BULL. AUSTRAL. MATH. SOC. 90C25, 90C30
VOL. 25 (1982), 71-80.

ON CONVERSE DUALITY FOR A NONDIFFERENTIABLE PROGRAM

T.R. GULATI

A nonlinear nondifferentiable program with linear constraints is considered and a converse duality theorem is discussed. First we weaken an assumption previously made by Bhatia, and then give a simple proof under this weaker hypothesis, using the Fritz John conditions. Finally, defining a generalized Slater constraint qualification which implies Abadie's constraint qualification, we give a simple condition for the dual problem to satisfy this constraint qualification.

1. Introduction

Consider the pair of problems:

PRIMAL PROBLEM: Maximize $f(x) = p'x - \sum_{i=1}^{t} \left(x'D^{i}x\right)^{\frac{1}{2}}$

Subject to $Ax \le b$,
$x \ge 0$;

and

Received 5 August 1981. The author is thankful to Dr B.D. Craven for helpful discussions.

DUAL PROBLEM:

$$\text{Minimize} \qquad h(y) = b'y$$

$$\text{Subject to} \quad A'y + \sum_{i=1}^{t} D^i w_i \geq p ,$$

$$w_i' D^i w_i \leq 1 , \quad i = 1, 2, \ldots, t ,$$

$$y \geq 0 ,$$

where A is an $m \times n$ matrix, b is an m-dimensional vector, p is an n-dimensional vector and D^i $(i = 1, 2, \ldots, t)$ are $n \times n$ symmetric positive semidefinite matrices. These problems were first considered by Sinha [13], who proved these results:

THEOREM 1 (Weak Duality Theorem). $\operatorname{Sup} f(x) \leq \operatorname{Inf} h(y)$.

THEOREM 2 (Direct Duality Theorem). *Assume that the constraint set of the primal problem is bounded. If* $\bar{x}$ *is an optimal solution of the primal problem, then there exists an optimal solution* $(\bar{y}, \bar{w}_i)$, $i = 1, 2, \ldots, t$, *of the dual problem and the two optimal values are equal.*

THEOREM 3 (Converse Duality Theorem). *Assume that the constraint set of the primal problem is bounded. If* $(\bar{y}, \bar{w}_i)$, $i = 1, 2, \ldots, t$, *is an optimal solution of the dual problem, then there exists a vector* $\tilde{x}$, *which is optimal for the primal problem and the two optimal values are equal.*

To prove Theorem 3, Sinha used Eisenberg's duality in homogeneous programming [5]. Bhatia [3] proved Theorem 2 without the boundedness restriction on the primal constraint set. She also observed that if this assumption is removed, Sinha's proof of Theorem 3 is still valid under the less restrictive Eisenberg's hypothesis, namely

$$(1) \qquad Ax \leq 0 , \quad x \geq 0 , \quad f(x) \geq 0 \Rightarrow x = 0 .$$

Mond [9] studied duality for a complex version (with $t = 1$) of the above problems. He proved a converse duality theorem assuming the Kuhn-Tucker constraint qualification for the dual problem.

This paper is divided into four sections. In the second section we prove a converse duality theorem under an assumption weaker than (1). This proof depends on Sinha's proof of Theorem 3 and on some of the results of Bhatia [3]. A simpler proof, using the well-known Fritz John necessary optimality conditions, is then given in Section 3. In the last section, defining a generalized Slater constraint qualification which implies Abadie's constraint qualification, we give a simple condition for the dual problem to satisfy this constraint qualification.

For notations and definitions of convex-like functions we refer to Mangasarian [8].

2. Converse duality theorem

We shall need the following lemmas:

LEMMA 1 [3]. *Let* $D \in R^{n\times n}$ *be a positive semidefinite matrix. Then*

$$[(x+\bar{x})'D(x+\bar{x})]^{\frac{1}{2}} \le (x'Dx)^{\frac{1}{2}} + (\bar{x}'D\bar{x})^{\frac{1}{2}} .$$

LEMMA 2. *If the dual problem is feasible and* $h(y)$ *is bounded below, then*

(a) the primal problem is feasible, and

(b) the set $S = \{x \mid Ax \le 0,\ x \ge 0,\ f(x) > 0\}$ *is empty.*

Proof. *(a)* The proof is given in [3].

