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ON A CONJECTURE OF ERDÖS, FABER, AND LOVÁSZ ABOUT n-COLORINGS

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1. Introduction. Let $\mathscr{A}$ be a finite family of finite sets with the property that $|A \cap B| \leqq 1$ whenever A and B are distinct members of $\mathscr{A}$ and let $n = |\cup \mathscr{A}|$. It is a conjecture of Erdös, Faber, and Lovász ([**1**] a 50 pound problem and [**2**] a 100 dollar problem) that there is an n-coloring of $\mathscr{A}$ (i.e., a function $f:\mathscr{A} \rightarrow \{0, 1, 2, \ldots, n-1\}$ such that $A \cap B = \emptyset$ whenever A and B are distinct members of $\mathscr{A}$ with $f(A) = f(B)$). They actually state the conjecture in a different form. Namely, if n is a positive integer and $\mathscr{B} = \{B_p: 1 \leqq p \leqq n\}$ is a family of n sets satisfying (1) $|B_p| = n$ for each p and (2) $|B_p \cap B_q| \leqq 1$ when $p \neq q$, then there is an n-coloring of the elements of $\cup \mathscr{B}$ so that each set B_p gets all n colors. The equivalence of the two forms is easily seen by interchanging the roles of elements and sets.

Thus, given $\mathscr{B}$, we may let $A_k = \{p: k \in B_p\}$ for each $k \in \cup \mathscr{B}$. Then, letting $\mathscr{A} = \{A_k: k \in \cup \mathscr{B}\}$, we have $|\cup \mathscr{A}| = n$ and $|A_i \cap A_k| \leqq 1$ when $i \neq k$. Now let f be an n-coloring of $\mathscr{A}$. Given $k \in \cup \mathscr{B}$, if $|A_k| \geqq 2$, let $g(k) = f(A_k)$. Then g colors all elements of $\cup \mathscr{B}$ except those which lie in only one member of $\mathscr{B}$. Then g may be easily extended to the rest of $\cup \mathscr{B}$ so that each member of $\mathscr{B}$ gets all n colors.

That the original form of the conjecture implies our form is established in a similar fashion. We shall be interested throughout in our formulation and a particular strengthening of it.

It will be helpful to view the problem as a board game on an $n \times n$ chessboard. Given a family $\mathscr{A}$ with $|\cup \mathscr{A}| = n$ and $|A \cap B| \leqq 1$ whenever $A, B \in \mathscr{A}$ with $A \neq B$, (we shall call such a family $\mathscr{A}$ a *small intersection family*), label the columns of the board by the elements of $\cup \mathscr{A}$ and assign names to the members of $\mathscr{A}$. The object is to assign each set in $\mathscr{A}$ to a row of the chessboard, writing the name of that set in each column which is labeled with one of its elements. No two names may occupy the same square. For example let $A = \{1, 2, 3\}$, $B = \{2, 4, 5\}$, $C = \{1, 4\}$, $D = \{1, 5\}$, $E = \{3, 4\}$, and $F = \{3, 5\}$ and let $\mathscr{A} = \{A, B, C, D, E, F\}$. The following diagram illustrates a successful completion of the game.

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1	2	3	4	5
A	A	A		
	B		B	B
C			C	
D		E	E	D
		F		F

This solution corresponds to the coloring f of $\mathscr{A}$ where $f(A) = 0$, $f(B) = 1, f(C) = 2, f(D) = f(E) = 3$, and $f(F) = 4$.

We note that we can (and will) confine our attention to small intersection families $\mathscr{A}$ without singletons (since coloring the singletons is a trivial matter once all larger members of $\mathscr{A}$ have been assigned; there must necessarily be at least one gap in each column).

