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QUOTIENT SUPERMANIFOLDS

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A necessary and sufficient condition for the existence of a supermanifold structure on a quotient defined by an equivalence relation is established. Furthermore, we show that an equivalence relation R on a Berezin–Leĭtes–Kostant supermanifold X determines a quotient supermanifold X/R if and only if the restriction R_0 of R to the underlying smooth manifold X_0 of X determines a quotient smooth manifold X_0/R_0.

1. INTRODUCTION

The necessity of taking quotients of supermanifolds arises in a great variety of cases; just to mention a few examples, we recall the notion of supergrassmannian, the definition of the Teichmüller space of super Riemann surfaces, or the procedure of super Poisson reduction. These constructions play a crucial rôle in superstring theory as well as in supersymmetric field theories.

The first aim of this paper is to prove a necessary and sufficient condition ensuring that an equivalence relation in the category of supermanifolds gives rise to a quotient supermanifold; analogous results, in the setting of Berezin-Leĭtes-Kostant (BLK) supermanifolds, were already stated in [**6**].

Secondly and rather surprisingly at that, it turns out that for Berezin–Leĭtes–Kostant supermanifolds any obstacles to the existence of a quotient supermanifold can exist only in the even sector and are therefore purely topological. We demonstrate that an equivalence relation R on a BLK supermanifold X determines a quotient BLK supermanifold X/R if and only if the restriction of R to the underlying manifold X_0 of X determines a quotient smooth manifold.

The definition of supermanifold we adopt here encompasses both the BLK supermanifold theory [5] and, in a certain sense, the DeWitt-Rogers approach. The basic idea is to introduce an axiomatics yielding a category of supermanifolds where the usual

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geometric constructions (products, bundles, characteristic classes, ...) work reasonably well; for a more detailed presentation the reader is referred to [**1, 2, 3**].

Let $B = B_0 \oplus B_1$ be a $\mathbb{Z}_2$-graded-commutative associative unital $\mathbb{R}$-algebra (or $\mathbb{C}$-algebra); in other words, for $\alpha, \beta \in \mathbb{Z}_2$, we have

$$B_\alpha B_\beta \subset B_{\alpha+\beta}, \qquad ab = (-1)^{\alpha\beta} ba \qquad \text{if } a \in B_\alpha,\ b \in B_\beta .$$

We denote by $B^{m,n}$ the direct sum $B_0^m \oplus B_1^n$. For the sake of simplicity we shall assume that B is finite-dimensional and that it is generated, as a unital algebra, by the odd part B_1. (Equivalently: B is a graded factor-algebra of a finite-dimensional Grassmann algebra, for example, the exterior algebra $\bigwedge B_1$.) The latter assumption easily implies that every element $x \in B$ is uniquely decomposed as the sum of a (real or complex) number $\beta(x)$ and a nilpotent $\sigma(x)$.

By *superspace* over B we mean a triple $(X, \mathcal{A}, \mathrm{ev})$, where X is a paracompact topological space, $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$-graded-commutative B-algebras, and $\mathrm{ev}\colon \mathcal{A} \to \mathcal{C}_X$ is a morphism of sheaves of graded B-algebras (here $\mathcal{C}_X$ is the sheaf of B-valued continuous functions on X). We shall sometimes write $\tilde{\varphi}$ for $\mathrm{ev}(\varphi)$. A *morphism of superspaces* $\left(f, f^\sharp\right)\colon \left(X, \mathcal{A}, \mathrm{ev}^X\right) \to \left(Y, \mathcal{B}, \mathrm{ev}^Y\right)$ is a pair consisting of a continuous map $f\colon X \to Y$ and a morphism of sheaves of graded B-algebras $f^\sharp\colon \mathcal{B} \to f_*\mathcal{A}$ such that $\mathrm{ev}^X \circ f^\sharp = f^* \circ \mathrm{ev}^Y$.

By the morphism $\mathrm{ev}\colon \mathcal{A} \to \mathcal{C}_X$ one can evaluate germs of superfunctions — that is, sections of $\mathcal{A}$ — at a point $p \in X$. In this way, we can define the graded ideal $\mathfrak{L}_p$ of the stalk $\mathcal{A}_p$ formed by the germs of superfunctions vanishing at p:

$$\mathfrak{L}_p = \left\{\varphi \in \mathcal{A}_p \mid \tilde{\varphi}(p) = 0\right\}.$$

A *supermanifold* of dimension (m, n) is by definition a superspace $(X, \mathcal{A}, \mathrm{ev})$ satisfying the following four Axioms. (The supermanifolds we characterise here were called R^∞-supermanifolds in reference [**3**].)

AXIOM 1. The graded $\mathcal{A}$-dual of the sheaf of derivations, $\mathcal{D}er^*\mathcal{A}$, is a locally free graded $\mathcal{A}$-module of rank (m, n). Every point $p \in X$ has an open neighbourhood U with sections $x^1, \ldots, x^m \in \mathcal{A}(U)_0$, $y^1, \ldots, y^n \in \mathcal{A}(U)_1$ such that $\{dx^1, \ldots, dx^m, dy^1, \ldots, dy^n\}$ is a graded basis of $\mathcal{D}er^*\mathcal{A}(U)$ over $\mathcal{A}(U)$.

AXIOM 2. Given a coordinate chart $(U, x^1, \ldots, x^m, y^1, \ldots, y^n)$, the assignment

$$p \mapsto \left(\tilde{x}^1(p), \ldots, \tilde{x}^m(p), \tilde{y}^1(p), \ldots, \tilde{y}^n(p)\right)$$

defines a homeomorphism of U onto an open subset in $B^{m,n}$.

AXIOM 3 For every $p \in X$ the ideal $\mathfrak{L}_p$ is finitely generated.

In order to state the last remaining Axiom we need to topologise the algebras of sections $\mathcal{A}(U)$. Let $\mathcal{D}(\mathcal{A})$ denote the sheaf of differential operators over $\mathcal{A}$, that is, the graded $\mathcal{A}$-module generated multiplicatively by $\mathcal{D}er\,\mathcal{A}$ over $\mathcal{A}$. We define the family of prenorms

$$p_{L,K}(\varphi) = \max_{p\in K} \left\| \widetilde{L(\varphi)}(p) \right\|_B ,$$

with $L \in \mathcal{D}\left(\mathcal{A}_{|U}\right)$, $K \subset U$ compact, and $\|\ \|_B$ a Banach norm on B. In this way we induce in $\mathcal{A}(U)$ the structure of a locally convex graded B-algebra.

