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EHRLICH'S THEOREM FOR GROUPS

YUANLIN LI and W. K. NICHOLSON✉

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Abstract

A group G is called morphic if every endomorphism $\alpha : G \to G$ for which $G\alpha \lhd G$ satisfies $G/G\alpha \cong \ker(\alpha)$. Call an endomorphism $\alpha \in \text{end}(G)$ regular if $\alpha\beta\alpha = \alpha$ for some $\beta \in \text{end}(G)$, and call α unit regular if β can be chosen to be an automorphism of G. The main purpose of this paper is to prove the following group-theoretic analogue of a theorem of Ehrlich: if G is a morphic group, an endomorphism $\alpha : G \to G$ for which $G\alpha \lhd G$ is unit regular if and only if it is regular. As an application, a cancellation theorem is proved that characterizes the morphic groups among those with regular endomorphism monoids.

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If R is a ring, an endomorphism α of an R-module ${}_RM$ is called (von Neumann) *regular* if $\alpha\beta\alpha = \alpha$ for some endomorphism β. In 1976 Gertrude Ehrlich called α *unit regular* if β can be chosen to be an automorphism of the module M. She showed that α is unit regular if and only if it is regular and has the property that $M/M\alpha \cong \ker(\alpha)$, and she went on to relate these endomorphisms to certain cancellation theorems. In this paper we will extend these notions to the category of groups, and prove the analogue of Ehrlich's theorem.

If G is a group, we write $\text{end}(G)$ for the monoid of endomorphisms $\alpha : G \to G$, and we write $\text{aut}(G)$ for the group of automorphisms of G. Group homomorphisms will be written on the right. As usual, we write $H \lhd G$ to indicate that H is a normal subgroup of G, and we write C_n for the cyclic group of order n. If H and K are subgroups of G, we write $G = H \odot K$ to mean that $H \lhd G$, $K \lhd G$, $G = HK$ and $H \cap K = 1$; and in this case we say that H and K are *direct factors* of the group G. We say that G is a *semidirect product* of K by H (and write $G = K \rtimes H$) if $K \lhd G$, $G = KH$ and $H \cap K = 1$; in this case we say that K is a *semidirect factor* of G.

We begin with a few facts about the groups of interest here. The first result characterizes the group endomorphisms under discussion.

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LEMMA 1. *If G is a group, the following are equivalent for $\alpha \in \mathrm{end}(G)$.*

(1) $G\alpha \lhd G$ *and* $G/G\alpha \cong \ker(\alpha)$.
(2) *There exists* $\beta \in \mathrm{end}(G)$ *with* $\ker(\alpha) = G\beta$ *and* $G\alpha = \ker(\beta)$.
(3) *There exists* $\beta \in \mathrm{end}(G)$ *with* $\ker(\alpha) \cong G\beta$ *and* $G\alpha = \ker(\beta)$.

PROOF.

(1) $\Rightarrow$ (2). If $\sigma : G/G\alpha \to \ker(\alpha)$ is an isomorphism, let $\varphi : G \to G/G\alpha$ denote the coset map and define $\beta = \varphi\sigma$. Then $\beta \in \mathrm{end}(G)$, $G\beta = (G/G\alpha)\sigma = \ker(\alpha)$, and $\ker(\beta) = G\alpha$ because σ is one-to-one.

(2) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (1). Suppose that we are given β such that $\ker(\alpha) \cong G\beta$ and $G\alpha = \ker(\beta)$. Then $G\alpha = \ker(\beta) \lhd G$ and then $G/G\alpha = G/\ker(\beta) \cong G\beta \cong \ker(\alpha)$. □

An endomorphism $\alpha \in \mathrm{end}(G)$ is called *morphic* if it satisfies the conditions in Lemma 1. We say that α is *normal* if $G\alpha \lhd G$, so α is morphic if and only if it is normal and $G/G\alpha \cong \ker(\alpha)$.