There are two possible extremes for a small intersection family with $\cup\mathscr{A} = \{1, 2, \ldots, n\}$. One is that $\mathscr{A} = \{\{1, 2, \ldots, n\}\}$ and the other is that $\mathscr{A} = [\{1, 2, \ldots, n\}]^2$, the set of two element subsets of $\{1, 2, \ldots, n\}$. The first extreme is completely trivial and the second extreme can be colored by letting

$$f(\{i, j\}) \equiv (i + j) \bmod n.$$

The difficulty in the problem obviously lies somewhere between the two extremes when $\mathscr{A}$ includes several "large" sets, that is sets with at least 3 elements. Given a small intersection family $\mathscr{A}$, we write

$$\mathscr{L}(\mathscr{A}) = \{A \in \mathscr{A}: |A| \geqq 3\}.$$

Our principal result is that, given a positive integer k, a finite computation suffices to determine the truth of the Erdös-Faber-Lovász conjecture for all small intersection families $\mathscr{A}$ (with $n = |\cup\mathscr{A}|$ arbitrarily large) such that $|\cup\mathscr{L}(\mathscr{A})| \leqq k$. We present this result in Section 2.

Since, at this writing, it is conceivable that the reason the Erdös-Faber-Lovász conjectures has eluded proof is that it is false, one is interested in more than the finiteness of the computation. That is, in a search for a counterexample one is interested also in feasibility. The computation we describe in Section 2 is not really feasible for $k > 8$.

In Section 3 we describe a computation which we have used to verify the Erdös-Faber-Lovász conjecture for all small intersection families $\mathscr{A}$ with $|\cup\mathscr{L}(\mathscr{A})| \leqq 10$. (According to Erdös in conversation, this result is new.) The procedure described in Section 3 is logically weaker however, because it depends on the verification of a property which does not always hold.

2. Reduction to a finite computation. The board game described in the introduction is relatively easy if $\mathscr{A}$ includes a high proportion of large sets, for then fewer total squares are needed. In attempting to construct a counterexample, one is tempted to begin with a small intersecting family $\mathscr{A}$ with sufficiently many large sets to disrupt the modular coloring of the pairs, and then add additional elements to $\cup \mathscr{A}$ and add all available pairs to $\mathscr{A}$ so as to force occupancy of a large proportion of the squares. We shall see here that if $|\cup \mathscr{A}| = |\cup \mathscr{L}(\mathscr{A})| = n$ and this approach has not worked by the addition of $n + 1$ or fewer elements, it will not work at all. We (recall and) introduce some notation.

2.1 *Definition.* $\mathscr{A}$ is a *small intersection family* if and only if $|\cup \mathscr{A}|$ is finite, $|A \cap B| \leqq 1$ whenever A and B are distinct members of $\mathscr{A}$, and each member of $\mathscr{A}$ has at least two elements.

2.2 *Definition.* Let k and n be positive integers with $k \leqq n$ and let $\mathscr{A}$ be a small intersection family with $\cup \mathscr{A} = \{1, 2, \ldots, k\}$.

(a) $\mathscr{L}(\mathscr{A}) = \{A \in \mathscr{A} : |A| \geqq 3\}$

(b) $\mathscr{C}(\mathscr{A}, n) = \mathscr{A} \cup \{\{i, j\} : 1 \leqq i < j \leqq n$ and $\{i, j\}$ is contained in no member of $\mathscr{A}\}$

(c) $\mathscr{A}$ *can be colored* if and only if there is a function $f: \mathscr{A} \to \{0, 1, 2, \ldots, k - 1\}$ such that $A \cap B = \emptyset$ whenever A and B are distinct members of $\mathscr{A}$ with $f(A) = f(B)$. The function f is called a *coloring* of $\mathscr{A}$.

The family $\mathscr{C}(\mathscr{A}, n)$ is a "completed" intersection family. Note that $\mathscr{C}(\mathscr{A}, n)$ is a small intersection family and that $\mathscr{C}(\mathscr{L}(\mathscr{A}), n) = \mathscr{C}(\mathscr{A}, n)$.

2.3 *Definition.* Let $\mathscr{A}$ and $\mathscr{B}$ be small intersection families with $|\cup \mathscr{A}| = |\cup \mathscr{B}|$. $\mathscr{A}$ and $\mathscr{B}$ are *isomorphic* if and only if there is a one to one function $f: \cup \mathscr{A} \to \cup \mathscr{B}$ such that $\mathscr{B} = \{f[A] : A \in \mathscr{A}\}$.