AXIOM 4. For every open subset $U \subset X$, the topological algebra $\mathcal{A}(U)$ is complete Hausdorff.

We notice that Axiom 2 implies that X is locally Euclidean, while by Axiom 3 one can prove — via a graded version of Nakayama's lemma — the existence of local Taylor expansions.

One can prove that any (m, n) dimensional supermanifold is locally isomorphic to the standard supermanifold $(B^{m,n}, \mathcal{G}_{m,n})$ [**3**].

Quite obviously, the above definition of supermanifold includes the usual notion of smooth or complex analytic manifold; in this case, the graded algebra B reduces to $\mathbb{R}$ or $\mathbb{C}$ and the sheaf $\mathcal{A}$ is the sheaf of germs of the appropriate (smooth or holomorphic) class of functions on X. When B is $\mathbb{R}$ or $\mathbb{C}$, but $\mathcal{A}$ is a genuine sheaf of graded algebras (that is $\mathcal{A}_1 \neq 0$), the notion of BLK supermanifold is recovered. Finally, in the case of a Grassmann algebra B over $\mathbb{R}$ or $\mathbb{C}$, the four Axioms we have stated are equivalent to the definition of G-supermanifold; a detailed treatment of this equivalence, together with an analysis of the axiomatics, can be found in [**1**]. The case of an infinite-dimensional ground algebra B is examined in [**3**].

In the next Section we collect some preliminary results concerning sub-supermanifolds and morphisms of supermanifolds, while in Section 3 we shall prove our main theorem. In Section 4 we study quotients on BLK supermanifolds.

2. SUB-SUPERMANIFOLDS, TRANSVERSALITY, AND FIBRE PRODUCT

We introduce the notions of immersion, submersion in the category of supermanifolds as straightforward generalisations of the corresponding properties of smooth manifolds.

DEFINITION 2.1: A morphism of supermanifolds $f\colon (Y, \mathcal{B}) \to (X, \mathcal{A})$ is

(1) a *closed immersion* if $f\colon Y \to X$ is a closed topological embedding, and $f^\sharp\colon \mathcal{A} \to f_*\mathcal{B}$ is an epimorphism;

(2) an *open immersion* if $f\colon Y \to X$ is an open topological embedding, and $\mathcal{B} \simeq f^\sharp\mathcal{A}$.

In local coordinates any closed immersion reduces to an immersion of open sets of model superspaces.

PROPOSITION 2.2. *Let $f\colon (Y, \mathcal{B}) \to (X, \mathcal{A})$ be a closed immersion, with* $\dim(Y, \mathcal{B}) = (m, n)$ *and* $\dim(X, \mathcal{B}) = (r, s)$. *For each $y \in Y$ there are a neighbourhood U*

of y and a neighbourhood V of $f(y)$, together with isomorphisms $\left(U,\mathcal{B}_{|U}\right) \xrightarrow{\sim} \left(\tilde{U},\mathcal{G}_{m,n}\right)$ and $\left(V,\mathcal{A}_{|V}\right) \xrightarrow{\sim} \left(\tilde{V},\mathcal{G}_{r,s}\right)$, such that the induced morphism $\left(\tilde{U},\mathcal{G}_{m,n}\right) \to \left(\tilde{V},\mathcal{G}_{r,s}\right)$ is the natural immersion.

PROOF: After fixing a coordinate system $(x^1,\ldots,x^r,\xi^1,\ldots,\xi^s)$ in a neighbourhood of $f(y)$, one must show that the image in $T_y^*(Y,\mathcal{B})$ of the differentials dx, $d\xi$ contains a basis. This is a direct consequence of simple algebraic facts about free graded modules. $\square$

Notice that in fact the property formulated in the above Proposition 2.2 is not characteristic of closed immersions (as simple examples at the level of smooth manifolds show), but rather of arbitrary sub-supermanifolds, see Definition 2.5 below.

DEFINITION 2.3: A morphism of supermanifolds $f\colon(Y,\mathcal{B}) \to (X,\mathcal{A})$ is a *submersion* if for all $y \in Y$ the morphism of graded B-modules $f^\sharp_{f(y)}\colon T^*_{f(y)}(X,\mathcal{A}) \to T_y^*(Y,\mathcal{B})$ is injective.

The following Proposition is dual to Proposition 2.2, and is proved analogously.

PROPOSITION 2.4. *Let $f\colon(Y,\mathcal{B}) \to (X,\mathcal{A})$ be a supermanifold morphism, with $\dim(Y,\mathcal{B}) = (m,n)$ and $\dim(X,\mathcal{A}) = (r,s)$. For each $y \in Y$ let $(x^1,\ldots,x^r,\xi^1,\ldots,\xi^s)$ be local coordinates for $(X,\mathcal{A})$ in a neighbourhood of $f(y)$. The morphism f is a submersion if and only if the image of the coordinates $(x^1,\ldots,x^r,\xi^1,\ldots,\xi^s)$ via $f^\sharp$ can be completed to a coordinate system in a neighbourhood of y, for all y.*

An immediate corollary is the fact that a morphism $f\colon(Y,\mathcal{B}) \to (X,\mathcal{A})$ is a submersion if and only if it admits local sections, that is, if and only if for each point $x \in X$ there are a neighbourhood U and a morphism $s\colon\left(U,\mathcal{A}_{|U}\right) \to (Y,\mathcal{B})$ such that $f\circ s = \mathrm{id}_{\left(U,\mathcal{A}_{|U}\right)}$.

DEFINITION 2.5: A supermanifold $(Y,\mathcal{B})$ is a *sub-supermanifold* of $(X,\mathcal{A})$ if there is a morphism $f\colon(Y,\mathcal{B}) \to (X,\mathcal{A})$ which is the composition of a closed immersion with an open immersion.

If $f\colon(Y,\mathcal{B}) \to (X,\mathcal{A})$ is a morphism of supermanifolds, and $(W,\mathcal{C})$ is a sub-supermanifold of $(X,\mathcal{A})$, the inverse image of $(W,\mathcal{C})$ via f is defined as the supermanifold $\left(f^{-1}(W), f^*i_*\mathcal{C}\right)$. Here i is the imbedding $i\colon(W,\mathcal{C}) \to (X,\mathcal{A})$, and f^* is the inverse image of sheaves of rings, that is

$$f^*i_*\mathcal{C} = f^{-1}i_*\mathcal{C} \otimes_{f^{-1}\mathcal{A}} \mathcal{B}\,.$$

The proof of the following Lemma is straightforward, so that we omit it.

