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Abstract

A special linear Grassmann variety $\mathrm{SGr}(k,n)$ is the complement to the zero section of the determinant of the tautological vector bundle over $\mathrm{Gr}(k,n)$. For an SL-oriented representable ring cohomology theory $A^*(-)$ with invertible stable Hopf map η, including Witt groups and $\mathrm{MSL}_\eta^{*,*}$, we have $A^*(\mathrm{SGr}(2,2n+1)) \cong A^*(pt)[e]/(e^{2n})$, and $A^*(\mathrm{SGr}(k,n))$ is a truncated polynomial algebra over $A^*(pt)$ whenever $k(n-k)$ is even. A splitting principle for such theories is established. Using the computations for the special linear Grassmann varieties, we obtain a description of $A^*(\mathrm{BSL}_n)$ in terms of homogeneous power series in certain characteristic classes of tautological bundles.

1. Introduction

The basic and most fundamental computation for oriented cohomology theories is the projective bundle theorem (see [PS03, Theorem 3.9]) claiming $A^*(\mathbb{P}^n)$ to be a truncated polynomial ring over $A^*(pt)$ with an explicit basis in terms of the powers of a Chern class. Having this result at hand, one can define higher characteristic classes and compute the cohomology of Grassmann and flag varieties. In particular, the fact that the cohomology of a full flag variety is a truncated polynomial algebra gives rise to a splitting principle, which states that from the viewpoint of an oriented cohomology theory every vector bundle is in a certain sense a sum of linear bundles. For a representable cohomology theory one can deal with the infinite-dimensional Grassmannian which is a model for the classifying space BGL_n, and obtain, even more neatly, formal power series in the characteristic classes of the tautological vector bundle.

There are analogous computations for symplectically oriented cohomology theories [PW11a] with appropriately chosen varieties: quaternionic projective spaces HP^n instead of the ordinary ones and quaternionic Grassmannian and flag varieties. The answers are essentially the same: algebras of truncated polynomials in characteristic classes. In [PW11a] these classes were referred to as 'Pontryagin classes', but it was noted by Buchstaber that it would be more convenient to name them 'Borel classes' since they correspond to symplectic Borel classes in topology. We prefer to adopt this modification of the terminology.

These computations have a variety of applications, including motivic versions of theorems by Conner and Floyd [CF66] describing K-theory and hermitian K-theory as quotients of certain universal cohomology theories [PPR09a, PW11d].

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In the present paper we establish analogous results for SL-oriented cohomology theories. The notion of such orientation was introduced in [PW11c, Definition 5.1]. In the same preprint there was constructed the universal SL-oriented cohomology theory, namely the algebraic special linear cobordisms MSL [PW11c, Definition 4.2]. A more down-to-earth example are the derived Witt groups defined by Balmer [Bal99] and oriented via Koszul complexes [Nen07]. A comprehensive survey on the Witt groups can be found in [Bal05]. Of course, every oriented cohomology theory admits a special linear orientation, but it will turn out that we are not interested in such examples. We will deal with representable cohomology theories and work in the unstable $H_\bullet(k)$ and stable $\mathcal{SH}(k)$ motivic homotopy categories introduced by Morel and Voevodsky [MV99, Voe98]. We recall all the necessary constructions and notions in §§ 2–4 and provide preliminary calculations with special linear orientations.

We need to choose an appropriate version of 'projective space' analogous to $\mathbb{P}^n$ and HP^n. Natural candidates are SL_{n+1}/SL_n and $\mathbb{A}^{n+1}-\{0\}$. The choice makes no difference since the former is an affine bundle over the latter,, so they have the same cohomology. We take $\mathbb{A}^{n+1}-\{0\}$ since it looks prettier from the geometric point of view. There is a calculation for the Witt groups of this space [BG05, Theorem 8.13] claiming that $W^*(\mathbb{A}^{n+1}-\{0\})$ is a free module of rank two over $W^*(pt)$ with an explicit basis. The fact that it is a free module of rank two is not surprising since $\mathbb{A}^{n+1}-\{0\}$ is a sphere in the stable homotopy category $\mathcal{SH}(k)$ and $W^*(-)$ is representable [Hor05]. The interesting part is the basis. Let $\mathcal{T}=\mathcal{O}^{n+1}/\mathcal{O}(-1)$ be the tautological rank n bundle over $\mathbb{A}^{n+1}-\{0\}$. Then for $n=2k$ the basis consists of the element 1 and the class of the Koszul complex of $\mathcal{T}$. The latter is the Euler class $e(\mathcal{T})$ in the Witt groups. Unfortunately, for odd n the second term of the basis looks more complicated. Moreover, for an oriented cohomology theory even in the case of $n=2k$ the corresponding Chern class vanishes, so one cannot expect that 1 and $e(\mathcal{T})$ form a basis for every cohomology theory with a special linear orientation.