2.4 LEMMA. *Let k and n be positive integers such that $k \leqq n$ and n is odd. Let $\mathscr{A}$ be a small intersection family with $\cup \mathscr{A} = \{1, 2, \ldots, k\}$. If $\mathscr{C}(\mathscr{A}, n)$ can be colored, then each of $\mathscr{C}(\mathscr{A}, 2n)$, $\mathscr{C}(\mathscr{A}, 2n + 1)$, $\mathscr{C}(\mathscr{A}, 2n + 2)$, and $\mathscr{C}(\mathscr{A}, 2n + 3)$ can be colored.*

Proof. Let f be a coloring of $\mathscr{C}(\mathscr{A}, n)$ (so that $f: \mathscr{C}(\mathscr{A}, n) \to \{0, 1, 2, \ldots, n - 1\}$) and let $t \in \{0, 1, 2, 3\}$. We define a coloring g of $\mathscr{C}(\mathscr{A}, 2n + t)$ as follows. Note that

$$\mathscr{C}(\mathscr{A}, 2n + t) = \mathscr{C}(\mathscr{A}, n) \cup \{\{i, j\} : 1 \leqq i \leqq n < j \leqq 2n + t\} \cup \{\{i, j\} : n + 1 \leqq i < j \leqq 2n + t\}.$$

For $A \in \mathscr{C}(\mathscr{A}, n)$, we let $g(A) = f(A)$. For $1 \leqq i \leqq n < j \leqq 2n + t$ we let $g(\{i, j\})$ be that member of $\{n, n + 1, n + 2, \ldots, 2n + t - 1\}$ which is congruent to $(i + j)$ modulo $(n + t)$. For $n + 1 \leqq i < j \leqq 2n + t$ we split into cases depending on the value of t.

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Case 1. $t = 0$. Let $g(\{i, j\})$ be that member of $\{0, 1, 2, \ldots, n-1\}$ which is congruent to $(i + j)$ modulo n.

Case 2. $t = 1$. Let $g(\{i, j\}) \in \{0, 1, 2, \ldots, n-1\}$. If $j \leqq 2n$, let

$$g(\{i, j\}) \equiv (i + j) \bmod n.$$

If $j = 2n + 1$, let

$$g(\{i, j\}) \equiv 2i \bmod n.$$

Case 3. $t = 2$. If $j = i + 1$, let $g(\{i, j\}) = i$. Let

$$g(\{n + 1, 2n + 2\}) = n.$$

For all remaining possibilities we will have $g(\{i, j\}) \in \{0, 1, 2, \ldots, n-1\}$. If $n + 1 \leqq i < i + 1 < j \leqq 2n$, let

$$g(\{i, j\}) \equiv (i + j) \bmod n.$$

If $n + 2 \leqq i \leqq 2n$, let

$$g(\{i, 2n + 2\}) \equiv 2i \bmod n.$$

If $n + 1 \leqq i \leqq 2n - 1$, let

$$g(\{i, 2n + 1\}) \equiv (2i + 1) \bmod n.$$

Case 4. $t = 3$. If $j = i + 1$, let $g(\{i, j\}) = i$. Let

$$g(\{n + 1, 2n + 3\}) = n.$$

For all remaining possibilities we will have $g(\{i, j\}) \in \{0, 1, 2, \ldots, n-1\}$. If $n + 1 \leqq i < i + 1 < j \leqq 2n$, let

$$g(\{i, j\}) \equiv (i + j) \bmod n.$$

If $n + 1 \leqq i \leqq 2n$, let

$$g(\{i, 2n + 2\}) \equiv 2i \bmod n.$$

If $n + 1 \leqq i \leqq 2n - 1$, let

$$g(\{i, 2n + 1\}) \equiv (2i + 1) \bmod n.$$

If $n + 2 \leqq i \leqq 2n + 1$, let

$$g(\{i, 2n + 3\}) \equiv (2i - 1) \bmod n.$$

Note that in each case we have defined

$$g: \mathscr{C}(\mathscr{A}, 2n + t) \rightarrow \{0, 1, 2, \ldots, 2n + t - 1\}.$$

We shall verify that g is a coloring only for the case $t = 3$, the other cases being similar.