(b) From *(a)* the primal problem is feasible. Suppose $\bar{x} \in S$. Then for a feasible solution x of the primal problem and any nonnegative number λ , $x + \lambda\bar{x}$ is feasible for the primal problem. Also, using Lemma 1,

$$\begin{aligned} f(x+\lambda\bar{x}) &= p'(x+\lambda\bar{x}) - \sum_{i=1}^{t} \left[(x+\lambda\bar{x})'D^{i}(x+\lambda\bar{x})\right]^{\frac{1}{2}} \\ &\ge p'(x+\lambda\bar{x}) - \sum_{i=1}^{t} \left[(x'D^{i}x)^{\frac{1}{2}}+\lambda\left(\bar{x}'D^{i}\bar{x}\right)^{\frac{1}{2}}\right] \\ &= f(x) + \lambda f(\bar{x}) . \end{aligned}$$

Since $f(\bar{x}) > 0$, the above inequality implies that $f(x+\lambda\bar{x}) \to \infty$ as $\lambda \to \infty$. Therefore, from Theorem 1, the dual problem is infeasible. This contradicts the hypothesis. Hence the set S is empty.

THEOREM 4. *Assume that*

(2) $$Ax \leq 0 \ , \quad x \geq 0 \ , \quad f(x) = 0 \Rightarrow x = 0 \ .$$

If $(\bar{y}, \bar{w}_i)$, $i = 1, 2, \ldots, t$, *is an optimal solution of the dual problem, then there exists a vector* $\tilde{x}$, *which is optimal for the primal problem and the two optimal values are equal.*

Proof. Since the dual problem has an optimal solution, by Lemma 2, the set S is empty. This, with (2), implies condition (1). The proof then follows from Bhatia [3] and Sinha [13].

3. A simple proof using Fritz John conditions

The above proof of Theorem 4 depends on some of the results of Bhatia [3] and on Sinha's proof of Theorem 3. This makes the whole proof lengthy and complicated. We now give a simpler proof using the well-known Fritz John necessary optimality conditions. In fact, observations of the last section are outcomes of this section. Mond [10] has also used the Fritz John conditions to prove a converse duality theorem for a more general class of problems but his hypothesis is not satisfied by our problems. See [4], for a discussion of the advantages of using the Fritz John conditions rather than the Kuhn-Tucker conditions to prove converse duality.

We first state the following lemma:

LEMMA 3 [6], [10]. *Let* $D \in R^{n \times n}$ *be a symmetric positive semi-definite matrix. Then*

(3) $$x'Dw \leq (x'Dx)^{\frac{1}{2}}(w'Dw)^{\frac{1}{2}} \ .$$

Equality holds if, for some $\lambda \geq 0$, $Dx = \lambda Dw$.

An Alternative Proof of Theorem 4. Since $(\bar{y}, \bar{w}_i)$, $i = 1, 2, \ldots, t$, is an optimal solution of the dual problem, by the Fritz John Theorem (Theorem 7.3.2 in [8]), there exist $\bar{r} \in R$, $\bar{x} \in R^n$, $\bar{u} \in R^m$, $\bar{v}_i \in R$, $i = 1, 2, \ldots, t$, satisfying

https://doi.org/10.1017/S0004972700005049 Published online by Cambridge University Press

(4) $$-A\bar{x} + \bar{r}b = \bar{u} \geq 0 ,$$

(5) $$D^i\bar{x} - \bar{v}_i D^i \bar{w}_i = 0 , \quad i = 1, 2, \ldots, t ,$$

(6) $$\left(p - A'\bar{y} - \sum_{i=1}^{t} D^i \bar{w}_i\right)' \bar{x} = 0 ,$$

(7) $$\left(\bar{w}_i' D^i \bar{w}_i - 1\right) \bar{v}_i = 0 , \quad i = 1, 2, \ldots, t ,$$