Here the following observation comes into play. The maximal compact subgroup of $SL_n(\mathbb{R})$ is $\mathrm{SO}_n(\mathbb{R})$, so over $\mathbb{R}$ the notion of a special linear orientation of a vector bundle leads to the usual topological orientation of a bundle. The Euler classes of oriented vector bundles in topology behave well only after inverting 2 in the coefficients, so we want to invert in the algebraic setting something analogous to 2. There are two interesting elements in the stable cohomotopy groups $\pi^{*,*}(pt)$ that go to 2 after taking $\mathbb{R}$-points: the usual $2\in\pi^{0,0}(pt)$ and the stable Hopf map $\eta\in\pi^{-1,-1}(pt)$ arising from the tautological morphism $\mathbb{A}^2-\{0\}\to\mathbb{P}^1$. In general 2 is not invertible in Witt groups, so we will invert η. Moreover, recall a result due to Morel [Mor12, Corollary 6.43] claiming that for a perfect field k there is an isomorphism $\bigoplus_n \pi^{n,n}(\operatorname{Spec} k)[\eta^{-1}]\cong W^0(k)[\eta,\eta^{-1}]$, so in a certain sense η is invertible in Witt groups. In §§ 5–6 we carry out some computations justifying the choice of η.

In this paper we deal mainly with the cohomology theories obtained as follows. Take a commutative monoid $(A,m,e:\mathbb{S}\to A)$ in the stable homotopy category $\mathcal{SH}(k)$ and fix a special linear orientation on the cohomology theory $A^{*,*}(-)$. The unit $e:\mathbb{S}\to A$ of the monoid (A,m,e) induces a morphism of cohomology theories $\pi^{*,*}(-)\to A^{*,*}(-)$ making $A^{*,*}(X)$ an algebra over stable cohomotopy groups. Inverting the stable Hopf map η, we obtain a cohomology theory

$$A^{*,*}_\eta(X)=A^{*,*}(X)[\eta^{-1}].$$

This theory is periodic in the $(1,1)$-direction via cup product $(-\cup\eta^n)$, thus without loss of generality we may focus on

$$A^*(X)=A^{*,0}_\eta(X).$$

This is still a cohomology theory. One can regard it as a $(1,1)$-periodic cohomology theory $A_\eta^{*,*}(-)$ collapsed in the $(1,1)$-direction. For these cohomology theories we have a result analogous to the case of Witt groups.

Theorem. $A^*(\mathbb{A}^{2n+1}-\{0\}) = A^*(pt) \oplus A^{*-2n}(pt)e(\mathcal{T})$.

A relative version of this statement is obtained in Theorem 3 in § 7. Note that there is no similar result for $\mathbb{A}^{2n}-\{0\}$.

In the next section we consider another family of varieties, called special linear Grassmannians $\mathrm{SGr}(2,n) = SL_n/P_2'$, where P_2' stands for the derived group of the parabolic subgroup P_2, i.e. P_2' is the stabilizer of the bivector $e_1 \wedge e_2$ in the exterior square of the regular representation of SL_n. There are tautological bundles $\mathcal{T}_1$ and $\mathcal{T}_2$ over $\mathrm{SGr}(2,n)$ of ranks 2 and $n-2$, respectively. We have the following theorem which seems to be the correct version of the projective bundle theorem in a special linear setting.

Theorem. *For the special linear Grassmann varieties we have the equalities*

$$A^*(\mathrm{SGr}(2,2n)) = \bigoplus_{i=0}^{2n-2} A^{*-2i}(pt)e(\mathcal{T}_1)^i \oplus A^{*-2n+2}(pt)e(\mathcal{T}_2),$$
$$A^*(\mathrm{SGr}(2,2n+1)) = \bigoplus_{i=0}^{2n-1} A^{*-2i}(pt)e(\mathcal{T}_1)^i.$$

Recall that there is a recent computation of twisted Witt groups of Grassmannians [BC12]. The twisted groups are involved since the authors use pushforwards that exist only in the twisted case. We deal with varieties with a trivialized canonical bundle and closed embeddings with a special linear normal bundle in order to avoid these difficulties. In fact we are interested in relative computations that could be extended to Grassmannian bundles, so we look for a basis consisting of characteristic classes rather than pushforwards of certain elements. It turns out that such bases exist only for the special linear flag varieties with all but at most one dimension step being even, i.e. we can handle $\mathrm{SGr}(1,7)$, $\mathcal{SF}(2,4,6)$ and $\mathcal{SF}(2,5,7)$ but not $\mathrm{SGr}(3,6)$. Nevertheless it seems that one can construct a basis for the latter case in terms of pushforwards.