Let A and B be distinct members of $\mathscr{C}(\mathscr{A}, 2n + 3)$ and assume $g(A) = g(B)$. If $A, B \in \mathscr{C}(\mathscr{A}, n)$, then $f(A) = f(B)$ so $A \cap B = \emptyset$.

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We thus may assume $A = \{i, j\}$ with $i < j$, $1 \leqq i \leqq 2n + 2$ and $n + 1 \leqq j \leqq 2n + 3$. We can't have both $B \in \mathscr{C}(\mathscr{A}, n)$ and $i \leqq n$ for then $g(B) < n \leqq g(A)$. Further if $B \in \mathscr{C}(\mathscr{A}, n)$ and $n < i$, then $A \cap B = \emptyset$ as desired. We thus may assume $B = \{l, m\}$ with $l < m$, $1 \leqq l \leqq 2n + 2$ and $n + 1 \leqq m \leqq 2n + 3$. Using the fact that $g(A) = g(B)$ (so that for example one can't have $g(A) < n$ and $g(B) \geqq n$) there are a total of eleven cases to consider depending on how $g(A)$ and $g(B)$ were defined. These cases involve however only five different arguments, so we will present one example of each argument, leaving the rest to the reader.

Case 1. $i \leqq n$ and $l \leqq n$. Then $(i + j) \equiv (l + m) \bmod (n + 3)$. If $A \cap B \neq \emptyset$, then either $i = l$ or $j = m$. If $i = l$, then $j \equiv m \bmod (n + 3)$ and since $j, m \in \{n + 1, n + 2, \ldots, 2n + 3\}$ we conclude that $j = m$ and hence $A = B$. Similarly if $j = m$, then $i = l$.

Case 2. $i \leqq n$ and $m = l + 1 > n + 1$. Then $(i + j) \equiv l \bmod (n + 3)$. If $A \cap B = \emptyset$, then either $j = m$ or $j = l$. If $j = l$, then $i \equiv 0 \bmod (n + 3)$. If $j = m$, then $(i + 1) \equiv 0 \bmod (n + 3)$. But $1 \leqq i \leqq n$ so either conclusion is impossible.

All remaining cases which we will consider involve both $n + 1 \leqq i$ and $n + 1 \leqq l$.

Case 3. $j = m = 2n + 2$ and $i, l \leqq 2n$. Then $2i \equiv 2l \bmod n$ and hence, since n is odd, $i \equiv l \bmod n$. Since $i, l \in \{n + 1, n + 2, \ldots, 2n\}$ we have $i = l$ and hence $A = B$.

Case 4. $n + 1 \leqq i < i + 1 < j \leqq 2n$, $m = 2n + 3$, and $n + 2 \leqq l \leqq 2n + 1$. Then $(i + j) \equiv (2l - 1) \bmod n$. Suppose $A \cap B \neq \emptyset$. Then $i = l$ or $j = l$. If $i = l$, then $j \equiv (i - 1) \bmod n$ and hence $j = 2n$ and $i = n + 1$. But this is impossible since $i = l \geqq n + 2$. If $j = l$, then $i \equiv (j - 1) \bmod n$ and hence $j = i + 1$, a contradiction.

Case 5. $j = 2n + 2$, $n + 1 \leqq i \leqq 2n$, $m = 2n + 3$, and $n + 2 < l \leqq 2n + 1$. Then $2i \equiv (2l - 1) \bmod n$. If $A \cap B \neq \emptyset$, then $i = l$ and hence $0 \equiv -1 \bmod n$, a contradiction.

2.5 THEOREM. *Let n be a positive integer and let $\mathscr{A}$ be a small intersection family with $\cup \mathscr{A} = \{1, 2, \ldots, n\}$. If (1) n is odd and $\mathscr{C}(\mathscr{A}, m)$ can be colored for $n \leqq m \leqq 2n - 1$ or (2) n is even and $\mathscr{C}(\mathscr{A}, m)$ can be colored for $n \leqq m \leqq 2n + 1$, then $\mathscr{C}(\mathscr{A}, m)$ can be colored for all $m \geqq n$.*

Proof. Suppose not and pick the least such m. Then $m = 2r + t$ where $t \in \{0, 1, 2, 3\}$ and r is an odd number such that $\mathscr{C}(\mathscr{A}, r)$ can be colored, contradicting Lemma 2.4.

