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Restriction of the Tangent Bundle of G/P to a Hypersurface

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Abstract. Let P be a maximal proper parabolic subgroup of a connected simple linear algebraic group G, defined over $\mathbb{C}$, such that $n := \dim_{\mathbb{C}} G/P \geq 4$. Let $\iota\colon Z \hookrightarrow G/P$ be a reduced smooth hypersurface of degree at least $(n-1)\cdot \text{degree}(T(G/P))/n$. We prove that the restriction of the tangent bundle $\iota^* TG/P$ is semistable.

1 Introduction

Given a semistable vector bundle E over a polarized smooth projective variety X, the restrictions of E to smooth hypersurfaces in X of sufficiently large degree remain semistable; see [Fl] for general estimates of how large the degree should be. However, for the case of $X = \mathbb{CP}^n$, much sharper results are known for some vector bundles E of special interest [Pa]. Our aim here is to consider the restrictions to the hypersurfaces of the tangent bundle of a rational homogeneous space of Picard number one.

Let G be a connected simple linear algebraic group defined over the field of complex numbers and $P \subset G$ a maximal proper parabolic subgroup. Let ξ be the ample generator of $\text{Pic}(G/P) \cong \mathbb{Z}$. The degree of a hypersurface on G/P lying in the linear system $|\xi^{\otimes j}|$ is defined to be j. Similarly, the degree of a vector bundle V on G/P is defined to be ℓ if $\bigwedge^{\text{top}} V \cong \xi^{\otimes \ell}$.

We prove the following (see Theorem 2.2).

Theorem 1.1 *Assume that $n := \dim_{\mathbb{C}} G/P \geq 4$. Let $\iota\colon Z \hookrightarrow G/P$ be a reduced smooth hypersurface with* $\text{degree}(Z) \geq \text{degree}(T(G/P))(n-1)/n$*. Then the pull back $\iota^* T(G/P)$ is semistable.*

The key inputs in the proof of Theorem 1.1 are a result of [Br] and the Akizuki–Nakano vanishing theorem.

2 Semistability of Restriction of Tangent Bundle

Let G be a connected simple linear algebraic group defined over $\mathbb{C}$. Fix a maximal proper parabolic subgroup P of G. Therefore, the quotient $M := G/P$ is a smooth projective variety with $\text{Pic}(M) = \mathbb{Z}$.

Let ξ denote the ample generator of $\text{Pic}(G/P)$. This line bundle ξ is actually very ample. For any $m \in \mathbb{Z}$, set $\text{degree}(\xi^{\otimes m}) := m$. The *degree* of a hypersurface $Z \subset M$ is defined to be $\text{degree}(\mathcal{O}_M(Z))$.

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For notational convenience, for a coherent sheaf E on M and an integer j, the tensor product $E \otimes \xi^{\otimes j}$ will be denoted by $E(j)$.

Proposition 2.1 *Assume that $n := \dim_{\mathbb{C}} G/P \geq 3$. Then $H^{n-1}(M, \Omega^k_M(\ell)) = 0$ for all $k \in [1, n-2]$ and $\ell > 0$.*

Proof Since $\ell > 0$, the line bundle $\mathcal{O}_M(\ell) = \xi^{\otimes \ell}$ over M is ample. Therefore, if $k \geq 2$, then the Akizuki–Nakano vanishing theorem says that

$$H^{n-1}(M, \Omega^k_M(\ell)) = 0$$

(see [Ko, p. 74, Theorem 3.11; p. 68, (3.2)] for the Akizuki–Nakano vanishing theorem).