Sections 8 and 9 are devoted to computations of the cohomology rings of partial flag varieties. We obtain an analogue of the splitting principle in Theorem 6 and derive certain properties of characteristic classes. In particular, there is a complete description of the cohomology rings of maximal SL_2 flag varieties,

$$\mathcal{SF}(2n) = SL_{2n}/P'_{2,4,\ldots,2n-2}, \quad \mathcal{SF}(2n+1) = SL_{2n+1}/P'_{2,4,\ldots,2n}.$$

The result is as follows (see Remark 12).

Theorem. *For $n \geqslant 1$, consider*

$$s_i = \sigma_i(e_1^2, e_2^2, \ldots, e_n^2), \quad t = \sigma_n(e_1, e_2, \ldots, e_n),$$

with σ_i being the elementary symmetric polynomials in n variables. Then we have the following isomorphisms:

(i) $A^*(\mathcal{SF}(2n)) \cong A^*(pt)[e_1, e_2, \ldots, e_n] \big/ (s_1, s_2, \ldots, s_{n-1}, t)$;

(ii) $A^*(\mathcal{SF}(2n+1)) \cong A^*(pt)[e_1, e_2, \ldots, e_n] \big/ (s_1, s_2, \ldots, s_n)$.

Note that one can substitute $SL_n/(SL_2)^{[n/2]}$ instead of $\mathcal{SF}(n)$. These answers and the choice of commuting SL_2 in SL_n perfectly agree with our principle that $SL_n(\mathbb{R})$ stands for $\mathrm{SO}_n(\mathbb{R})$, since $SL_2(\mathbb{R})$ stands for the compact torus $S^1 \cong \mathrm{SO}_2(\mathbb{R})$, and our choice of maximal product of commuting SL_2 is parallel to a choice of a maximal compact torus. We get the coinvariants for the Weyl groups $W(B_n)$ and $W(D_n)$, and this is what one gets computing the ordinary cohomology of $\mathrm{SO}_n(\mathbb{R})/T$.

In § 12 we carry out a computation for the cohomology rings of the special linear Grassmannians $\mathrm{SGr}(m,n)$ with $m(n-m)$ even; see Theorem 9. The answer is a truncated polynomial algebra in certain characteristic classes. We assemble the calculations for the special linear Grassmannians and compute in Theorem 10 the cohomology of the classifying spaces BSL_n in terms of homogeneous formal power series.

Theorem. *We have the following isomorphisms:*

$$A^*(\mathrm{BSL}_{2n}) \cong A^*(pt)[[p_1,\ldots,p_{n-1},e]]_h,$$
$$A^*(\mathrm{BSL}_{2n+1}) \cong A^*(pt)[[p_1,\ldots,p_n]]_h.$$

Finally, we leave to a separate paper [Aan13] a careful proof of the fact that Witt groups arise from hermitian K-theory in the fashion described above, i.e. $W^*(X) \cong (BO_\eta^{*,*}(X))^{*,0}$. In the same paper we obtain the following special linear version of the motivic Conner and Floyd theorem.

Theorem. *Let k be a field of characteristic different from 2. Then for every smooth variety X over k there exists an isomorphism*

$$\mathrm{MSL}_\eta^{*,*}(X) \otimes_{\mathrm{MSL}^{4*,2*}(pt)} W^{2*}(pt) \cong W^*(X)[\eta,\eta^{-1}].$$

Another application of the technique developed lies in the field of the equivariant Witt groups, and we will address this topic in another paper.

2. Preliminaries on $\mathcal{SH}(k)$ and ring cohomology theories

In this section we recall some basic definitions and constructions in the nonstable and stable motivic homotopy categories $H_\bullet(k)$ and $\mathcal{SH}(k)$. We refer the reader to the foundational papers [MV99, Voe98] for a comprehensive treatment of the subject.

Let k be a field of characteristic different from 2 and let Sm/k be the category of smooth varieties over k.

A motivic space over k is a simplicial presheaf on Sm/k. Each $X \in \mathrm{Sm}/k$ defines an unpointed motivic space $\mathrm{Hom}_{\mathrm{Sm}/k}(-,X)$ constant in the simplicial direction. We will often write pt for $\mathrm{Spec}\, k$ regarded as a motivic space.

We use the injective model structure on the category of the pointed motivic spaces $M_\bullet(k)$. Inverting all the weak motivic equivalences in $M_\bullet(k)$, we obtain the pointed motivic unstable homotopy category $H_\bullet(k)$.

Let $T = \mathbb{A}^1/(\mathbb{A}^1-\{0\})$ be the Morel–Voevodsky object. A T-spectrum M [Jar00] is a sequence of pointed motivic spaces $(M_0, M_1, M_2, \ldots)$ equipped with structural maps $\sigma_n : T \wedge M_n \to M_{n+1}$. A map of T-spectra is a sequence of maps of pointed motivic spaces which is compatible with the structure maps. We write $MS(k)$ for the category of T-spectra. Inverting the stable motivic weak equivalences as in [Jar00], we obtain the motivic stable homotopy category $\mathcal{SH}(k)$.