LEMMA 2.6. *Let $(X,\mathcal{A})$ be a supermanifold, $Y \subset X$ a closed subset, $\mathcal{B}$ a quotient of $\mathcal{A}$ supported in Y; that is, there is an exact sequence*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

with $\operatorname{supp}\mathcal{B} \subset Y$. Then the pair $(Y,\mathcal{B})$ defines a closed sub-supermanifold of $(X,\mathcal{A})$ if and only if $\mathcal{I}$ is locally generated by a subset of a coordinate system of $(X,\mathcal{A})$.

We notice that the evaluation morphism $\mathrm{ev}^Y: \mathcal{B} \to C_Y^\infty \otimes B$ is well-defined because $\mathrm{ev}^X(\mathcal{I}) \subset \mathcal{I}_Y \otimes B$, where $\mathcal{I}_Y$ is the ideal of C_Y^∞ defined by Y as a smooth submanifold of X.

By Lemma 2.6 one readily proves the following basic result.

Proposition 2.7. *If $f: (Y, \mathcal{B}) \to (X, \mathcal{A})$ is a submersion, and $(W, \mathcal{C})$ is a sub-supermanifold of $(X, \mathcal{A})$, the inverse image $f^{-1}(W, \mathcal{C})$ is a sub-supermanifold of $(Y, \mathcal{B})$.*

We introduce now the notion of transversality in the category of supermanifolds.

Definition 2.8: Two supermanifold morphisms $f: (Y, \mathcal{B}) \to (X, \mathcal{A})$ and $g: (W, \mathcal{C}) \to (X, \mathcal{A})$ are said to be *transversal* if for any points $y \in Y$ and $w \in W$ such that $f(y) = g(w) = x$ one has $f_* T_y(Y, \mathcal{B}) + g_* T_w(W, \mathcal{C}) = T_x(X, \mathcal{A})$.

Let $f: (Y, \mathcal{B}) \to (X, \mathcal{A})$ be a supermanifold morphism, and $i: (W, \mathcal{C}) \to (X, \mathcal{A})$ a sub-supermanifold of $(X, \mathcal{A})$; if f and i are transversal (or in other words, if f is transversal to $(W, \mathcal{C})$), then $f^{-1}(W, \mathcal{C})$ is a sub-supermanifold of $(Y, \mathcal{B})$.

One can define the fibre product of two supermanifold morphisms in categorial terms. Let us fix two supermanifold morphisms $f: (Y, \mathcal{B}) \to (X, \mathcal{A})$ and $g: (W, \mathcal{C}) \to (X, \mathcal{A})$. Their fibre product — provided it exists — is the unique triple $((P, \mathcal{E}), q_1, q_2)$, where $(P, \mathcal{E})$ is a supermanifold, and $q_1: (P, \mathcal{E}) \to (Y, \mathcal{B})$ and $q_2: (P, \mathcal{E}) \to (W, \mathcal{C})$ are supermanifold morphisms, enjoying the following universal property: for any pair of morphisms

$$\phi_1: (S, \mathcal{F}) \to (Y, \mathcal{B}) \qquad \text{and} \qquad \phi_2: (S, \mathcal{F}) \to (W, \mathcal{C})$$

such that $f \circ \phi_1 = g \circ \phi_2$, there is a unique morphism

$$\phi: (S, \mathcal{F}) \to (P, \mathcal{E})$$

such that $q_i \circ \phi = \phi_i$. The supermanifold $(P, \mathcal{E})$ is usually denoted by $(Y, \mathcal{B}) \times_{(X, \mathcal{A})} (W, \mathcal{C})$.

Transversal morphisms enjoy the fundamental property of admitting fibre products, as it can be easily checked.

Proposition 2.9. *Let $f: (Y, \mathcal{B}) \to (X, \mathcal{A})$ and $g: (W, \mathcal{C}) \to (X, \mathcal{A})$ be transversal morphisms. Then they admit a fibre product, which is isomorphic, as a supermanifold, to the sub-supermanifold of $(Y, \mathcal{B}) \times (W, \mathcal{C})$ defined by the sheaf of ideals $(f \times g)^\sharp \mathcal{I}_\Delta$, where $(\Delta, \mathcal{I}_\Delta)$ is the diagonal of $(X, \mathcal{A})$.*

Let us notice the quite obvious fact that the fibre product of a submersion with any morphism always exists, since a submersive morphism is transversal to any other morphism.

3. Equivalence relations and quotient supermanifolds

Following the usual procedure in supermanifold theory, we borrow the definition of equivalence relation in the category of supermanifolds from algebraic geometry [4] (see also [6]).

Let $\rho: (R, \mathcal{C}) \to (X, \mathcal{A}) \times (X, \mathcal{A})$ be a sub-supermanifold of the cartesian product $(X, \mathcal{A}) \times (X, \mathcal{A})$, and set $\varpi_i = \pi_i \circ \rho$, where the π_i's are the canonical projections onto the factors of the cartesian product.

DEFINITION 3.1: We say that $(R, \mathcal{C})$ is an *equivalence relation* if the following conditions hold:

(i) there exists a supermanifold morphism $c: (X, \mathcal{A}) \to (R, \mathcal{C})$ such that $\rho \circ c = \delta$, where δ is the diagonal immersion of $(X, \mathcal{A})$.

(ii) There exists a morphism $\sigma: (R, \mathcal{C}) \to (R, \mathcal{C})$ that swaps ϖ_1 and ϖ_2, that is, $\varpi_1 \circ \sigma = \varpi_2$ and $\varpi_2 \circ \sigma = \varpi_1$.

(iii) Let $(R, \mathcal{C}) \times_{(X,\mathcal{A})} (R, \mathcal{C})$ be the fibre product of the morphisms ϖ_2 and ϖ_1, which exists by condition (i). Let ϕ_1, ϕ_2 be the projections of $(R, \mathcal{C}) \times_{(X,\mathcal{A})} (R, \mathcal{C})$ onto the first and the second factor, respectively. There is a morphism $\phi_0: (R, \mathcal{C}) \times_{(X,\mathcal{A})} (R, \mathcal{C}) \to (R, \mathcal{C})$ such that

$$\varpi_1 \circ \phi_0 = \varpi_1 \circ \phi_1 \quad \text{and} \quad \varpi_2 \circ \phi_0 = \varpi_2 \circ \phi_2 \,. \tag{3.1}$$

Condition (i) amounts to the reflexivity property, in that it states that the relation contains the diagonal of $(X, \mathcal{A})$. Conditions (ii) and (iii) express the symmetry and the transitivity properties. It is quite evident that an equivalence relation $(R, \mathcal{C})$ in $(X, \mathcal{A})$ induces an equivalence relation defined on the underlying differentiable manifold X.