(8) $$\bar{y}'\bar{u} = 0 ,$$

(9) $$\left(\bar{r}, \bar{x}, \bar{u}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_t\right) \geq 0 ,$$

(10) $$\left(\bar{r}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_t\right) \neq 0 .$$

We now show that $\bar{r} > 0$ and $\tilde{x} = \bar{x}/\bar{r}$ is feasible for the primal problem. If possible, let $\bar{r} = 0$. Then, from (4),

(11) $$A\bar{x} = -\bar{u} \leq 0 .$$

Also, from (4) and (8),

(12) $$\bar{y}'A\bar{x} = -\bar{y}'\bar{u} = 0 .$$

Since (5) is the condition of equality in Lemma 3,

(13) $$\bar{x}'D^i\bar{w}_i = \left(\bar{x}'D^i\bar{x}\right)^{\frac{1}{2}} \left(\bar{w}_i' D^i \bar{w}_i\right)^{\frac{1}{2}} , \quad i = 1, 2, \ldots, t .$$

From (7), for each i, either $\bar{v}_i = 0$ or $\bar{w}_i' D^i \bar{w}_i = 1$. In either case, from (5) and (13) we get

(14) $$\bar{x}'D^i\bar{w}_i = \left(\bar{x}'D^i\bar{x}\right)^{\frac{1}{2}} , \quad i = 1, 2, \ldots, t .$$

Now (14), (6) and (12) give

$$\begin{aligned} f(\bar{x}) &= p'\bar{x} - \sum_{i=1}^{t} \left(\bar{x}'D^i\bar{x}\right)^{\frac{1}{2}} \\ &= p'\bar{x} - \sum_{i=1}^{t} \left(\bar{x}'D^i\bar{w}_i\right) \\ &= \bar{y}'A\bar{x} = 0 . \end{aligned}$$

Thus we have $A\bar{x} \leq 0$, $\bar{x} \geq 0$ and $f(\bar{x}) = 0$. Therefore, from

https://doi.org/10.1017/S0004972700005049 Published online by Cambridge University Press

assumption (2), $\bar{x} = 0$. This, with (5) and (7), implies that $\bar{v}_i = 0$ for $i = 1, 2, \ldots, t$, contradicting (10). Hence $\bar{r} > 0$, and from (4) and $\bar{x} \geq 0$ we obtain that $\tilde{x} = \bar{x}/\bar{r}$ is feasible for the primal problem. Also, as above

$$\begin{aligned} f(\tilde{x}) &= p'\tilde{x} - \sum_{i=1}^{t} \left(\tilde{x}'D^i\tilde{x}\right)^{\frac{1}{2}} \\ &= p'\tilde{x} - \sum_{i=1}^{t} \left\{\tilde{x}'D^i\bar{w}_i\right\} \\ &= \bar{y}'A\tilde{x} \\ &= \bar{y}'b - \bar{y}'\bar{u}/\bar{r} \\ &= b'\bar{y} = h(\bar{y}) . \end{aligned}$$

Hence, from Theorem 1, $\tilde{x}$ is optimal for the primal problem.

REMARK. Note that the above proof gives also the relations $D^i\tilde{x} = \lambda_i D^i\bar{w}_i$, where $\lambda_i = \bar{v}_i/\bar{r}$, between the optimal solutions of the primal and dual problems. Thus if, for some i, D^i has an inverse, for example, if D^i is positive definite, then $\tilde{x} = \lambda_i\bar{w}_i$. This fact was also pointed out by Mond [9]. Sinha's proof does not provide these relations. However, he obtained them at the end of his paper.

4. Generalized Slater constraint qualification

Francis and Cabot [7] have given an application of Theorems 1 to 3 in a multifacility location problem wherein the objective function is the sum of costs which are directly proportional to the Euclidian distances. They use a converse duality theorem due to Mond [9], who assumed the Kuhn-Tucker constraint qualification for the dual problem in order to apply the Kuhn-Tucker necessary conditions. However, these conditions hold under several other constraint qualifications [1], [2], [8], [12] and sometimes it is easier to verify other constraint qualifications than that of Kuhn-Tucker. For example, in Francis and Cabot's problem the vector p is a zero vector. There may also be problems in which p is the negative of a cost vector (thus $p \leq 0$). In this section we define a generalized Slater constraint qualification and give a simple sufficient condition (implied by $p \leq 0$) for the dual problem to satisfy this constraint qualification. We also show that it implies Abadie's constraint qualification, which in turn implies ([1], Theorem 3) the Kuhn-Tucker constraint qualification (as defined in [8], p. 102).