2.6 COROLLARY. *Let n be a positive integer. A finite computation suffices to determine the truth (or falsity) of the statement "whenever $\mathscr{A}$ is a small intersection family with $|\cup \mathscr{L}(\mathscr{A})| \leqq n$, $\mathscr{A}$ can be colored".*

Proof. We shall first describe the computation (in gross terms). There are finitely many small intersection families $\mathscr{A}$ with $\cup \mathscr{A} \subseteq \{1, 2, \ldots, n\}$

and a finite computation suffices to find them all (although it will do to find one representative of each isomorphism class). Given such a small intersection family $\mathscr{A}$, a finite computation suffices to determine if each of $\mathscr{C}(\mathscr{A}, n), \mathscr{C}(\mathscr{A}, n+1), \ldots, \mathscr{C}(\mathscr{A}, 2n+1)$ can be colored.

If in the process described above, a small intersection family $\mathscr{A}$ and an integer m are found with $\bigcup \mathscr{A} \subseteq \{1, 2, \ldots, n\}$ and $n \leqq m \leqq 2n+1$ such that $\mathscr{C}(\mathscr{A}, m)$ cannot be colored, then the quoted statement is false. (One has $\mathscr{L}(\mathscr{C}(\mathscr{A}, m)) \subseteq \mathscr{A}$.)

Assume then that all such $\mathscr{C}(\mathscr{A}, m)$ can be colored and let $\mathscr{B}$ be a small intersection family with $|\bigcup \mathscr{L}(\mathscr{B})| \leqq n$. Let $k = |\bigcup \mathscr{B}|$ and let

$$f: \bigcup \mathscr{B} \xrightarrow{1-1} \{1, 2, \ldots, k\}$$

such that $f[\bigcup \mathscr{L}(\mathscr{B})] \subseteq \{1, 2, \ldots, n\}$. Let

$$\mathscr{A} = \{f[A] : A \in \mathscr{L}(\mathscr{B})\}.$$

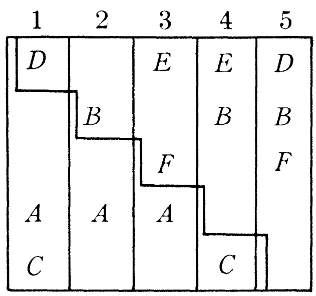
Since each of $\mathscr{C}(\mathscr{A}, n), \mathscr{C}(\mathscr{A}, n+1), \ldots, \mathscr{C}(\mathscr{A}, 2n+1)$ can be colored, we have by Theorem 2.5 that $\mathscr{C}(\mathscr{A}, k)$ can be colored. Since

$$\mathscr{B} \subseteq \{f^{-1}[A] : A \in \mathscr{C}(\mathscr{A}, k)\},$$

we have that $\mathscr{B}$ can be colored.

3. Split colorings. The procedure which we described in the proof of Corollary 2.6, while logically interesting, is computationally unrealistic. In this section we describe a procedure which has two big advantages and one big disadvantage. The advantages are that it significantly reduces the time needed to find a coloring and that it eliminates the need to check for colorings of several extensions. (The latter statement is made precise in Theorem 3.2.) The disadvantage is that it doesn't always work.

Consider again the board game described in the introduction. Now consider the following more difficult game. A diagonal fence is drawn across the board and the requirement is added that no set can cross the fence. Again letting $A = \{1, 2, 3\}$, $B = \{2, 4, 5\}$, $C = \{1, 4\}$, $D = \{1, 5\}$, $E = \{3, 4\}$, $F = \{3, 5\}$ and $\mathscr{A} = \{A, B, C, D, E, F\}$ a successful completion of the game is diagrammed below.



The first advantage which we mentioned can be seen here by noting that, if the sets are assigned in the order D, B, F, C, A, E, then no free choices are involved. Consequently coloring time is significantly reduced. (The exact reduction depends of course on the cleverness of the algorithms involved.) We call such a coloring a split coloring.