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A pointed motivic space X gives rise to a suspension T-spectrum $\Sigma_T^\infty X$. Set $\mathbb{S} = \Sigma_T^\infty(pt_+)$ for the spherical spectrum. Both $H_\bullet(k)$ and $\mathcal{SH}(k)$ are equipped with symmetric monoidal structures $(\wedge, pt_+)$ and $(\wedge, \mathbb{S})$ respectively, and

$$\Sigma_T^\infty : H_\bullet(k) \to \mathcal{SH}(k)$$

is a strict symmetric monoidal functor.

Recall that there are two spheres in $M_\bullet(k)$: the simplicial one $S^{1,0} = S_s^1 = \Delta^1/\partial(\Delta^1)$ and $S^{1,1} = (\mathbb{G}_m, 1)$. Here we follow the notation and indexing introduced in [MV99, p. 111]. For the integers $p, q \geqslant 0$ we write $S^{p+q,q}$ for $(\mathbb{G}_m, 1)^{\wedge q} \wedge (S_s^1)^{\wedge p}$ and $\Sigma^{p+q,q}$ for the suspension functor $-\wedge S^{p+q,q}$. This functor becomes invertible in the stable homotopy category $\mathcal{SH}(k)$, so we extend the notation to arbitrary integers p, q in an obvious way.

Any T-spectrum A defines a bigraded cohomology theory on the category of pointed motivic spaces. Namely, for a pointed space (X, x) one sets

$$A^{p,q}(X, x) = \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_T^\infty(X, x), \Sigma^{p,q} A)$$

and $A^{*,*}(X, x) = \bigoplus_{p,q} A^{p,q}(X, x)$. In the case of $i - j, j \geqslant 0$ one has a canonical suspension isomorphism $A^{p,q}(X, x) \cong A^{p+i,q+j}(\Sigma^{i,j}(X, x))$ induced by the shuffling isomorphism $S^{p,q} \wedge S^{i,j} \cong S^{p+i,q+j}$. In the motivic homotopy category there is a canonical isomorphism $T \cong S^{2,1}$, and we write

$$\Sigma_T : A^{*,*}(X, x) \xrightarrow{\simeq} A^{*+2,*+1}((X, x) \wedge T)$$

for the corresponding suspension isomorphism. See Definition 16 in § 5 for the details.

For an unpointed space X we set $A^{p,q}(X) = A^{p,q}(X_+, +)$ with $A^{*,*}(X)$ defined accordingly. Set $\pi^{i,j}(X) = \mathbb{S}^{i,j}(X)$ to be the stable cohomotopy groups of X.

We can regard smooth varieties as unpointed motivic spaces and obtain groups $A^{p,q}(X)$. Given a closed embedding $i : Z \to X$ of varieties, we write $\mathrm{Th}(i)$ for $X/(X - Z)$. For a vector bundle $E \to X$ set $\mathrm{Th}(E) = E/(E - X)$ to be the Thom space of E.

A commutative ring T-spectrum is a commutative monoid (A, m, e) in $(\mathcal{SH}(k), \wedge, \mathbb{S})$. The cohomology theory defined by a commutative T-spectrum is a ring cohomology theory satisfying a certain bigraded commutativity condition described by Morel.

We recall the essential properties of the cohomology theory represented by a commutative ring T-spectrum A.

(1) *Localization.* For a closed embedding of varieties $i : Z \to X$ with smooth X and an open complement $j : U \to X$, put $z : X \to \mathrm{Th}(i) = X/U$ for the canonical quotient map. Then we have a long exact sequence

$$\cdots \xrightarrow{\partial} A^{*,*}(\mathrm{Th}(i)) \xrightarrow{z^A} A^{*,*}(X) \xrightarrow{j^A} A^{*,*}(U) \xrightarrow{\partial} A^{*+1,*}(\mathrm{Th}(i)) \xrightarrow{z^A} \cdots$$

This is a long exact sequence associated to the cofibration $j : U \to X$.

(2) *Nisnevich excision.* Consider a Cartesian square of smooth varieties

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where i is a closed embedding, f is étale and f' is an isomorphism. Then for the induced morphism $g : \mathrm{Th}(i') \to \mathrm{Th}(i)$ the corresponding morphism $g^A : A^{*,*}(\mathrm{Th}(i)) \to A^{*,*}(\mathrm{Th}(i'))$ is

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an isomorphism. This follows from the fact that g is an isomorphism in the homotopy category (homotopy purity theorem, [MV99, § 3, Theorem 2.23]).