To define some of the constraint qualifications we consider the non-linear program:

NLP: Minimize $\theta(x)$

Subject to $x \in S = \{x \mid x \in X,\ g(x) \le 0\}$,

where X is an open set in R^n , and θ and g are respectively a numerical function and an m-dimensional vector function both defined on X .

Let $\bar{x} \in S$ and $I = \{i \mid g_i(\bar{x}) = 0\}$. The function g is said to satisfy

(i) Slater's weak constraint qualification [2], [8] at $\bar{x}$ if g_I is pseudoconvex at $\bar{x}$ and there exists an $\tilde{x} \in S$ such that $g_I(\tilde{x}) < 0$;

(ii) The generalized Slater constraint qualification I [11] on X if X is a convex set, g is a convex function on X and there exists an $\tilde{x} \in S$ such that $g_J(\tilde{x}) < 0$, where $J = \{i \mid g_i \text{ is nonlinear}\}$;

(iii) Abadie's constraint qualification [1], [12] at $\bar{x}$ if g_I is differentiable at $\bar{x}$ and if

$$\left\langle \begin{matrix} \nabla g_M(\bar{x})x \ge 0 \\ \nabla g_N(\bar{x})x > 0 \end{matrix} \right\rangle \text{ has a solution } x \in R^n ,$$

where $M = \{i \mid g_i(\bar{x}) = 0 \text{ and } g_i \text{ is linear}\}$ and $N = I \sim M$.

We combine the two generalizations of Slater's constraint qualification to define:

DEFINITION. The function g is said to satisfy the generalized Slater constraint qualification II at $\bar{x} \in S$ if g_N is pseudoconvex at $\bar{x}$ and there exists an $\tilde{x} \in S$ such that $g_N(\tilde{x}) < 0$, where

$N = \{i \mid g_i(\bar{x}) = 0 \text{ and } g_i \text{ is nonlinear}\}$.

We now show some relations among the above constraint qualifications.

THEOREM 5. *Let X, S, I and g be as defined above, and let g_I be differentiable at $\bar{x} \in S$.*

(a) If g satisfies Slater's weak constraint qualification at $\bar{x}$ or the generalized Slater constraint qualification I on X , then g satisfies the generalized Slater constraint qualification II at $\bar{x}$.

(b) If g satisfies the generalized Slater constraint qualification II at $\bar{x}$, then g satisfies Abadie's constraint qualification at $\bar{x}$.

Proof. *(a)* With sets N, I, J as defined above, N is a subset of both I and J , hence the proof is immediate.

(b) Since g satisfies the generalized Slater constraint qualification II at $\bar{x} \in S$ there exists an $\tilde{x} \in S$ such that

$$g_N(\tilde{x}) < 0 = g_N(\bar{x}) .$$

Since g_N is pseudoconvex at $\bar{x}$, the above inequality implies

$$\nabla g_N(\bar{x})(\tilde{x}-\bar{x}) < 0 .$$

Now let $M = I \sim N = \{i \mid g_i(\bar{x}) = 0 \text{ and } g_i \text{ is linear}\}$. Therefore

$$\nabla g_M(\bar{x})(\tilde{x}-\bar{x}) = g_M(\tilde{x}) - g_M(\bar{x}) \leq 0 .$$

By taking $x = \bar{x} - \tilde{x}$, we have that $\nabla g_M(\bar{x})x \geq 0$ and $\nabla g_N(\bar{x})x > 0$. Hence g satisfies Abadie's constraint qualification at $\bar{x}$.

The following examples respectively show that the converses of the implications in Theorem 5 are not true in general.

