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SELF-COMPLEMENTARY GENERALIZED ORBITS OF A PERMUTATION GROUP

BY
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Dedicated to Ronald C. Read on his 50th birthday

ABSTRACT. A permutation group A of degree n acting on a set X has a certain number of orbits, each a subset of X. More generally, A also induces an equivalence relation on $X^{(k)}$, the set of all k-subsets of X, and the resulting equivalence classes are called k-orbits of A, or *generalized orbits*. A *self-complementary k-orbit* is one in which for every k-subset S in it, $X-S$ is also in it. Our main results are two formulas for the number $s(A)$ of self-complementary generalized orbits of an arbitrary permutation group A in terms of its cycle index. We show that self-complementary graphs, digraphs, and relations provide special classes of self-complementary generalized orbits.

1. **Statement of the theorem.** Any permutation group A acting on a finite object set X partitions X into orbits in the usual way. It also induces a partition of $X^{(2)}$, the collection of all 2-subsets of X, and in general of $X^{(k)}$, for all k, $1 \leq k \leq |X| = p$. We call the subsets of $X^{(k)}$ so obtained the *k-orbits* of A, so that 1-orbits are the usual orbits of A. A *generalized orbit* of A is a k-orbit for some k. Then a subset S of X with $|S|=k$ belongs to a *self-complementary k-orbit* if $X-S$ is in the same k-orbit as S. Of course this can only happen when $k=p/2$, so that self-complementary generalized orbits cannot occur in permutation groups of odd degree.

Our main object is to derive a formula for the number $s(A)$ of self-complementary generalized orbits of an arbitrary permutation group A in terms of its cycle structure. In order to do this, we require the definition of the cycle index of A, which we now state for the sake of completeness. A permutation $\alpha \in A$ has *cycle type* $(j_1, j_2, \ldots, j_p)$, where $j_k = j_k(\alpha)$ is the number of k-cycles in α when expressed in the usual disjoint cycle form. The permutation group A of degree p has as its *cycle index*, defined by Pólya [7], the following polynomial in p indeterminates y_k:

$$Z(A) = Z(A; y_1, y_2, \ldots, y_p) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_k y_k^{j_k(\alpha)}. \tag{1}$$

We can now state the principal result.

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THEOREM 1. *The number of self-complementary generalized orbits of A is:*

$$s(A) = Z(A; 0, 2, 0, 2, \ldots), \tag{2}$$

which is obtained from $Z(A)$ on replacing every y_k with odd subscript by 0 *and every y_k with even subscript by* 2.

The formulas found by Read [8] for the number of self-complementary graphs and directed graphs provide a special case of this formula. The proof of this theorem to be given in the next section is based on that of Read but applies the process to an arbitrary permutation group instead of to the special permutation group known as the pair group of the symmetric group.

2. **Proof of Theorem 1.** We begin by reviewing the derivation by Read [8] of the formula for the number of self-complementary graphs with a given number of points; see also [3, p. 192]. The first step is to define a new equivalence relation on graphs in which $G_1 \sim G_2$ if either the two graphs are isomorphic or one of them is isomorphic to the complement of the other. Let c_p be the number of such equivalence classes of graphs with p points.

We need to compare this equivalence relation with that of isomorphism for a given graph G. There are two possibilities: Either G and its complement $\bar{G}$ are different or they are isomorphic. When they are different, the relation counts them once even though they are two graphs up to isomorphism, but when G is self-complementary, the relation $\sim$ still counts this just once. Let g_p be the number of different graphs with p points (up to isomorphism) and let s_p be the number of self-complementary graphs with p points. Then we have just observed that $g_p = 2c_p - s_p$ so that

$$s_p = 2c_p - g_p. \tag{3}$$

It has been shown [1] that the appropriate permutation group for counting graphs is $A = S_p^{(2)}$, which is called the *pair group* of the symmetric group S_p because it acts on the set of all point pairs:

$$g_p = Z(S_p^{(2)}; 2, 2, 2, 2, \ldots). \tag{4}$$

It follows from the Power Group Enumeration Theorem [4] that

$$c_p = \tfrac{1}{2}[Z(S_p^{(2)}; 2, 2, 2, 2, \ldots) + Z(S_p^{(2)}; 0, 2, 0, 2, \ldots)]. \tag{5}$$

Substituting (4) and (5) into (3), Read found that

$$s_p = Z(S_p^{(2)}; 0, 2, 0, 2, \ldots). \tag{6}$$

The enumeration of self-converse digraphs in [5] uses analogous considerations.