In Section 4 we shall need the following result, whose proof is a simple check of the commutativity of some diagrams.

LEMMA 3.2. *Let X be a sub-supermanifold of a supermanifold Y, and let R be an equivalence relation on Y. Then the restriction of R to X (defined as the inverse image of R under the direct product morphism $i \times i$, where $i: X \to Y$ is the immersion) is an equivalence relation on X.*

Whenever $(R, \mathcal{C})$ is a closed sub-supermanifold of $(X, \mathcal{A}) \times (X, \mathcal{A})$, the previous definition has some direct consequences about the ideal sheaf $\mathcal{J}$ of $(R, \mathcal{C})$ in $(X, \mathcal{A}) \times (X, \mathcal{A})$. Let $\iota: (X, \mathcal{A}) \times (X, \mathcal{A}) \to (X, \mathcal{A}) \times (X, \mathcal{A})$ be the interchange of factors; then, the symmetry property implies the inclusion

$$\iota(\mathcal{J}) \subseteq \mathcal{J} \,. \tag{3.2}$$

The interchange of factors $\gamma: (R, \mathcal{C}) \times_{(X,\mathcal{A})} (R, \mathcal{C}) \to (R, \mathcal{C}) \times_{(X,\mathcal{A})} (R, \mathcal{C})$ satisfies the conditions

$$\phi_1 \circ \gamma = \sigma \circ \phi_2, \qquad \phi_2 \circ \gamma = \sigma \circ \phi_1 \,. \tag{3.3}$$

PROPOSITION 3.3. *If $f \otimes 1 - 1 \otimes g \in \mathcal{J}$, $g \otimes 1 - 1 \otimes h \in \mathcal{J}$, then $f \otimes 1 - 1 \otimes h \in \mathcal{J}$. Moreover, if $f \otimes 1 - 1 \otimes g \in \mathcal{J}$, then $f \otimes 1 - 1 \otimes f \in \mathcal{J}$.*

PROOF: Notice first that the condition $f \otimes 1 - 1 \otimes g \in \mathcal{J}$ is equivalent to $\varpi_1^\sharp(f) = \varpi_2^\sharp(g)$. So we have $\varpi_1^\sharp(f) = \varpi_2^\sharp(g)$, $\varpi_1^\sharp(g) = \varpi_2^\sharp(h)$ and must prove that $\varpi_1^\sharp(f) = \varpi_2^\sharp(h)$. Since $\phi_1^\sharp$ is injective, it is enough to show that $\phi_1^\sharp \varpi_1^\sharp(f) = \phi_1^\sharp \varpi_2^\sharp(h)$. Now, taking into account the relations (3.1) and (3.3), together with the symmetry property and the relation $\varpi_2 \circ \phi_1 = \varpi_1 \circ \phi_2$ deduced from the definition of fibred product, we obtain

$$\begin{aligned} \phi_1^\sharp \varpi_1^\sharp(f) &= \phi_0^\sharp \varpi_1^\sharp(f) = \phi_0^\sharp \varpi_2^\sharp(g) = \phi_2^\sharp \varpi_2^\sharp(g) = \phi_2^\sharp \sigma^\sharp \varpi_1^\sharp(g) \\ &= \phi_2^\sharp \sigma^\sharp \varpi_2^\sharp(h) = \gamma^\sharp \phi_1^\sharp \varpi_2^\sharp(h) = \gamma^\sharp \phi_2^\sharp \varpi_1^\sharp(h) = \phi_1^\sharp \sigma^\sharp \varpi_1^\sharp(h) = \phi_1^\sharp \varpi_2^\sharp(h) . \end{aligned}$$

The second claim can be now readily proved. ∎

DEFINITION 3.4: Let $\rho\colon (R, \mathcal{C}) \to (X, \mathcal{A}) \times (X, \mathcal{A})$ be an equivalence relation. A supermanifold morphism $q\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ is said to be a quotient of $(X, \mathcal{A})$ by ρ if

(i) q is a submersion

(ii) the morphism ρ induces an isomorphism $(R, \mathcal{C}) \xrightarrow{\sim} (X, \mathcal{A}) \times_{(Y,\mathcal{B})} (X, \mathcal{A})$ as sub-supermanifolds of $(X, \mathcal{A}) \times (X, \mathcal{A})$.

The significance of the requirement (ii) is easily understood by recalling that in the ordinary set-theoretic notion of equivalence relation R in a set X, two points have the same image in the fibre product $X \times_{X/R} X$ if and only if they are R-related.

The very definition of a quotient implies its uniqueness.

PROPOSITION 3.5. *Two quotients $q\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ and $q'\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ of the same equivalence relation are isomorphic, in the sense that there is an isomorphism of supermanifolds $f\colon (Y, \mathcal{B}) \xrightarrow{\sim} (Y', \mathcal{B}')$ such that $q' = f \circ q$.*

EXAMPLE 3.6 As an example of equivalence relation admitting a quotient we consider the generalisation of the equivalence relation in $\mathbb{R}^m$ according to which two points are related if their first r coordinates coincide. The quotient manifold in this case is obviously $\mathbb{R}^{m-r}$. We define an equivalence relation $(R, \mathcal{C})$ in the model (m, n) dimensional supermanifold $(B^{m,n}, \mathcal{G}_{m,n})$ by letting

$$\begin{aligned} R &= \{(p_1, p_2) \in B^{m,n} \times B^{m,n} = B^{2m,2n} \text{ such that} \\ &\quad \tilde{x}^1(p_1) = \tilde{x}^1(p_2), \dots, \tilde{x}^r(p_1) = \tilde{x}^r(p_2), \tilde{\xi}^1(p_1) = \tilde{\xi}^1(p_2), \dots, \tilde{\xi}^s(p_1) = \tilde{x}^s(p_2)\} \end{aligned}$$

while $\mathcal{C}$ is the quotient of $\mathcal{G}_{m,n} \hat{\otimes} \mathcal{G}_{m,n} \simeq \mathcal{G}_{2m,2n}$ by the ideal generated by

$$\pi_1^\sharp\left(x^1\right) - \pi_2^\sharp\left(x^1\right), \dots, \pi_1^\sharp(x^r) - \pi_2^\sharp(x^r), \pi_1^\sharp\left(\xi^1\right) - \pi_2^\sharp\left(\xi^1\right), \dots, \pi_1^\sharp(\xi^s) - \pi_2^\sharp(\xi^s) .$$

Here π_1, π_2 are the projections onto the factors of $(B^{m,n}, \mathcal{G}_{m,n}) \times (B^{m,n}, \mathcal{G}_{m,n})$. It is not difficult to check that this indeed defines an equivalence relation which admits a quotient described by the projection $q\colon (B^{m,n}, \mathcal{G}_{m,n}) \to (B^{m-r,n-s}, \mathcal{G}_{m-r,n-s})$, which consists in taking the last $m - r$ even and the last $n - s$ odd coordinates (according to [**1**, Lemma

IV.1.1] a supermanifold morphism is fully described by its action on coordinate systems).