Clearly every automorphism is morphic by (1) of Lemma 1, as is the *trivial endomorphism* θ of G defined by $g\theta = 1$ for each $g \in G$. More generally, if $G = H \times K$ the projection $(h, k) \mapsto (h, 1)$ is morphic. If $\alpha \in \mathrm{end}(G)$ is morphic then it is easy to see that α is one-to one if and only if it is onto, so being morphic is a finiteness condition on α.

LEMMA 2. *The following are equivalent for a group G.*

(1) *Every normal endomorphism of G is morphic.*
(2) *If $K \lhd G$ is such that $G/K \cong N \lhd G$, then $G/N \cong K$.*

PROOF.

(1) $\Rightarrow$ (2). If $\tau : G/K \to N$ is an isomorphism, define $\alpha \in \mathrm{end}(G)$ by $\alpha = \varphi\tau$ where $\varphi : G \to G/K$ is the coset map. Then $G\alpha = N \lhd G$, so α is morphic by (1). Hence, by Lemma 1, $G/N = G/G\alpha \cong \ker\alpha = K$.

(2) $\Rightarrow$ (1). Let $G\alpha \lhd G$, $\alpha \in \mathrm{end}(G)$. Since $G/\ker(\alpha) \cong G\alpha$, (2) gives $G/G\alpha \cong \ker(\alpha)$. □

A group is called a *morphic group* if every normal endomorphism is morphic. It is routine to verify that $C_2 \times C_4$ is not morphic, so not every finite abelian group is morphic. However, an (additive) abelian group is morphic if and only if it is morphic as a $\mathbb{Z}$-module, so [4, Theorem 26] gives the following example.

EXAMPLE 3. A finitely generated abelian group is morphic if and only if it is finite and each p-primary component has the form $(C_{p^k})^n$ for some $n \geq 0$ and $k \geq 1$.

Thus the abelian groups $(C_{p^k})^n$ are morphic for each prime p and integer $k \geq 0$. Note that $C_4 \times C_4$ is morphic by Example 3, but its (normal) subgroup $C_2 \times C_4$ is *not* morphic.

While C_n is morphic for each $n \geq 2$, the infinite cyclic group $C_\infty = \langle a \rangle$ is not morphic by (2) of Lemma 2 because $C_\infty/\langle 1 \rangle \cong \langle a^2 \rangle$ but $C_\infty\langle a^2 \rangle \ncong \langle 1 \rangle$. On the other

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hand, the infinite (additive) group $\mathbb{Q}$ is morphic because $\mathbb{Q}/K \cong N \subseteq \mathbb{Q}$ implies that $K = 0$, $\mathbb{Q}$ because $\mathbb{Q}/K$ is torsion (if $0 \neq (m/n) \in K$ and $(a/d) \in \mathbb{Q}$ then $md((a/d) + K) = 0$).

A routine application of (2) of Lemma 1 gives the following example.

EXAMPLE 4. Every simple group G is morphic.

More generally, write $S = S_1 \times S_2 \times \cdots \times S_n$ where the S_i are simple nonabelian groups. Then S is morphic. Indeed, $S \times G$ is morphic if G is morphic with the descending chain condition on subgroups and no S_i is an image of a normal subgroup of G. In particular, $S \times G$ is morphic for any morphic, finite abelian group. These results, and many others, are discussed in another paper [3].

Our present interest is in formulating the group-theoretic version of Ehrlich's theorem. An endomorphism $\pi \in \operatorname{end}(G)$ is called an *idempotent* if $\pi^2 = \pi$.

LEMMA 5. *Let G be a group.*

(1) *If* $\pi^2 = \pi \in \operatorname{end}(G)$ *then* $G = \ker(\pi) \rtimes G\pi$.
(2) *If* $\pi^2 = \pi \in \operatorname{end}(G)$ *is normal then* π *is morphic; indeed* $G = \ker(\pi) \odot G\pi$.
(3) *An idempotent endomorphism need not be normal.*

PROOF.