It was shown in [2] that the substitution of $1+x$ into the cycle index $Z(A)$ of an arbitrary permutation group A results in a polynomial in which the coefficient of x^r is the number of r-orbits of A. By definition, this substitution results in

$$f_A(x) = Z(A; 1+x, 1+x^2, \ldots, 1+x^p). \tag{7}$$

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Therefore,

$$f_A(1) = Z(A; 2, 2, 2, 2, \ldots) \tag{8}$$

is the total number of equivalence classes of subsets of X. By precisely the same reasoning as in Read's derivation of the formula (6), the number $s(A)$ of self-complementary generalized orbits of A is given by $Z(A; 0, 2, 0, 2, \ldots)$, proving (2) and Theorem 1.

3. **Alternating sum formula.** Comparing (2) with (7), we see that

$$s(A) = f_A(-1). \tag{9}$$

This specializes at once to the equations (4) and (6) for graphs and self-complementary graphs. Writing g_{pq} for the number of graphs with p points and q lines, we have defined, see [3, p. 185], the counting polynomial $g_p(x)$ for p-point graphs by

$$g_p(x) = \sum_q g_{pq} x^q, \tag{10}$$

where the sum runs from $q=0$ to $p(p-1)/2$. The formula obtained by Pólya for $g_p(x)$, as in [1], is

$$g_p(x) = Z(S_p^{(2)}; 1+x, 1+x^2, 1+x^3, \ldots). \tag{11}$$

On taking (11) and (6) as special cases of (7) and (2), equation (9) expressed in terms of graphs becomes

$$s_p = g_p(-1), \tag{12}$$

which in view of (10) gives

$$s_p = \sum_q (-1)^q g_{pq}. \tag{13}$$

This simple but useful formula appears not to have been explicitly observed previously. We now illustrate (13) using the tables for g_{pq} in [3; p. 214]:

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1-1 = 0 \\
s_3 &= 1-1+1-1 = 0 \\
s_4 &= 1-1+2-3+2-1+1 = 1 \\
s_5 &= 1-1+2-4+6-6+6-4+2-1+1 = 2.
\end{aligned}$$

Clearly $s_p=0$ when $p\equiv 2$ or 3 (mod 4) because in these cases, the number $p(p-1)/2$ of lines in K_p is odd. The expression for s_3 above illustrates the corresponding behavior of the coefficients, and this is also seen for s_6, s_7, s_{10}, s_{11}, etc. In Figure 1, we show the self-complementary graphs for $p=4$ and 5.

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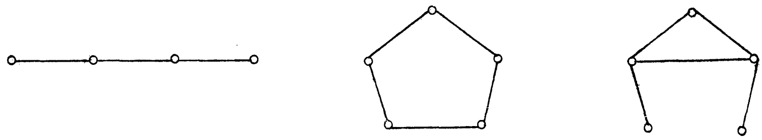

FIGURE 1. The smallest non-trivial self-complementary graphs

The structure of self-complementary graphs was developed in detail by Ringel [9] and Sachs [10].

We now apply (9) to write an equation which generalizes (13) to any permutation group A, of degree n. The proof proceeds exactly like that of (13).

THEOREM 2. *Let $a_r(A)$ be the number of r-orbits of A; then*

$$s(A) = \sum_{r=0}^{n} (-1)^r a_r(A). \tag{14}$$

The specialization of (13) to obtain the number of self-complementary digraphs is assisted by the data in [6] and in [3, Appendix 2]. Let $\bar{s}_p$ be the number of self-complementary p-point digraphs and d_{pq} be the number of digraphs with p points and q arcs (directed lines). Then (14) becomes

$$\bar{s}_p = \sum_q (-1)^q d_{pq}, \tag{15}$$

so that

$$\bar{s}_1 = 1$$

$$\bar{s}_2 = 1-1+1 = 1$$

$$\bar{s}_3 = 1-1+4-4+4-1+1 = 4.$$

Surprisingly all four digraphs with 3 points and 4 arcs are self-complementary, as a glance at Figure 2 can verify.