(3) *Homotopy invariance.* For an $\mathbb{A}^n$-bundle $p: E \to X$ over a variety X the induced homomorphism $p^A : A^{*,*}(X) \to A^{*,*}(E)$ is an isomorphism.

(4) *Mayer–Vietoris property.* If $X = U_1 \cup U_2$ is a union of two open subsets U_1 and U_2 then there exists a natural long exact sequence

$$\cdots \to A^{*,*}(X) \to A^{*,*}(U_1) \oplus A^{*,*}(U_2) \to A^{*,*}(U_1 \cap U_2) \to A^{*+1,*}(X) \to \cdots$$

(5) *Cup product.* For a motivic space Y we have a functorial graded ring structure

$$\cup : A^{*,*}(Y) \times A^{*,*}(Y) \to A^{*,*}(Y).$$

Moreover, let $i_1 : Z_1 \to X$ and $i_2 : Z_2 \to X$ be closed embeddings and put $i_{12} : Z_1 \cap Z_2 \to X$. Then we have a functorial, bilinear and associative cup product

$$\cup : A^{*,*}(\mathrm{Th}(i_1)) \times A^{*,*}(\mathrm{Th}(i_2)) \to A^{*,*}(\mathrm{Th}(i_{12})).$$

In particular, setting $Z_1 = X$, we obtain an $A^{*,*}(X)$-module structure on $A^{*,*}(\mathrm{Th}(i_2))$. All the morphisms in the localization sequence are homomorphisms of $A^{*,*}(X)$-modules. We will sometimes omit $\cup$ from the notation.

(6) *Module structure over stable cohomotopy groups.* For every motivic space Y we have a homomorphism of graded rings $\pi^{*,*}(Y) \to A^{*,*}(Y)$, which defines a $\pi^{*,*}(pt)$-module structure on $A^{*,*}(Y)$. For a smooth variety X the ring $A^{*,*}(X)$ is a graded $\pi^{*,*}(pt)$-algebra via $\pi^{*,*}(pt) \to \pi^{*,*}(X) \to A^{*,*}(X)$.

(7) *Graded ϵ-commutativity* [Mor04, Lemma 6.1.1]. Let $\epsilon \in \pi^{0,0}(pt)$ be the element corresponding under the suspension isomorphism to the morphism $T \to T, x \mapsto -x$. Then for every motivic space X and $a \in A^{i,j}(X), b \in A^{p,q}(X)$ we have

$$ab = (-1)^{ip}\epsilon^{jq}ba.$$

Recall that $\epsilon^2 = 1$.

3. Special linear orientation

In this section we recall the notion of a special linear orientation introduced in [PW11c] and establish some of its basic properties.

Definition 1. A *special linear bundle* over a variety X is a pair (E, λ) with $E \to X$ a vector bundle and $\lambda : \det E \xrightarrow{\simeq} \mathcal{O}_X$ an isomorphism of line bundles. An isomorphism $\phi : (E, \lambda) \xrightarrow{\simeq} (E', \lambda')$ of special linear vector bundles is an isomorphism $\phi : E \xrightarrow{\simeq} E'$ of vector bundles such that $\lambda' \circ (\det \phi) = \lambda$. For a special linear bundle $\mathcal{T} = (E, \lambda)$ we usually denote by the same letter $\mathcal{T}$ the total space of the bundle E.

Remark 1. Isomorphism classes of rank n special linear vector bundles over X are in a canonical bijection with the pointed set $H^1(X, SL_n)$.

Definition 2. Consider the trivialized rank n bundle $\mathcal{O}_X^n$ over a smooth variety X. There is a canonical trivialization $\det \mathcal{O}_X^n \xrightarrow{\simeq} \mathcal{O}_X$. We denote the corresponding special linear bundle by $(\mathcal{O}_X^n, 1)$ and refer to it as the *trivialized special linear bundle.*

LEMMA 1. *Let (E, λ) be a special linear bundle over a smooth variety X such that $E \cong \mathcal{O}_X^n$. Then there exists an isomorphism of special linear bundles*

$$\phi : (E, \lambda) \xrightarrow{\simeq} (\mathcal{O}_X^n, 1).$$

Proof. Let $\psi : E \xrightarrow{\simeq} \mathcal{O}_X^n$ be an isomorphism of vector bundles. Then $\lambda \det \psi^{-1} \in \mathcal{O}_X^*$ is an invertible regular function on X. Consider the diagonal matrix $D = \mathrm{diag}(\lambda \det \psi^{-1}, 1, 1, \ldots, 1)$ of size $n \times n$. One can easily check that $\phi = D\psi$ is the required isomorphism of special linear bundles. $\square$

LEMMA 2. *Let E_1 be a subbundle of a vector bundle E over a smooth variety X. Then there are canonical isomorphisms:*

(i) $\det E_1 \otimes \det(E/E_1) \cong \det E$;
(ii) $\det E^\vee \cong (\det E)^\vee$.