(1). If $\pi^2 = \pi$ and $g \in G$ then $[g(g^{-1}\pi)]\pi = g\pi \cdot (g^{-1})\pi^2 = 1$ and it follows that $G = \ker(\pi) \cdot G\pi$. Moreover, $\ker(\pi) \cap G\pi = 1$ because $g\pi \in \ker(\pi)$ implies that $1 = (g\pi)\pi = g\pi$.

(2). If $G\pi \lhd G$, then $G = G\pi \odot \ker(\pi)$ by (1). But then $G/G\pi \cong \ker(\pi)$ so π is morphic by Lemma 1.

(3). Consider the dihedral group D_3 with presentation $D_3 = \langle a, b\rangle$ where $|a| = 3$, $|b| = 2$ and $aba = b$. If we write $A = \langle a\rangle$ then $A \lhd G$ and $G/A \cong C_2 \cong \langle b\rangle$. Define $\pi : G \to G$ by

$$g\pi = \begin{cases} 1 & \text{if } g \in A \\ b & \text{if } g \notin A. \end{cases}$$

Then π is an idempotent endomorphism, but π is not normal because $G\pi = \langle b\rangle \ntriangleleft G$. $\square$

We note in passing that D_3 is a morphic group; in fact it can be shown [3] that the dihedral group D_n is morphic if and only if n is odd.

The converse of (1) in Lemma 5 is also true: if $G = K \rtimes H$ then $G = KH$ and we define $\pi : G \to G$ by $(kh)\pi = h$, then π is well defined because $K \cap H = 1$, and π is an endomorphism of G because $K \lhd G$. Now it is clear that $\pi^2 = \pi$, $\ker(\pi) = K$ and $G\pi = H$, and hence that $G/K \cong H$. Note that $H \lhd G$ if and only if π is normal.

Following the terminology for modules, a group endomorphism $\alpha \in \operatorname{end}(G)$ will be called *regular* if $\alpha\beta\alpha = \alpha$ for some $\beta \in \operatorname{end}(G)$. Hence, every idempotent is regular, so Lemma 5 presents a regular endomorphism that is not normal. Automorphisms are examples of normal, regular endomorphisms, as is the trivial endomorphism. Our interest here is primarily in normal regular endomorphisms.

A theorem of Azumaya [1] asserts that if α is an endomorphism of a module ${}_RM$ over a ring R, then α is regular if and only if both $M\alpha$ and $\ker(\alpha)$ are direct summands of M. The group-theoretic version of Azumaya's theorem seems to be the following (where we extend the result to an arbitrary group homomorphism with no extra effort).

THEOREM 6. *Let $\alpha : G \to H$ be a group homomorphism. The following are equivalent.*

(1) *α is normal and $\alpha\beta\alpha = \alpha$ for some $\beta : H \to G$.*
(2) $\ker(\alpha)$ *is a semi-direct factor of G and $G\alpha$ is a direct factor of H.*

PROOF.

(1) $\Rightarrow$ (2). Let $\alpha\beta\alpha = \alpha$ as in (1). We may assume that also $\beta\alpha\beta = \beta$ (replace β by $\beta' = \beta\alpha\beta$). Consider the idempotents $\pi = \alpha\beta \in \operatorname{end}(G)$ and $\tau = \beta\alpha \in \operatorname{end}(H)$. Then $H\tau = G\alpha \lhd H$ by hypothesis, so τ is a normal idempotent in $\operatorname{end}(H)$. Thus Lemma 5 shows that $G\alpha = H\tau$ is a direct factor of H.

Turning to $\ker(\alpha)$, we show that $G = \ker(\alpha) \rtimes H\beta$. If $g \in G$ then $g\alpha = (g\alpha\beta)\alpha$ so we have $g(g\alpha\beta)^{-1} \in \ker(\alpha)$. It follows that $G = \ker(\alpha) \cdot H\beta$. Next, suppose that $g \in \ker(\alpha) \cap H\beta$, say $g = h\beta$, $h \in H$. Then $1 = g\alpha = h\beta\alpha$ so, since $\beta\alpha\beta = \beta$, we have $1 = 1\beta = (h\beta\alpha)\beta = h\beta = g$. This shows that $\ker(\alpha) \cap H\beta = 1$.