We now prove our first main theorem. The proof follows the same lines as in [**6**].

THEOREM 3.6. *An equivalence relation $\rho\colon (R,\mathcal{C}) \to (X,\mathcal{A}) \times (X,\mathcal{A})$ admits a quotient supermanifold $q\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ if and only if the following conditions are fulfilled:*

(i) $(R,\mathcal{C})$ is a closed sub-supermanifold of $(X,\mathcal{A}) \times (X,\mathcal{A})$;

(ii) the canonical projections $\varpi_1, \varpi_2\colon (R,\mathcal{C}) \to (X,\mathcal{A})$ are submersive.

PROOF:

1. NECESSITY. (i) Let $(\Delta,\mathcal{I})$ be the diagonal in $(R,\mathcal{C})\times(R,\mathcal{C})$. We have an identification

$$(R,\mathcal{C}) \simeq (q\times q)^{-1}(\Delta,\mathcal{I})\,.$$

Since q is submersive, $(R,\mathcal{C})$ is closed in $(X,\mathcal{A}) \times (X,\mathcal{A})$ by Proposition 2.4.
(ii) Again since q is submersive, any point in Y has an open neighbourhood which supports a section σ of q. Then the morphism $(\mathrm{id}, \sigma\circ q)$ is a local section of $\varpi_1\colon (R,\mathcal{C}) \to (X,\mathcal{A})$. The same holds for ϖ_2.

2. SUFFICIENCY. We prove the claim in two steps.

Step 1. We show that we may choose local coordinates $(x^1,\dots,x^m,\xi^1,\dots,\xi^n)$ on open sets U in X which allow one to make the identification

$$\begin{aligned} R\cap(U\times U) = \{&(p_1,p_2)\in X\times X \text{ such that} \\ &\tilde{x}^1(p_1)=\tilde{x}^1(p_2),\dots,\tilde{x}^r(p_1)=\tilde{x}^r(p_2),\tilde{\xi}^1(p_1)=\tilde{\xi}^1(p_2),\dots,\tilde{\xi}^s(p_1)=\tilde{x}^s(p_2)\}\end{aligned}$$

while the structure sheaf $\mathcal{C}$ may be identified with the restriction to R of the quotient of the structure sheaf $\left(\mathcal{A}\widehat{\otimes}\mathcal{A}\right)_{|U\times U}$ by the ideal generated by the sections

$$\pi_1^\sharp\left(x^1\right)-\pi_2^\sharp\left(x^1\right),\dots,\pi_1^\sharp(x^r)-\pi_2^\sharp(x^r),\pi_1^\sharp\left(\xi^1\right)-\pi_2^\sharp\left(\xi^1\right),\dots,\pi_1^\sharp(\xi^s)-\pi_2^\sharp(\xi^s)\,.$$

Here r and s are suitable natural numbers, $r\leqslant m$, $s\leqslant n$.

Let us prove this claim. Let $\mathcal{J}$ be the ideal sheaf of $(R,\mathcal{C})$ in $(X,\mathcal{A})\times(X,\mathcal{A})$. By condition (i) there exist coordinates $(u^1,\dots,u^{2m},v^1,\dots,v^{2n})$ on an open set $U\times U$ such that $\mathcal{J}(U\times U)=\langle u^1,\dots,u^r,v^1,\dots,v^s\rangle$, where $\dim(R,\mathcal{C})=(2m-r,2n-s)$. Denote by $(R_U,\mathcal{C}_U)$ the restriction of $(R,\mathcal{C})$ to $U\times U$; then the restricted morphism $\varpi_{1,U}\colon(R_U,\mathcal{C}_U)\to(U,\mathcal{A})$ is submersive, so that for every point $p\in U$ the fibre $\varpi_{1,U}^{-1}(p)$ is a sub-supermanifold of $(R_U,\mathcal{C}_U)$. Moreover, taking U as small as needed, we can assume that there is a coordinate system $(y^1,\dots,y^m,\theta^1,\dots,\theta^n)$ centred in p such that the fibre $\varpi_{1,U}^{-1}(p)$ takes the form $(p,B)\times(S,\mathcal{C}')$, where $(S,\mathcal{C}')$ is the sub-supermanifold of $(U,\mathcal{A})$ generated by the ideal $\langle y^1,\dots,y^r,\theta^1,\dots,\theta^s\rangle$.

One now proves that, taking again U as small as needed, the functions

$$\Big(\varpi_1^\sharp(y^1),\dots,\varpi_1^\sharp(y^m),u^1,\dots,u^r,\qquad \varpi_2^\sharp(y^{r+1}),\dots,\varpi_2^\sharp(y^m),\\ \varpi_1^\sharp(\theta^1),\dots,\varpi_1^\sharp(\theta^n),\qquad v^1,\dots,v^s,\varpi_2^\sharp(\theta^{s+1}),\dots\varpi_2^\sharp(\theta^n)\Big)$$

form a coordinate system on a $U\times U$. Therefore, if $(V,\mathcal{F})$ is the closed sub-supermanifold of $(U,\mathcal{A})\times(U,\mathcal{A})$ defined by the ideal

$$\mathcal{I}=\langle u^1,\dots,u^r,\varpi_2^\sharp(y^{r+1}),\dots,\varpi_2^\sharp(y^m),v^1,\dots,v^s,\varpi_2^\sharp(\theta^{s+1}),\dots\varpi_2^\sharp(\theta^n)\rangle\,,$$

π_1 induces an isomorphism $\beta\colon(V,\mathcal{F})\xrightarrow{\sim}(U,\mathcal{A})$, so that $(V,\mathcal{F})$ can be regarded as the graph of the morphism $\psi=\beta^{-1}\circ\pi_2\colon(U,\mathcal{A})\to(U,\mathcal{A})$. We set $x^i=\psi^\sharp(y^i)$ and $\xi^\alpha=\psi^\sharp(\theta^\alpha)$.