To illustrate (14) with binary relations, we show in Figure 3 all of these with only two points.

We denote the number of relations with p points and q arcs (with loops permitted) by r_{pq} and we write

$$r_p(x) = \sum_{q=0}^{p^2} r_{pq} x^q. \tag{16}$$

Then by (9), the number of self-complementary p-point relations is $r_p(-1)$. Thus

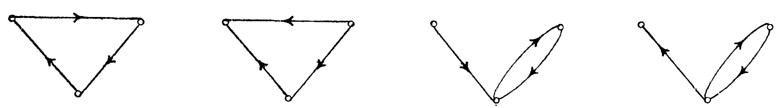

FIGURE 2. The self-complementary 3-point diagraphs

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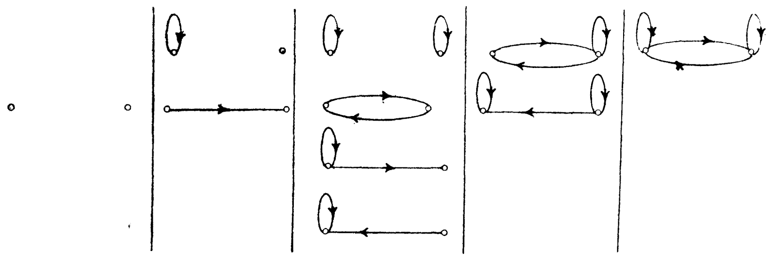

FIGURE 3. The relations with 2 points

using Figure 3 we find that there are

$$r_2(-1) = 1-2+4-2+1 = 2$$

self-complementary relations with just two points, and these are the last two of the four relations shown with two arcs.

4. **Special permutation groups.** We now apply equation (2) of Theorem 1 to some special permutation groups of even degree $2m$, including the cyclic group C_{2m}, the dihedral group D_{2m}, the identity group E_{2m}, the symmetric group, and the alternating group. It is well known (see [3, p. 184]) that

$$Z(C_{2m}) = \frac{1}{2m}\sum_{k|2m}\phi(k)y_k^{2m/k}, \tag{17}$$

$$Z(D_{2m}) = \tfrac{1}{2}Z(C_{2m}) + \tfrac{1}{4}(y_2^m+y_1^2y_2^{m-1}), \tag{18}$$

$$Z(E_{2m}) = y_1^{2m}. \tag{19}$$

Applying (2) to (17), we find at once that the number of self-complementary generalized orbits of a cyclic group is

$$s(C_{2m}) = \frac{1}{2m}\sum_{d|m}\phi(2d)2^{m/d}, \tag{20}$$

which reduces for $m=r$, an *odd* prime, to the pleasing result that

$$s(C_{2r}) = 1+\frac{2^{r-1}-1}{r}. \tag{21}$$

When the vertices of a polygon are colored green or red, the result has been called a *necklace* [3, p. 192]. If green is considered presence and red absence of a bead, we can speak of the *complement of a necklace*. Then the number of self-complementary necklaces becomes precisely the number of self-complementary generalized orbits of the dihedral group:

$$s(D_{2m}) = \tfrac{1}{2}s(C_{2m}) + 2^{m-2}. \tag{22}$$

As a consequence of the way in which $s(C_{2m})$ occurs in the right side of (22), we

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note that it is always an even number. For the identity group, we note that

$$s(E_{2m}) = 0, \tag{23}$$

since y_1 in (19) has an odd subscript. Finally, for the symmetric group S_{2m} and the alternating group A_{2m}, we find just one self-complementary generalized orbit in each case.

5. **Conclusion.** Just as for many other enumeration results, we can calculate $s(A)$, provided we know the cycle index $Z(A)$, but this gives no clue to the process of constructing the self-complementary generalized orbits of A.

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