(2) $\Rightarrow$ (1). Let $G = \ker(\alpha) \rtimes X$ and $H = G\alpha \odot Y$ where $X \subseteq G$ and $Y \subseteq H$ are subgroups. Then $G\alpha = X\alpha$, so $H = X\alpha \odot Y$. Define

$$\beta : H = X\alpha \odot Y \to G \quad \text{by} \quad [(x\alpha)y]\beta = x \text{ where } x \in X \text{ and } y \in Y.$$

This is well defined because $H = X\alpha \odot Y$ and $\ker(\alpha) \cap X = 1$. With this we can show that β is a group homomorphism:

$$\begin{aligned}((x\alpha)y \cdot (x_1\alpha)y_1)\beta &= ((x\alpha)(x_1\alpha) \cdot yy_1)\beta = ((xx_1)\alpha \cdot yy_1)\beta \\ &= xx_1 = ((x\alpha)y)\beta \cdot ((x_1\alpha)y_1)\beta.\end{aligned}$$

Finally, we verify that $\alpha\beta\alpha = \alpha$. If $g \in G$, write $g\alpha = x\alpha$ where $x \in X$. Then

$$g(\alpha\beta\alpha) = (x\alpha)\beta\alpha = ((x\alpha \cdot 1)\beta)\alpha = (x)\alpha = g\alpha.$$

Since $g \in G$ was arbitrary, this shows that $\alpha\beta\alpha = \alpha$. □

Note that the idempotent π in (3) of Lemma 5 is a regular endomorphism that is not normal.

Again following module terminology, $\alpha \in \operatorname{end}(G)$ is called *unit regular* if $\alpha\sigma\alpha = \alpha$ for some automorphism σ of the group G. Hence, the trivial endomorphism is unit regular, as are all automorphisms and idempotents. Note that part (3) of Lemma 5 shows that a unit regular endomorphism need not be normal. We need the following characterization of these unit regular maps.

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LEMMA 7. *If* $\alpha \in \operatorname{end}(G)$ *then the following are equivalent.*

(1) α *is unit regular.*
(2) $\alpha = \pi\sigma$ *where* $\pi^2 = \pi \in \operatorname{end}(G)$ *and* $\sigma \in \operatorname{aut}(G)$.
(3) $\alpha = \sigma\pi$ *where* $\pi^2 = \pi \in \operatorname{end}(G)$ *and* $\sigma \in \operatorname{aut}(G)$.

PROOF. We prove that (1) is equivalent to (2); the proof that (1) and (3) are equivalent is analogous.

(1) $\Rightarrow$ (2). If $\alpha = \alpha\sigma\alpha$ where $\sigma \in \operatorname{aut}(G)$, write $\pi = \alpha\sigma$. Then $\pi^2 = \pi$ and $\alpha = \pi\sigma^{-1}$.

(2) $\Rightarrow$ (1). If $\alpha = \pi\sigma$ where $\pi^2 = \pi$ and $\sigma \in \operatorname{aut}(G)$, then $\alpha\sigma^{-1}\alpha = \pi\alpha = \alpha$. □

Before proceeding, we need a technical lemma.

LEMMA 8. *If* $\alpha \in \operatorname{end}(G)$ *is morphic, so also are* $\alpha\sigma$ *and* $\sigma\alpha$ *for every* $\sigma \in \operatorname{aut}(G)$.

PROOF. Since $G/G\alpha \cong \ker(\alpha)$, these follow from:

$$G/G\alpha\sigma = G\sigma/(G\alpha)\sigma \cong G/G\alpha \cong \ker(\alpha) = \ker(\alpha\sigma),$$
$$G/G\sigma\alpha = G/G\alpha \cong \ker(\alpha) \cong [\ker(\alpha)]\sigma^{-1} = \ker(\sigma\alpha). \qquad \square$$

The study of morphic rings was motivated by a result of Ehrlich [2]: a regular endomorphism of a module ${}_RM$ is unit regular if and only if it is morphic. Here is the analogue of Ehrlich's theorem for groups.