By construction, the ideal $\mathcal{I}$ contains $\mathcal{J}$; since $(V,\mathcal{F})$ is the graph of ψ we have $\mathcal{I}=\ker\left(\delta^\sharp\circ\left(\mathrm{id}\times\psi^\sharp\right)\right)$, and therefore

$$\mathcal{I}=\langle x^i\otimes 1-1\otimes y^i,\xi^\alpha\otimes 1-1\otimes\theta^\alpha\rangle,\quad i=1,\dots,m,\ \alpha=1,\dots,n\,.$$

Then $\mathcal{J}(U\times U)$ and $\mathcal{I}$ define closed sub-supermanifolds of $(U,\mathcal{A})\times(U,\mathcal{A})$ of the same dimension. Since one of these contains the other, they coincide. By Proposition 3.3,

$$\mathcal{J}(U\times U)=\langle x^i\otimes 1-1\otimes x^i,\xi^\alpha\otimes 1-1\otimes\xi^\alpha\rangle\,.$$

We have thus proved Step 1.

Step 2. We construct explicitly a supermanifold $(Y,\mathcal{B})$, which will then be shown to be the required quotient. We notice that the current hypotheses imply the validity of the corresponding assumptions for the equivalence relation R in X. The topological manifold Y is therefore defined as the quotient X/R. Now, given an open set $U\subset Y$, we have morphisms of graded B-algebras

$$\pi_1^\sharp\colon\mathcal{A}\big(q^{-1}(U)\big)\to\mathcal{C}\Big(\pi^{-1}\big(q^{-1}(U)\big)\Big),\qquad \pi_2^\sharp\colon\mathcal{A}\big(q^{-1}(U)\big)\to\mathcal{C}\Big(\pi_2^{-1}\big(q^{-1}(U)\big)\Big).$$

Since $\pi_1^{-1}\big(q^{-1}(U)\big)=\pi_2^{-1}\big(q^{-1}(U)\big)$, we may define a sheaf $\mathcal{B}$ on Y by setting

$$\mathcal{B}=\ker\left(\pi_1^\sharp-\pi_2^\sharp\right).$$

If we use the local coordinates introduced in Step 1, this reduces (locally) to Example 3. This proves that $(Y,\mathcal{B})$ is a supermanifold and that $q\colon(X,\mathcal{A})\to(Y,\mathcal{B})$ is the quotient of $(X,\mathcal{A})$ by ρ. ∎

4. Quotients of Berezin–Leĭtes–Kostant supermanifolds

It was observed in **[6]** that every equivalence relation R on a Berezin–Leĭtes–Kostant supermanifold determines in a unique way a smooth equivalence relation R_0 on the underlying smooth manifold, X_0: namely, R_0 is just the underlying smooth manifold of R.

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If R determines a quotient supermanifold of X, say Y, then R_0 determines a quotient smooth manifold of X_0, which is merely Y_0.

It turns out that the converse is also true, and thus for a Berezin–Leĭtes–Kostant supermanifold all obstructions to the existence of a quotient supermanifold are purely topological and dwell in the even sector. (See Theorem 4.4.)

To establish the theorem, we shall first obtain a result that is of interest on its own (Theorem 4.3): every equivalence relation on a Berezin–Leĭtes–Kostant supermanifold is submersive in the odd sector in the following sense.

DEFINITION 4.1: Say that a morphism of supermanifolds $f: (Y, \mathcal{B}) \to (X, \mathcal{A})$ is *submersive in the even* (resptively odd) sector if for all $y \in Y$ the restriction of the graded B-module morphism $f^{\sharp}_{f(y)}: T^*_{f(y)}(X, \mathcal{A}) \to T^*_y(Y, \mathcal{B})$ to $\left(T^*_{f(y)}(X, \mathcal{A})\right)_i$, $i = 0$ or 1 respectively, is injective.

Clearly, a morphism $f: (Y, \mathcal{B}) \to (X, \mathcal{A})$ between two supermanifolds is submersive if and only if it is submersive both in the even and in the odd sector.

LEMMA 4.2. *Let $f: (Y, \mathcal{B}) \to (X, \mathcal{A})$ be a morphism between BLK supermanifolds. Then f is submersive in the even sector if and only if the underlying morphism of the reduced superspaces (underlying smooth manifolds), $f_0: X_0 \to Y_0$, is submersive.*

PROOF: Indeed, it is well known and easily proved that the even sector of the graded tangent space to a BLK supermanifold at a point $x \in X$ is canonically isomorphic to the tangent space to the underlying manifold X_0 at x. Consequently, the similar statement is true of cotangent spaces. $\square$

THEOREM 4.3. *Let $\rho: R \to X \times X$ be an equivalence relation on a Berezin–Leĭtes–Kostant supermanifold X. Then each projection $\varpi_1, \varpi_2: (R, \mathcal{C}) \to (X, \mathcal{A})$ is submersive in the odd sector.*

PROOF: Assume the contrary. This means that for some $x, y \in X$ the restriction of the graded vector space morphism $\left(\varpi_1^{\sharp}\right)_x: T^*_x(X, \mathcal{A}) \to T^*_{(x,y)}(R, \mathcal{C})$ to $(T^*_x(X, \mathcal{A}))_1$ is not an injection. Choose a local coordinate system $x_1, \ldots, x_m, \xi_1, \ldots, \xi_n$ at x on X so that $\left(\varpi_1^{\sharp}\right)_x(d\xi_1) = 0$.

Let Z be a supermanifold of dimension $(0,1)$ having $\{x, y\}$ as the underlying topological space. Fix an odd generator for each stalk of Z, and denote such a generator, for x and y, respectively, by ξ and ζ. Now define a morphism $\iota: Z \to X$ as follows: the underlying map ι_0 is the embedding $\{x, y\} \subseteq X$, and the dual sheaf morphism $\iota^{\sharp}$ has the property that $\iota^{\sharp}_x(\xi_1) = \xi$, $\iota^{\sharp}_x(\xi_i) = 0$ for $i > 1$, while $\iota^{\sharp}_y$ is just any graded algebra epimorphism onto $\bigwedge(1)$. (Such an epimorphism exists because the odd sector of X is, by the very assumption of non-submersivity, nontrivial.) The morphism $\iota: Z \to X$ is a (closed) immersion of supermanifolds, and by Lemma 3.2, R induces an equivalence relation, $\widetilde{R}$, on Z. Notice that $\widetilde{R}$ is not submersive in the odd sector: indeed, $\left(\varpi_1|_{\widetilde{R}}\right)^{\sharp}_x(d\xi) = 0$.