Proof. These isomorphisms are induced by the corresponding vector space isomorphisms. In the first case we have $\Lambda^m V_1 \otimes \Lambda^n (V/V_1) \xrightarrow{\simeq} \Lambda^{m+n} V$ with

$$(v_1 \wedge \cdots \wedge v_m) \otimes (\overline{w}_1 \wedge \cdots \wedge \overline{w}_n) \mapsto v_1 \wedge \cdots \wedge v_m \wedge w_1 \wedge \cdots \wedge w_n.$$

For the second isomorphism consider the perfect pairing

$$\phi : \Lambda^n V \times \Lambda^n V^\vee \to k$$

defined by

$$\phi(v_1 \wedge \cdots \wedge v_n, f_1 \wedge \cdots \wedge f_n) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) f_{\sigma(1)}(v_1) \cdots f_{\sigma(n)}(v_n). \qquad \square$$

DEFINITION 3. Let $\mathcal{T} = (E, \lambda_E)$ be a special linear bundle over a smooth variety X and let $\mathcal{T}' = (E', \lambda_{E'})$ with $E' \leqslant E$ be a subbundle. By Lemma 2 we have canonical trivializations $\lambda_{E^\vee} : \det E^\vee \xrightarrow{\simeq} \mathcal{O}_X$ and $\lambda_{E/E'} : \det(E/E') \xrightarrow{\simeq} \mathcal{O}_X$. The special linear bundle $\mathcal{T}^\vee = (E^\vee, \lambda_{E^\vee})$ is called the *dual special linear bundle* and the special linear bundle $\mathcal{T}/\mathcal{T}' = (E/E', \lambda_{E/E'})$ is called the *quotient special linear bundle*. For a pair $\mathcal{T}_1 = (E_1, \lambda_{E_1}), \mathcal{T}_2 = (E_2, \lambda_{E_2})$ of special linear bundles over a smooth variety X we put $\mathcal{T}_1 \oplus \mathcal{T}_2 = (E_1 \oplus E_2, \lambda_{E_1} \otimes \lambda_{E_2})$ and refer to it as the *direct sum of special linear bundles*.

DEFINITION 4. Let $A^{*,*}(-)$ be the ring cohomology theory represented by a commutative ring T-spectrum A. A *(normalized) special linear orientation* on $A^{*,*}(-)$ is a rule which assigns to every special linear bundle $\mathcal{T}$ of rank n over a smooth variety X a class $th(\mathcal{T}) \in A^{2n,n}(\mathrm{Th}(\mathcal{T}))$ satisfying the following conditions [PW11c, Definition 5.1]:

(i) for an isomorphism $f : \mathcal{T} \xrightarrow{\simeq} \mathcal{T}'$ we have $th(\mathcal{T}) = f^A th(\mathcal{T}')$;
(ii) for a morphism $r : Y \to X$ we have $r^A th(\mathcal{T}) = th(r^* \mathcal{T})$;
(iii) the maps $- \cup th(\mathcal{T}) : A^{*,*}(X) \to A^{*+2n,*+n}(\mathrm{Th}(\mathcal{T}))$ are isomorphisms;
(iv) we have
$$th(\mathcal{T}_1 \oplus \mathcal{T}_2) = q_1^A th(\mathcal{T}_1) \cup q_2^A th(\mathcal{T}_2),$$
where q_1, q_2 are projections from $\mathcal{T}_1 \oplus \mathcal{T}_2$ onto its summands. Moreover, for the zero bundle $\mathbf{0} \to pt$ we have $th(\mathbf{0}) = 1 \in A^{0,0}(pt)$.

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(v) (Normalization.) For the trivialized special linear line bundle over a point we have $th(\mathcal{O}_{pt}, 1) = \Sigma_T 1 \in A^{2,1}(T)$.

The isomorphism $-\cup th(\mathcal{T})$ is the *Thom isomorphism*. The class $th(\mathcal{T})$ is the *Thom class* of the special linear bundle, and

$$e(\mathcal{T}) = z^A th(\mathcal{T}) \in A^{2n,n}(X)$$

with natural $z : X \to \mathrm{Th}(\mathcal{T})$ is its *Euler class*. A ring cohomology theory with a fixed (normalized) special linear orientation is called an *SL-oriented cohomology theory.*

Lemma 3. *Let $A^{*,*}(-)$ be an SL-oriented cohomology theory, let $\mathcal{T}$ be a special linear bundle over a smooth variety X and let $\mathcal{T}_1 \leqslant \mathcal{T}$ be a special linear subbundle. Then $e(\mathcal{T}) = e(\mathcal{T}_1)e(\mathcal{T}/\mathcal{T}_1)$.*