THEOREM 9. *Let* G *be a group and let* $\alpha \in \operatorname{end}(G)$ *be normal. Then* α *is unit regular if and only if it is both regular and morphic.*

PROOF. Let α be unit regular. By Lemma 7, write $\alpha = \sigma\pi$ where $\pi^2 = \pi$ and $\sigma \in \operatorname{aut}(G)$. Then π is normal because $G\pi = G\alpha \lhd G$, and so π is morphic by Lemma 5. But then α is morphic by Lemma 8.

Conversely, assume that α is regular and morphic. By Theorem 6, let $G = \ker(\alpha) \rtimes X$ and $G = G\alpha \odot Y$ where X and Y are subgroups of G. Then $G\alpha = X\alpha$, so $G = X\alpha \odot Y$. Since α is morphic, we have $\ker(\alpha) \cong G/G\alpha = G/X\alpha \cong Y$, so let $\gamma : Y \to \ker(\alpha)$ be an isomorphism. With this, define

$$\sigma : G = X\alpha \odot Y \to G \quad \text{by} \quad (x\alpha \cdot y)\sigma = x \cdot y\gamma.$$

As in the proof of Theorem 6, this is a well-defined endomorphism and $\alpha\sigma\alpha = \alpha$. Finally, $\ker(\sigma) = 1$ because γ is one-to-one and $\ker(\alpha) \cap X = 1$, and $G\sigma = X \cdot Y\gamma = X \cdot \ker(\alpha) = G$. Hence, σ is an automorphism of G. □

We can now prove the group-theoretic analogue of a theorem of Ehrlich that characterizes the morphic groups among the groups with a regular endomorphism monoid that enjoy a certain cancellation property.

THEOREM 10. *Let* G *be a group and assume that every endomorphism in* $\operatorname{end}(G)$ *is normal and regular. Then the following conditions are equivalent.*

(1) G *is morphic.*
(2) *If* $G = K \odot Y = K_1 \rtimes Y_1$ *where* $K \lhd G$, $K_1 \lhd G$, *and if* $K \cong Y_1$, *then* $Y \cong K_1$.

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PROOF.

(1) $\Rightarrow$ (2). Given the set-up in (2), we have $G/K_1 \cong Y_1 \cong K$. Hence, $G/K \cong K_1$ by (1). But then (2) follows because $Y \cong G/K$.

(2) $\Rightarrow$ (1). Let $\alpha \in \mathrm{end}(G)$. Since α is normal and regular by hypothesis, Theorem 6 gives

$$G = G\alpha \odot Y = \ker(\alpha) \rtimes X$$

where Y and X are subgroups of G. Then $G\alpha \cong G/\ker(\alpha) \cong X$ so, by (2), we obtain $Y \cong \ker(\alpha)$. But $Y \cong G/G\alpha$, and (1) follows. □

We conclude with a slight variation on the cancellation in Theorem 10.

THEOREM 11. *Let G be a morphic group. If $G = K \odot H = K_1 \rtimes H_1$ where $K \lhd G$, $K_1 \lhd G$ and $K \cong K_1$ then $H \cong H_1$.*

PROOF. Let $\sigma : K \to K_1$ be an isomorphism, and use it to define $\alpha : G = K \odot H \to G$ by $(kh)\alpha = k\sigma$ for all $k \in K$ and $h \in H$. As in the proof of Theorem 6, α is a well-defined group homomorphism, and α is normal because $G\alpha = K\sigma = K_1 \lhd G$. Since G is morphic, then $H_1 \cong G/K_1 = G/G\alpha \cong \ker(\alpha) = H$. □

QUESTION. If $\mathrm{end}(G)$ is regular, does the condition in Theorem 11 imply that G is morphic?

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YUANLIN LI, Department of Mathematics, Brock University, St Catharines, Ontario, Canada L2S 3A1
e-mail: Yuanlin.Li@brocku.ca

W. K. NICHOLSON, Department of Mathematics, University of Calgary, Calgary, Canada T2N 1N4
e-mail: wknichol@ucalgary.ca

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