) *Let i be in I_0. Then it holds that*

$$d < \dim \overline{V_\sigma + ou} \quad \textit{for } i^*.$$

(b) *Let i be in I_1. Then u is good for i.*

Proof. First we prove the case (a). Let $i \in I_0$. Take k in $K \cap \{i, i^*\}$. Then we have $\bar{u} = \bar{u}_k$ for i^*. Note $I_0 = I_{00} + I_{01}$. If $i \in I_{00}$ then $i^* \in I_{00}$ and so $k \in I_{00}$. Therefore by the property (P_3) for u_k we have (a) of the lemma. If $i \in I_{01}$ then $i^* \in I_{10}$, consequently $i^* = k$, because $I_{10} \subset K$ and $I_{01} \not\subset K$. Therefore, by the property (P_2) for u_k we have also (a) of the lemma.

Next we prove the case (b). Let $i \in I_1$. Take k in $K \cap \{i, i^*\}$. We have $I_1 = I_{11} + I_{10}$. If $i \in I_{11}$ then $i^* \in I_{11}$, consequently $k \in I_{11}$. If $i \in I_{10}$ then $i = k$. Hence, in each case, by the properties (P_1) and (P_2) we see that u_k is good for i. Let (a_i, y_i, c_i, u_k) be a good foursome for i. Then by (C_1) and (C_3) we have

$$c_i(\sigma - 1)u = a_i\left(\alpha_k y_i + \sum \alpha_j w_j\right)$$

https://doi.org/10.4153/CJM-1981-092-1 Published online by Cambridge University Press

for some w_j in V where $\sum$ is the sum for j in $K - \{k\}$. By (C_2) we have $y_i u_k \notin A_i$. Hence, putting

$$w = \alpha_k y_i + \sum_j \alpha_j w_j,$$

we have $\overline{wu} = \overline{y_i u_k} \neq 0$ for i. Thus (a_i, w, c_i, u) is a good foursome for i. That is, u is good for i.

LEMMA 3.10. *If $i \in I_0$, then $\bar{V} = \overline{v^\perp}$ for i^* (here $v = (\sigma - 1)u$).*

Proof. Since $i \in I_0$, for some c in $o - A_i$ we have $c(\sigma - 1)V = \{0\}$. Hence $cv = 0$. Therefore, for all w in V we have $0 = (cv)w = v(c^*w)$, i.e., $c^*w \subset v^\perp$ and so $c^*V \subset v^\perp$. On the other hand, since $c \notin A_i$, we have $c^* \notin A_i^*$. Thus it holds that $\bar{V} = \overline{v^\perp}$ for i^*.

LEMMA 3.11. *If $i \in I_0$, then u is admissible for i^*.*

Proof. By (a) of Lemma 3.9 we have

$$d < \dim \overline{V_\sigma + ou} \quad \text{for } i^*.$$

By Lemma 3.10 we have $\bar{V} = \overline{v^\perp + ou}$ for i^*.

LEMMA 3.12. *Let $i \in I$. For y in V if $yu \notin A_i$ then we have $\bar{V} = \overline{y^\perp + ou}$ for i^*.*

Proof. We use $-$ for π_i^*. Take any z in V. Put $a = zy$ and $c = uy$. We note that $yu \neq A_i$ if and only if $uy \notin A_i^*$. Hence $\bar{c} \neq 0$. Since $cz - au \in y^\perp$, we have $cz \in y^\perp + ou$. This implies $\bar{V} = \overline{y^\perp + ou}$ for i^*.

LEMMA 3.13. *If $i \in I_1$ then u is admissible for i^*.*

Proof. We write $i^* = j$ and use $-$ for π_j. Since $i \in I_1$, by (b) of Lemma 3.9 we have a good foursome (a, y, c, u) for i. Therefore it holds that $cv = ay$ with $c \notin A_i$, $yu \notin A_i$ and $c(\sigma - 1)V \subset aV$.

First, we show $\bar{V} = \overline{v^\perp + ou}$. Since $yu \notin A_i$, by Lemma 3.12, we have $\bar{V} = \overline{y^\perp + ou}$ for j. We show $\overline{y^\perp} \subset \overline{v^\perp}$. Take any z in $y^\perp$. Then $cvz = ayz = 0$, which implies $vc^*z = 0$, i.e., $c^*z \in v^\perp$. On the other hand, by $c \notin A_i$ we have $c^* \notin A_j$, hence $\overline{y^\perp} \subset \overline{v^\perp}$. Thus, $\bar{V} = \overline{v^\perp + ou}$.