Proof. There is an $\mathbb{A}^r$-bundle $p : Y \to X$ such that

$$p^*\mathcal{T} \cong p^*\mathcal{T}_1 \oplus p^*(\mathcal{T}/\mathcal{T}_1),$$

so the claim follows from homotopy invariance and multiplicativity of the Euler class with respect to direct sums. The variety Y can be constructed in the following way. Denote by $k = \operatorname{rank} \mathcal{T}_1$ and $n = \operatorname{rank} \mathcal{T}$ the ranks of the vector bundles. Consider the Grassmannian bundle $\pi : \mathrm{Gr}(k, \mathcal{T}) \times_X \mathrm{Gr}(n-k, \mathcal{T}) \to X$ and denote by $\mathcal{E}_1$ and $\mathcal{E}_2$ the tautological vector bundles of rank k and $n-k$ induced from the factors. Let U be an open subvariety of $\mathrm{Gr}(k, \mathcal{T}) \times_X \mathrm{Gr}(n-k, \mathcal{T})$ corresponding to the direct sum decompositions of $\mathcal{T}$, i.e. the complement to the support of the kernel of the canonical morphism $\mathcal{E}_1 \oplus \mathcal{E}_2 \to \pi^*\mathcal{T}$. The subbundle $\mathcal{T}_1 \leqslant \mathcal{T}$ corresponds to a section $z : X \to \mathrm{Gr}(k, \mathcal{T})$ of the Grassmannian bundle. We can take $Y = U \cap (z(X) \times_X \mathrm{Gr}(n-k, \mathcal{T}))$. The fibre of Y over a point $x \in X$ is the variety of the vector subspaces $V \leqslant \mathcal{T}|_x$ such that $\mathcal{T}|_x = \mathcal{T}_1|_x \oplus V$ which is an affine space of dimension $k(n-k)$. □

Remark 2. For a rank $2n$ special linear bundle $\mathcal{T}$ over a variety X we have $th(\mathcal{T}) \in A^{4n,2n}(\mathrm{Th}(\mathcal{T}))$ and $e(\mathcal{T}) \in A^{4n,2n}(X)$, so these classes are central, i.e. $e(\mathcal{T}) \cup \alpha = \alpha \cup e(\mathcal{T})$ and $th(\mathcal{T}) \cup \alpha = \alpha \cup th(\mathcal{T})$ for every $\alpha \in A^{*,*}(X)$.

Recall that a symplectic bundle is a special linear bundle in a natural way, so, having a special linear orientation, we have the Thom classes also for symplectic bundles, thus an *SL*-oriented cohomology theory is also symplectically oriented. We recall the definition of a Borel classes theory (cf. [PW11a, Definition 14.1]) that could be developed for a symplectically oriented cohomology theory. Note that our terminology is slightly different from that used in [PW11a]: we refer to the 'Pontryagin classes' in the sense of [PW11a] as 'Borel classes'.

Definition 5. Let $A^{*,*}(-)$ be the cohomology theory represented by a commutative ring T-spectrum A. A *Borel classes theory* on $A^{*,*}(-)$ is a rule which assigns to every symplectic bundle (E, ϕ) over every smooth variety X a system of *Borel classes* $b_i(E, \phi) \in A^{4i,2i}(X)$ for all $i \geqslant 1$ satisfying the following:

(i) for $(E_1, \phi_1) \cong (E_2, \phi_2)$ we have $b_i(E_1, \phi_1) = b_i(E_2, \phi_2)$ for all i;

(ii) for a morphism $r : Y \to X$ and a symplectic bundle (E, ϕ) over X we have $r^A(b_i(E, \phi)) = b_i(r^*(E, \phi))$ for all i;

(iii) for the tautological rank 2 symplectic bundle (E, ϕ) over $HP^1 = Sp_4/(Sp_2 \times Sp_2)$ the elements $1, b_1(E, \phi)$ form an $A^{*,*}(pt)$-basis of $A^{*,*}(HP^1)$;

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(iv) for a rank 2 symplectic bundle (V, ϕ) over pt we have $b_1(V, \phi) = 0$;
(v) for an orthogonal direct sum of symplectic bundles $(E, \phi) \cong (E_1, \phi_1) \perp (E_2, \phi_2)$ we have

$$b_i(E, \phi) = b_i(E_1, \phi_1) + \sum_{j=1}^{i-1} b_{i-j}(E_1, \phi_1) b_j(E_2, \phi_2) + b_i(E_2, \phi_2)$$

for all i;
(vi) for (E, ϕ) of rank $2r$ we have $b_i(E, \phi) = 0$ for $i > r$.

We set $b_*(E, \phi) = 1 + \sum_{i=1}^{\infty} b_i(E, \phi) t^i$ to be the *total Borel class*.