Now assume that $k = 1$. We have a canonical isomorphism

$$H^{n-1}(M, \Omega^1_M(\ell)) = H^{1,n-1}(M, \mathcal{O}_M(\ell)),$$

where $H^{1,n-1}(M, \mathcal{O}_M(\ell))$ is a Dolbeault cohomology. Since $n - 1 > 1$ and $k > 0$, it follows from [Br, p. 161, Theorem 1(i)] that $H^{1,n-1}(M, \mathcal{O}_M(\ell)) = 0$ (see also [Br, p. 155, lines 14–15]). This completes the proof of the proposition. ■

For a torsionfree coherent sheaf F on a reduced smooth hypersurface $\iota\colon Z \hookrightarrow M$ of degree d, we define $\operatorname{degree}(F) := \operatorname{degree}(F|_C)$, where $C \subset Z$ is a general complete intersection curve obtained by intersecting hyperplanes from the complete linear system $|\iota^*\xi|$ on Z. In particular, $\operatorname{degree}(\iota^*\xi^{\otimes m}) = mdc_1(\xi)^n \cap [M]$ for all $m \in \mathbb{Z}$.

We recall that a vector bundle E over a smooth projective variety X equipped with a polarization is called *semistable* if

$$\frac{\operatorname{degree}(F)}{\operatorname{rank}(F)} \leq \frac{\operatorname{degree}(E)}{\operatorname{rank}(E)}$$

for all nonzero coherent subsheaves F of E.

Theorem 2.2 *Assume that $n := \dim_{\mathbb{C}} G/P \geq 4$. Let $\iota\colon Z \hookrightarrow M := G/P$ be a reduced smooth hypersurface of degree d. If*

$$d \geq \frac{\operatorname{degree}(TM)(n-1)}{n}, \tag{2.1}$$

then the pullback ι^*TM is semistable.

Proof Let $\tau := \operatorname{degree}(TM)$. Take any reduced smooth hypersurface $\iota\colon Z \hookrightarrow M$ of degree d satisfying (2.1). Assume that ι^*TM is not semistable. Therefore, there is a nonzero coherent subsheaf

$$0 \neq F \subset \iota^*TM =: W, \tag{2.2}$$

such that

$$\frac{\delta}{k} = \frac{\operatorname{degree}(F)}{\operatorname{rank}(F)} > \frac{\operatorname{degree}(\iota^*TM)}{\operatorname{rank}(\iota^*TM)} = \frac{\operatorname{degree}(\iota^*\xi)\tau}{n}, \tag{2.3}$$

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where $\delta := \text{degree}(F)$ and $k = \text{rank}(F) \in [1, n-1]$; both τ and n are defined above.

Let $\det F = \bigwedge^k F$ be the determinant line bundle of F; see [Ko, Chapter V, § 6] for the construction of a determinant line bundle. Since $\dim G/P \geq 4$ and H is a reduced smooth ample hypersurface, from Grothendieck's Lefschetz theory it follows that the homomorphism $\text{Pic}(M) \to \text{Pic}(Z)$ defined by $L \mapsto \iota^* L$ is an isomorphism; see [Gr, Exposé X]. In particular, the determinant line bundle $\det F$ is the restriction of $\xi^{\otimes \delta'}$ to Z for some $\delta' \in \mathbb{Z}$. Since $\text{degree}(F) = \delta$, it follows that

$$\delta' = \frac{\delta}{\text{degree}(\iota^* \xi)}. \tag{2.4}$$

The rank of the subsheaf $F \subset W$ in (2.2) being k, from the properties of a determinant line bundle it follows that we have a nonzero homomorphism

$$\phi\colon\ \det F \longrightarrow \bigwedge^k W$$

(the existence of ϕ follows from [Ko, p. 166, Proposition 6.10]). This homomorphism ϕ gives a nonzero section

$$0 \neq \sigma \in H^0(Z, (\iota^* \xi^{\otimes \delta'})^* \otimes \bigwedge^k W) = H^0(Z, \iota^*(\xi^{\otimes -\delta'} \otimes \bigwedge^k TM)). \tag{2.5}$$

Since $\text{degree}(Z) = d$, the canonical line bundle K_Z of Z is isomorphic to $\iota^* \xi^{\otimes (d-\tau)}$. Therefore, the Serre duality gives

$$H^0(Z, \iota^*(\xi^{\otimes -\delta'} \otimes \bigwedge^k TM)) = H^{n-1}(Z, \iota^*(\xi^{\otimes(\delta'+d-\tau)} \otimes \Omega^k_M))^*. \tag{2.6}$$

The theorem will be proved by showing that the left-hand side in (2.6) vanishes.