Next we show $d < \dim \overline{V_\sigma + ou}$. Suppose the inequality does not hold, i.e., $d = \dim \overline{V_\sigma + ou}$. Then $\bar{u} \in \overline{V_\sigma}$ and so we may write $u = z + s$ for some z in V_σ and s in A_jV. Since $yu \notin A_i$, we have $uy \notin A_j$. Thus $zy \notin A_j$. Hence $yz \notin A_i$. Put $b = yz$, whence $b \notin A_i$. Then

$$ab = ayz = cvz = 0,$$

because $v = (\sigma - 1)u$ and $((\sigma - 1)V)V_\sigma = \{0\}$. Thus, we have

$$bc(\sigma - 1)V \subset baV \subset \{0\}.$$

But, since $bc \notin A_i$, this would imply $i \in I_0$, a contradiction.

LEMMA 3.14. *u is an admissible vector in V.*

Proof. We show that u is admissible for all i in I. Take any i in I. Then $i^* \in I$. We have $I = I_0 + I_1$. If $i^* \in I_0$ then u is admissible for i by Lemma 3.11. If $i^* \in I_1$ then u is admissible for i by Lemma 3.13.

Now, by the lemma we have for all i in I

(1) $\bar{V} = \overline{v^{\perp} + ou}$

and

(2) $d < \dim \overline{V_\sigma + ou}$.

Further, since $v = (\sigma - 1)u$ and $V_\sigma((\sigma - 1)V) = \{0\}$, we have

(3) $V_\sigma \subset v^{\perp}$.

Here, we may assume $\bar{u} \neq 0$ without loss of generality. Because, if necessary, we can choose z in V_σ with $\overline{u + z} \neq 0$ and take $u + z$ for u above.

Thus, by (1) $\sim$ (3), we can choose a base $\{\bar{u}_{i1}, \ldots, \bar{u}_{i(n-1)}, \bar{u}\}$ for $\bar{V}$ for each i in I such that $u_{i1}, u_{i2}, \ldots, u_{id}$ are in V_σ and $u_{i(d+1)}, \ldots, u_{i(n-1)}$ are in $v^{\perp}$. Put

$$u_\mu = \sum_{i \in I} \epsilon_i u_{i_\mu}$$

for each $\mu \in \{1, \ldots, n - 1\}$, where ϵ_i has been defined before as an element in o with $\pi_j(\epsilon_i) = \delta_{ij}$ (Kronecker δ) for i, j in I. It is clear that $\{\bar{u}_1, \ldots, \bar{u}_{n-1}, \bar{u}\}$ is a base for $\bar{V}$ for each i in I.

LEMMA 3.15. *Let $u_1, \ldots, u_n$ be n vectors in V. For each i in I, if $\bar{V} = \bigoplus_{\mu=1}^{n} \bar{o}\bar{u}_\mu$, then $V = \bigoplus_{\mu=1}^{n} ou_\mu$.*

Proof. By Lemma 3.3, we know

$$V = \sum_{\mu=1}^{n} ou_\mu.$$

Hence we show the linear independence of $\{u_\mu\}$ over o. Suppose

$$a_1u_1 + \ldots + a_nu_n = 0, \quad a_\mu \in o,$$

with at least one nonzero coefficient, say a_1. We take a maximal ideal A_i which contains the annilator of a_1. Then $a_1' \neq 0$ in o_{A_i}. Since o_{A_i} is a valuation ring, we may assume a_1' divides all a_μ'. So we have

$$a_1'(u_1' + (b_2'/c_2')u_2' + \ldots) = 0, \quad b_\mu \in o, \; c_\mu \in o - A_i.$$

Hence, we have

$$a_1(e_1u_1 + e_2u_2 + \ldots) = 0, \quad e_\mu \in o, \quad e_1 \in o - A_i.$$

Since $\bar{V} = \bigoplus_{\mu=1}^{n} \bar{o}\bar{u}_\mu$ is nonsingular, we have a vector v in V with $\bar{u}_1\bar{v} = 1$ and $\bar{u}_\mu\bar{v} = 0$ for $\mu \neq 1$. Put

$$b = (e_1u_1 + e_2u_2 + \ldots)v.$$

Then $b \in o - A_i$ and $a_1b = 0$, which contradicts the choice of A_t.