The dimension of $\tilde{R}$ can neither exceed $(0,2) = \dim (Z \times Z)$ nor be $(0,0)$ because $\tilde{R}$ contains the $(0,1)$-dimensional diagonal of Z. But $\dim \tilde{R} \neq (0,2)$ either: since $\tilde{R}$ is a sub-supermanifold of $Z \times Z$, the graded cotangent modules at each point of $\tilde{R}$ are spanned by $\varpi_i^\sharp(d\xi)$, $i = 1,2$, yet at (x,y) one has $\varpi_1^\sharp(d\xi) = 0$. We conclude that $\dim \tilde{R} = (0,1)$ and $\left(\varpi_2^\sharp\right)_y(d\zeta) \neq 0$.

This enables us to describe the equivalence relation $\tilde{R}$ on Z. We already know that the underlying set of $\tilde{R}$ coincides with $Z \times Z = \{(x,x),(x,y),(y,x),(y,y)\}$. The algebra of superfunctions over each of the four open singletons is isomorphic to the Grassmann algebra $\bigwedge(1)$ of rank one, and we shall fix a generator for each copy of such an algebra, denoted respectively by $\eta_{(x,x)}, \eta_{(x,y)}, \eta_{(y,x)}$, and $\eta_{(y,y)}$. In what follows we shall sometimes suppress the indices and denote each fixed generator simply by η, since it never leads to a confusion. The underlying map of the supermanifold morphism $\rho: \tilde{R} \to Z \times Z$ is the identity, $\mathrm{Id}_{Z\times Z}$. The dual sheaf morphism $\rho^\sharp$ is fully described by the following:

$$\begin{aligned}
\rho^\sharp_{(x,x)}(\xi \otimes 1) &= \eta_{(x,x)} = \rho^\sharp_{(x,x)}(1 \otimes \xi),\\
\rho^\sharp_{(y,y)}(\zeta \otimes 1) &= \eta_{(y,y)} = \rho^\sharp_{(y,y)}(1 \otimes \zeta),\\
\rho^\sharp_{(x,y)}(\xi \otimes 1) &= 0, \quad \rho^\sharp_{(x,y)}(1 \otimes \zeta) = \eta_{(x,y)},\\
\rho^\sharp_{(y,x)}(\zeta \otimes 1) &= \eta_{(y,x)}, \quad \rho^\sharp_{(y,x)}(1 \otimes \xi) = 0.
\end{aligned}$$

Since ρ must be reflexive and symmetric, the above description is unique up to a renormalisation of the selected odd generator in each of the four participating Grassmann algebras.

This leads us to an explicit form for the projections $\varpi_i: \tilde{R}: Z$, $i = 1,2$:

$$\begin{aligned}
&\left(\varpi_1^\sharp\right)_{(x,x)}(\xi) = \eta, \left(\varpi_1^\sharp\right)_{(x,y)}(\xi) = 0,\\
&\left(\varpi_1^\sharp\right)_{(y,x)}(\zeta) = \eta, \left(\varpi_1^\sharp\right)_{(y,y)}(\zeta) = \eta,\\
&\left(\varpi_2^\sharp\right)_{(x,x)}(\xi) = \eta, \left(\varpi_2^\sharp\right)_{(x,y)}(\zeta) = \eta,\\
&\left(\varpi_2^\sharp\right)_{(y,x)}(\xi) = 0, \left(\varpi_2^\sharp\right)_{(y,y)}(\zeta) = \eta.
\end{aligned}$$

Let us now compute the fibre product $\tilde{R} \times_Z \tilde{R}$ of the morphisms $\varpi_2, \varpi_1: R \to Z$. The underlying topological space of it is the topological fibre product of the two standard projections $Z \times Z \to Z$, and as such, it can be identified with $Z \times Z \times Z$. The underlying mappings to the projections ϕ_1, ϕ_2 of the fibre product $\tilde{R} \times_Z \tilde{R}$ onto the first and the second factor respectively are of the form $\phi_{1,0} = \pi_1 \times \pi_2$, $\phi_{2,0} = \pi_2 \times \pi_3$, where $\pi_i: Z \times Z \times Z \to Z$, $i = 1,2,3$ are the standard projections onto the i-th factor.

The rule $\varpi_2 \circ \phi_1 = \varpi_1 \circ \phi_2$ from the definition of the fibre product is equivalent to the following collection of constraints upon the dual sheaf morphisms $\phi_i^\sharp$, $i = 1,2$:

$$\left(\phi_1^\sharp\right)_{(x,x,x)}(\eta) = \left(\phi_2^\sharp\right)_{(x,x,x)}(\eta),$$

$$\begin{aligned}
&\left(\phi_1^\sharp\right)_{(x,x,y)}(\eta) = 0,\\
&\left(\phi_1^\sharp\right)_{(x,y,y)}(\eta) = \left(\phi_2^\sharp\right)_{(x,y,y)}(\eta),\\
&\left(\phi_2^\sharp\right)_{(y,x,x)}(\eta) = 0,\\
&\left(\phi_1^\sharp\right)_{(y,y,x)}(\eta) = \left(\phi_2^\sharp\right)_{(y,y,x)}(\eta),\\
&\left(\phi_1^\sharp\right)_{(y,y,y)}(\eta) = \left(\phi_2^\sharp\right)_{(y,y,y)}(\eta).
\end{aligned}$$