On M, we have the following short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{O}_M(-d) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M|_Z = \iota_* \mathcal{O}_Z \longrightarrow 0,$$

which is obtained from the fact that $\mathcal{O}_M(-Z) = \mathcal{O}_M(-d)$. Tensoring this exact sequence with $\Omega^k_M(\delta' + d - \tau)$ we obtain the short exact sequence of sheaves on M

$$0 \longrightarrow \Omega^k_M(\delta' - \tau) \longrightarrow \Omega^k_M(\delta' + d - \tau) \longrightarrow \Omega^k_M(\delta' + d - \tau)|_Z \longrightarrow 0. \tag{2.7}$$

The short exact sequence in (2.7) gives the following long exact sequence of cohomologies:

$$H^{n-1}(M, \Omega^k_M(\delta' + d - \tau)) \longrightarrow H^{n-1}(Z, \iota^* \Omega^k_M(\delta' + d - \tau)) \longrightarrow H^n(M, \Omega^k_M(\delta' - \tau)). \tag{2.8}$$

From (2.3) and (2.4) we have

$$\delta' > \frac{k\tau}{n}. \tag{2.9}$$

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Combining this with (2.1) we have

$$\delta' + d > \frac{\tau(k+n-1)}{n}.$$

Also, we have $\tau > 0$ and $k \geq 1$. Therefore, $\delta' + d - \tau > 0$. Consequently, from Proposition 2.1 it follows that

$$H^{n-1}(M, \Omega_M^k(\delta' + d - \tau)) = 0. \tag{2.10}$$

Since $K_M = \mathcal{O}_M(-\tau)$, the Serre duality gives

$$H^n(M, \Omega_M^k(\delta' - \tau)) = H^0(M, \bigwedge^k TM(-\delta'))^*. \tag{2.11}$$

We have

$$\frac{\text{degree}(\bigwedge^k TM(-\delta'))}{\text{rank}(\bigwedge^k TM(-\delta'))} = \frac{k\tau}{n} - \delta'.$$

Therefore, from (2.9) it follows immediately that degree$(\bigwedge^k TM(-\delta')) < 0$. We also know that the tangent bundle TM is semistable; this follows from [Um, p. 136, Theorem 2.4] and the fact that the Harder–Narasimhan filtration of TM being canonical is left invariant by the left-translation action of G on G/P. Since TM is semistable, we conclude that $\bigwedge^k TM$ is also semistable [RR, p. 285, Theorem 3.18]. Therefore, the vector bundle $\bigwedge^k TM(-\delta')$ is also semistable. Now using the definition of semistability, it can be shown that a semistable vector bundle of negative degree does not admit any nonzero sections. Indeed, if $f\colon \mathcal{O}_M \to V$ is a nonzero section of a semistable vector bundle of negative degree, then consider the image $V' := f(\mathcal{O}_M) \subset V$. Since V is semistable, degree$(V')/\,$rank$(V') \leq$ degree$(V)/\,$rank$(V) < 0$. This is a contradiction because the degree of $V' = \mathcal{O}_M$ is zero. Therefore, V does not admit any nonzero sections. In particular, we have $H^0(M, \bigwedge^k TM(-\delta')) = 0$. Therefore, (2.11) yields that

$$H^n(M, \Omega_M^k(\delta' - \tau)) = 0. \tag{2.12}$$

Now using (2.10) and (2.12), from the exact sequence in (2.8) we conclude that

$$H^{n-1}(Z, \iota^*\Omega_M^k(\delta' + d - \tau)) = 0.$$

Consequently, from (2.6) we have

$$H^0(Z, \iota^*(\xi^{\otimes -\delta'} \otimes \bigwedge^k TM)) = 0.$$

But this contradicts that the section σ in (2.5) is nonzero. Therefore, we conclude that the vector bundle ι^*TM is semistable. This completes the proof of the theorem. ■

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