By the lemma, we see that $\{u_1, \ldots, u_{n-1}, u\}$ is a base for V.
We write

$$U = \bigoplus_{\mu=1}^{n-1} u_\mu.$$

Then it holds that

(4) $V = U \oplus ou$

(5) $U \subset v^\perp$

(6) $d \leqq \dim \overline{U \cap V_\sigma}$ for all i in I.

By (4) we can define a linear map τ on V by defining $\tau = 1$ on U and $\tau u = u + v$. Since $v = (\sigma - 1)u$ and $U \subset v^\perp$ by (5), τ preserves the form on V. In fact, we shall see that τ is in $U_n(V)$ by the following lemma.

LEMMA 3.16. $V = U \oplus o\sigma u$.

Proof. We shall show that $\bar{V} = \bar{U} \oplus \bar{o}\overline{\sigma u}$ for all i in I, which will imply $V = U \oplus o\sigma x$ by Lemma 3.15. Since U is a hyperplane of V, we have $\bar{U} \subsetneqq \bar{V}$. Therefore it suffices to show that $\bar{V} = \overline{U + o\sigma u}$.

First let $i \in I_0$. Hence we have c in $o - A_i$ with $cv = 0$. We note $\overline{co} = \bar{o} = \overline{oc}$. Hence

$$\overline{U + o\sigma u} = \overline{U + o(u + v)} = \overline{U + oc(u + v)} = \overline{U + ou} = \bar{V}$$

by (4).

Next let $i \in I_1$. Then by (b) of Lemma 3.9 we have a good foursome (a, w, c, u) for i. Hence we have

$$\overline{U + o\sigma u} = \overline{U + oc(u + v)} = \overline{U + o(cu + aw)}.$$

Therefore if $\bar{a} = 0$ then the right hand side equals $\overline{U + ou} = \bar{V}$ and we have the lemma. So we treat the case $\bar{a} \neq 0$. In this case, we suppose $\overline{U + o\sigma u} \subsetneqq \bar{V}$ which will imply a contradiction. Since $\bar{U}$ is a hyperplane of $\bar{V}$, our assumption means $\overline{\sigma u} \in \bar{U}$. Since $U \subset v^\perp$, we have $\overline{(\sigma u)v} = 0$. Therefore

$$0 = \overline{(\sigma u)v} = \overline{\sigma u(\sigma u - u)} = \overline{(u - \sigma u)u} = -\overline{vu}.$$

Hence $0 = \overline{cvu} = \overline{awu} = \bar{a}\overline{wu}$, a contradiction, because we have $\overline{wu} \neq 0$.

By the lemma τ is an automorphism on V and so τ is contained in $U_n(V)$. Write $D = ou_1 + \ldots + ou_d$. Then $D \subset V_\sigma$. Since $D \subset U$, we have $D \subset V_\tau$. Therefore $D \subset V_{\tau^{-1}\sigma}$. Further, since $\tau u = \sigma u$, we have $\tau^{-1}\sigma u = u$. From these two, now we have

$$D \oplus ou \subset V_{\tau^{-1}\sigma}.$$

Finally, since $D \oplus ou$ is a subspace of V with

$$\dim (D \oplus ou) = d + 1,$$

we have

$$d + 1 \leqq \dim \overline{V_{\tau^{-1}\sigma}} \quad \text{for all } i \text{ in } I.$$

Thus we have completed the proof of the theorem.

References

1. C. Chang, *Unitary groups over semilocal domain*, J. Algebra *39* (1976), 160–173.
2. J. Dieudonné, *Sur les groupes classiques*, Actual. Scient. et ind., n *1040* (Hermann, Paris, 1948).
3. ——— *Sur les générateurs des groupes classiques*, Summa Brasil. Math. *3* (1955), 149–179.
4. E. W. Ellers, *Decomposition of orthgonal symplectic, and unitary isometries into simple isometries*, Abh. Math. Sem. Univ. Hamburg *46* (1977), 97–127.
5. H. Ishibashi, *Generators of $O_n(V)$ over a quasi semilocal semhereditary domain*, Comm. in Algebra *7* (1979), 1043–1064.
6. ——— *Generators of $Sp_n(V)$ over a quasi semilocal semihereditary domain*, Comm. in Algebra *6* (1979), 1673–1683.
7. ——— *Generators of $U_n(V)$ over a quasi semilocal semihereditary domain*, J. Algebra *60* (1979), 199–203.
8. D. G. James, *Unitary groups over local rings*, J. Algebra *52* (1978), 354–363.

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