One way to have those satisfied is to assume that as a supermanifold, $\tilde{R} \times_Z \tilde{R}$ has dimension $(0, 2)$ and the morphisms ϕ_i are defined as follows, where θ_1 and θ_2 (or, in full, $\theta_{1,(x,x,x)}$ et cetera) stand for arbitrary but fixed odd generators of the Grassmann algebras of superfunctions over each singleton $(a, b, c) \in Z \times Z \times Z$, $a, b, c \in Z$, isomorphic to $\bigwedge(2)$:

$$\begin{aligned}
&\left(\phi_1^\sharp\right)_{(x,x,x)}(\eta) = \theta_1 = \left(\phi_2^\sharp\right)_{(x,x,x)}(\eta),\\
&\left(\phi_1^\sharp\right)_{(x,x,y)}(\eta) = 0, \left(\phi_2^\sharp\right)_{(x,x,y)}(\eta) = \theta_1,\\
&\left(\phi_1^\sharp\right)_{(x,y,x)}(\eta) = \theta_1, \left(\phi_2^\sharp\right)_{(x,y,x)}(\eta) = \theta_2,\\
&\left(\phi_1^\sharp\right)_{(x,y,y)}(\eta) = \theta_1 = \left(\phi_2^\sharp\right)_{(x,y,y)}(\eta),\\
&\left(\phi_1^\sharp\right)_{(y,x,x)}(\eta) = \theta_1, \left(\phi_2^\sharp\right)_{(y,x,x)}(\eta) = 0,\\
&\left(\phi_1^\sharp\right)_{(y,x,y)}(\eta) = \theta_1, \left(\phi_2^\sharp\right)_{(y,x,y)}(\eta) = \theta_2,\\
&\left(\phi_1^\sharp\right)_{(y,y,x)}(\eta) = \theta_1 = \left(\phi_2^\sharp\right)_{(y,y,x)}(\eta),\\
&\left(\phi_1^\sharp\right)_{(y,y,y)}(\eta) = \theta_1 = \left(\phi_2^\sharp\right)_{(y,y,y)}(\eta).
\end{aligned}$$

It is easy to see that this choice of the structure sheaf on $\tilde{R} \times_Z \tilde{R}$ and the morphisms ϕ_i is in fact 'universal,' that is, the triple $\left(\tilde{R} \times_Z \tilde{R}, \phi_1, \phi_2\right)$ satisfies the universality condition from our definition of the fibre product of supermanifolds with $q_i = \phi_i$, $i = 1, 2$.

The transitivity of $\tilde{R}$ implies the existence of a morphism $\phi_0 \colon \tilde{R} \times_Z \tilde{R} \to \tilde{R}$ such that

$$\varpi_1 \circ \phi_0 = \varpi_1 \circ \phi_1 \quad \text{and} \quad \varpi_2 \circ \phi_0 = \varpi_2 \circ \phi_2 \,. \tag{4.1}$$

The underlying set-theoretic map of ϕ_0 is $\pi_1 \times \pi_3 \colon Z \times Z \times Z \to Z \times Z$. Therefore, at the point $(y, x, y) \in Z \times Z \times Z$ one must have:

$$(\phi_0)^\sharp_{(y,x,y)} \circ (\varpi_1)^\sharp_{(y,y)}(\zeta) = (\phi_1)^\sharp_{(y,x,y)} \circ \left(\varpi_1^\sharp\right)_{(y,x)}(\zeta) \tag{4.2}$$

and

$$(\phi_0)^\sharp_{(y,x,y)} \circ (\varpi_2)^\sharp_{(y,y)}(\zeta) = (\phi_2)^\sharp_{(y,x,y)} \circ \left(\varpi_2^\sharp\right)_{(x,y)}(\zeta) \,. \tag{4.3}$$

The equation (4.2) implies $(\phi_0)^\sharp_{(y,x,y)}(\eta) = \theta_1$, and the equation (4.3) yields modulo this observation that $\theta_1 = (\phi_2)^\sharp_{(y,x,y)}(\eta) = \theta_2$, a contradiction. ∎

THEOREM 4.4. *Let X be a Berezin–Leĭtes–Kostant supermanifold (BLK supermanifold). An equivalence relation $\rho: (R, \mathcal{C}) \to (X, \mathcal{A}) \times (X, \mathcal{A})$ on X admits a quotient supermanifold if and only if the underlying equivalence relation R_0 on X_0 admits a quotient smooth manifold.*

PROOF: The easier implication $\Rightarrow$ was observed in [6]. We shall therefore concentrate on $\Leftarrow$. Assume that R_0 admits a quotient smooth manifold. As is known in differential topology (and follows from our Theorem 3.6 in the case where the ground algebra $B = \mathbb{R}$ and the dimension is purely even), the embedding $R_0 \hookrightarrow X_0 \times X_0$ is then a closed immersion, and the canonical projections $\varpi_1, \varpi_2: R_0 \to X_0$ are smooth submersions.

Since by the definition of an equivalence relation ρ is an immersion, it is then a composition of a closed immersion with an open immersion, and therefore $\rho^\sharp$ is an epimorphism. Since in addition ρ is a closed homeomorphic embedding of R into $X_0 \times X_0$, one concludes that ρ is a closed immersion of supermanifolds and thus R is a closed sub-supermanifold of $X \times X$ and the condition (i) from Theorem 3.6 holds.

To establish the condition (ii) about the canonical projections $\varpi_1, \varpi_2: (R, \mathcal{C}) \to (X, \mathcal{A})$ being submersive it is enough now to apply Theorem 4.3 and Lemma 4.2. ∎

5. FINAL DISCUSSION

The underlying topological space of a supermanifold X supports a natural structure of a smooth manifold of dimension $m \dim B_0 + n \dim B_1$, which we shall denote by X_0. (Recall that the ground algebra, B, was assumed finite dimensional as a real vector space.) The correspondence $X \mapsto X_0$ is functorial. The tangent space to X_0 at a point $x \in X_0$ is canonically isomorphic (as a real vector space) to the even sector $(T_x X)_0$, of the tangent B-module to the supermanifold X at x, and a similar statement holds for the cotangent spaces and B-modules. Any equivalence relation R on X determines an equivalence relation R_0 on X_0, and it is not difficult to check that the morphisms ϖ_i are immersive in the even sector if and only if the projections from R_0 to X_0 are immersions.

It is therefore most natural to state the following conjecture: *an equivalence relation $\rho: (R, \mathcal{C}) \to (X, \mathcal{A}) \times (X, \mathcal{A})$ on a supermanifold X admits a quotient supermanifold if and only if the underlying equivalence relation R_0 on X_0 admits a quotient smooth manifold.*

Unfortunately, the proof of Theorem 4.3 does not work in this more general setting because in that case simple examples show that one cannot in general choose a coordinate system $x_1, \ldots, x_m, \xi_1, \ldots, \xi_n$ at a point x on X so as to satisfy $\left(\varpi_1^\sharp\right)_x (d\xi_1) = 0$, as we did in the first paragraph of the proof. It is, however, easy to establish the following partial

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result, backing our conjecture: *every equivalence relation on a supermanifold of purely odd dimension determines a quotient supermanifold.*

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