3.1 *Definition.* Let n be a positive integer and let $\mathscr{A}$ be a small intersection family with $\cup \mathscr{A} = \{1, 2, \ldots, n\}$. A *split coloring* of $\mathscr{A}$ is a coloring f of $\mathscr{A}$ such that, whenever $A \in \mathscr{A}$, either $\min A > f(A)$ or $\max A \leqq f(A)$.

3.2 THEOREM. *Let n be a positive integer and let $\mathscr{A}$ be a small intersection family with $\cup \mathscr{A} = \{1, 2, \ldots, n\}$. If $\mathscr{C}(\mathscr{A}, n)$ has a split coloring so does $\mathscr{C}(\mathscr{A}, n + 1)$.*

Proof. Let f be a split coloring of $\mathscr{C}(\mathscr{A},n)$. Define a split coloring g of $\mathscr{C}(\mathscr{A}, n + 1)$ as follows. First let $A \in \mathscr{C}(\mathscr{A}, n)$. If $\max A \leqq f(A)$, let $g(A) = f(A)$. If $f(A) = 0$, let $g(A) = n$. If $\min A > f(A) \geqq 1$, let $g(A) = f(A) - 1$. For $i \in \{1, 2, \ldots, n\}$, let $g(\{i, n + 1\}) = i - 1$.

3.3 COROLLARY. *Let $\mathscr{A}$ be a small intersection family. If some isomorphic copy of $\mathscr{C}(\mathscr{L}(\mathscr{A}))$ has a split coloring, then $\mathscr{A}$ can be colored.*

We have verified (by computer) that some isomorphic copy of each small intersection family $\mathscr{A}$ with $\cup \mathscr{A} \subseteq \{1, 2, \ldots, 10\}$ has a split coloring. Consequently, if $\mathscr{A}$ is a small intersection family with $|\cup \mathscr{L}(\mathscr{A})| \leqq 10$, then $\mathscr{A}$ can be colored.

The smallest small intersection family which we know of with the property that no isomorphic copy has a split coloring is the following.

$$\begin{aligned}\mathscr{A} = \{&\{1, 2, 3, 4, 5, 6\}, \{1, 7, 8, 9, 10, 11\}, \{1, 12, 13, 14, 15, 16\},\\ &\{1, 17, 18, 19, 20, 21\}, \{1, 22, 23, 24, 25, 26\}, \{1, 27, 28, 29, 30, 31\},\\ &\{2, 7, 12, 17, 22, 27\}, \{2, 8, 13, 18, 23, 28\}, \{2, 9, 14, 19, 24, 29\},\\ &\{2, 10, 15, 20, 25, 30\}, \{2, 11, 16, 21, 26, 31\}, \{3, 7, 13, 19, 25, 31\},\\ &\{3, 8, 14, 20, 26, 27\}, \{3, 9, 15, 21, 22, 28\}, \{3, 10, 16, 17, 23, 29\},\\ &\{3, 11, 12, 18, 24, 30\}, \{4, 7, 14, 21, 23, 30\}, \{4, 8, 15, 17, 24, 31\},\\ &\{4, 9, 16, 18, 25, 27\}, \{4, 10, 12, 19, 26, 28\}, \{4, 11, 13, 20, 22, 29\},\\ &\{5, 7, 15, 18, 26, 29\}, \{5, 8, 16, 19, 22, 30\}, \{5, 9, 12, 20, 23, 31\},\\ &\{5, 10, 13, 21, 24, 27\}, \{5, 11, 14, 17, 25, 28\}, \{6, 7, 16, 20, 24, 28\},\\ &\{6, 8, 12, 21, 25, 29\}, \{6, 9, 13, 17, 26, 30\}, \{6, 10, 14, 18, 22, 31\},\\ &\{6, 11, 15, 19, 23, 27\}\}.\end{aligned}$$

(The verification that no isomorphic copy of $\mathscr{A}$ has a split coloring was accomplished with the aid of a computer.) We have (without computer assistance) used Theorem 2.5 to show that $\mathscr{C}(\mathscr{A}, n)$ for this particular $\mathscr{A}$ can nevertheless be colored for all $n \geqq 31$.

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