EXAMPLE 1. $X = R^2$, $g(x) = (x_1+x_2-1, -x_1-x_2+1, -x_1, -x_2)$, $\theta(x) = x_1 + 2x_2$, $\bar{x} = (1, 0)$.

EXAMPLE 2. $X = R$, $g(x) = x^3 + x$, $\theta(x) = -x$, $\bar{x} = 0$.

EXAMPLE 3. $X = R$, $g(x) = -x^2 + x$, $\theta(x) = x^2$, $\bar{x} = 0$.

In view of Theorem 5 above and Theorem 2 in [*1*], the Kuhn-Tucker necessary conditions hold for NLP if θ and g are differentiable, and g satisfies the generalized Slater constraint qualification II at the optimal point. We give below a simple sufficient condition for our dual problem to satisfy the generalized Slater constraint qualification I (and hence II).

THEOREM 6. *If the system*

$$A'y \geq p \text{ , } \quad y \geq 0 \quad \textit{has a solution,} \tag{15}$$

then the dual problem satisfies the generalized Slater constraint qualification I.

Proof. Let there exist a $\bar{y} \geq 0$ such that $A'\bar{y} \geq p$. Then $(\bar{y}, \bar{w}_i = 0)$, $i = 1, 2, \ldots, t$, is a feasible solution of the dual problem. Moreover, all the nonlinear constraints, which are differentiable and convex, hold as strict inequalities. This proves the theorem.

Therefore, using the Kuhn-Tucker necessary condition, we can obtain a result similar to Theorem 4 with assumption (2) replaced by (15). Also, note that if $p \leq 0$, then (15) holds. Since in the multifacility location problem of Francis and Cabot [7] the vector $p = 0$, it follows from Theorems 6 and 5 above and Theorem 3 in [1] that their dual satisfies the Kuhn-Tucker constraint qualification which, therefore, need not be assumed.

References

[1] J. Abadie, "On the Kuhn-Tucker theorem", *Nonlinear programming*, 19-36 (North Holland, Amsterdam, 1967).

[2] M.S. Bazaraa, C.M. Shetty, *Foundations of optimization* (Lecture Notes in Economics and Mathematical Systems, 122. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[3] Davinder Bhatia, "A note on duality theorem for a nonlinear programming problem", *Management Sci.* 16 (1969/70), 604-606.

[4] B.D. Craven and B. Mond, "On converse duality in nonlinear programming", *Oper. Res.* 19 (1971), 1075-1078.

[5] E. Eisenberg, "Duality in homogeneous programming", *Proc. Amer. Math. Soc.* 12 (1961), 783-787.

[6] E. Eisenberg, "Supports of a convex function", *Bull. Amer. Math. Soc.* **68** (1962), 192-195.

[7] Richard L. Francis and A. Victor Cabot, "Properties of a multifacility location problem involving Euclidian distances", *Naval Res. Logist. Quart.* **19** (1972), 335-353.

[8] Olvi L. Mangasarian, *Nonlinear programming* (McGraw-Hill, New York, London, Sydney, 1969).

[9] Bertram Mond, "Nonlinear nondifferentiable programming in complex space", *Nonlinear programming*, 385-400 (Proc. Sympos. Mathematics Research Centre, University of Wisconsin, Madison, 1970. Academic Press, New York, London, 1970).

[10] Bertram Mond, "A class of nondifferentiable mathematical programming problems", *J. Math. Anal. Appl.* **46** (1974), 169-174.

[11] Bertram Mond and Murray Schechter, "On a constraint qualification in a nondifferentiable programming problem", *Naval Res. Logist. Quart.* **23** (1976), 611-613.

[12] David W. Peterson, "A review of constraint qualifications in finite-dimensional spaces", *SIAM Rev.* **15** (1973), 639-654.

[13] S.M. Sinha, "A duality theorem for nonlinear programming", *Management Sci.* **12** (1966), 385-390.

Department of Mathematics,
University of Roorkee,
Roorkee 247672,
India.

Present address:
Department of Mathematics,
University of Melbourne,
Parkville,
Victoria 3052,
Australia.