Remark 3. For a symplectically oriented cohomology theory $A^{*,*}(-)$ the canonical Borel classes theory can be defined in the following way. For a symplectic vector bundle (E, ϕ) of rank $2n$ over a smooth variety X one puts

$$b_n(E, \phi) = z^A th(E, \phi) \in A^{4n,2n}(X)$$

for the natural map $z : X \to \mathrm{Th}(E)$. Then one may define the lower Borel classes using the symplectic version of the projective bundle theorem; see [PW11a] for the details. Note that since these Borel classes are similar to the symplectic Borel classes in topology and not to the Pontryagin classes, we omit the sign in the above formula for the top Borel class.

Every oriented cohomology theory possesses a special linear orientation via $th(E, \lambda) = th(E)$, so one can consider K-theory or algebraic cobordism represented by MGL as examples. We have two main instances of the theories with a special linear orientation but without a general one. The first one is hermitian K-theory [Sch10] represented by the spectrum BO [PW11b]. The special linear orientation of $BO^{*,*}$ via Koszul complexes could be found in [PW11b]. The second one is universal in the sense of [PW11c, Theorem 5.9] and represented by the algebraic special linear cobordism spectrum MSL [PW11c, Definition 4.2].

Definition 6. From now on $A^{*,*}(-)$ is an SL-oriented ring cohomology theory represented by a commutative monoid in $\mathcal{SH}(k)$.

Lemma 4. *For a smooth variety X we have*

$$th(\mathcal{O}_X^n, 1) = \Sigma_T^n 1, \quad th(\mathcal{O}_X, -1) = \Sigma_T \epsilon.$$

Proof. The first equality follows from conditions (iv) and (v). For the second equality consider the morphism $f^A : \mathrm{Th}(\mathcal{O}_X) \to \mathrm{Th}(\mathcal{O}_X)$ corresponding to the isomorphism of the special linear bundles $(\mathcal{O}_X, -1) \to (\mathcal{O}_X, 1)$, $v \mapsto -v$. By the very definition we have $f^A(\Sigma_T 1) = \Sigma_T \epsilon = \Sigma_T 1 \cup \epsilon$. On the other hand, functoriality of Thom classes together with normalization yields

$$th(\mathcal{O}_X, -1) = f^A(th(\mathcal{O}_X, 1)) = f^A(\Sigma_T 1) = \Sigma_T 1 \cup \epsilon = \Sigma_T \epsilon. \qquad \square$$

Lemma 5. *Let (E, λ_E) be a special linear bundle over a smooth variety X. Then*

$$e(E, \lambda_E) = \epsilon \cup e(E, -\lambda_E).$$

Proof. Consider the bundle $E \oplus \mathcal{O}_X$ and denote the projections onto the summands by q_1, q_2. We have

$$(E \oplus \mathcal{O}_X, \lambda_E \otimes 1) = (E \oplus \mathcal{O}_X, (-\lambda_E) \otimes -1),$$

hence
$$q_1^* th(E,\lambda_E)\cup q_2^*\Sigma_T 1 = q_1^* th(E,-\lambda_E)\cup q_2^*\Sigma_T\epsilon.$$
By the suspension isomorphism we obtain
$$th(E,\lambda_E) = th(E,-\lambda_E)\cup\epsilon,$$
hence $e(E,\lambda_E)=\epsilon\cup e(E,-\lambda_E)$. □

Definition 7. Let E be a vector bundle over a smooth variety X. The *hyperbolic bundle associated to* E is the symplectic bundle
$$H(E) = \left(E\oplus E^\vee, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$
Denote by $p_i(E)=(-1)^i b_{2i}(H(E))$ the signed even Borel classes of $H(E)$ and refer to them as *Pontryagin classes*. The *total Pontryagin class* is $p_*(E)=\sum p_i(E)t^{2i}$.

Remark 4. This definition is parallel to the definition of the Pontryagin classes in topology with the Borel classes substituted for the Chern ones and using hyperbolization instead of complexification.

We defined Pontryagin classes for arbitrary vector bundles without any additional structure. For special linear bundles there is an interconnection between the top Pontryagin class and the Euler class. The following lemma shows this in the case of rank 2 bundles. The general case would be dealt with in Corollary 3, but only for a cohomology theory with invertible stable Hopf element. Note that in general Pontryagin classes behave quite badly, for example, the total Pontryagin class may even lack multiplicativity, but things become much better if we invert the stable Hopf element; see Corollary 3 and Lemma 16 for the details.

Lemma 6. *Let $\mathcal{T}=(E,\lambda)$ be a rank 2 special linear bundle. Then*
$$b_*(H(E)) = 1+(1+\epsilon)e(\mathcal{T})t+\epsilon e(\mathcal{T})^2t^2,\quad p_*(\mathcal{T}) = 1-\epsilon e(\mathcal{T})^2t^2.$$

Proof. Let ϕ be the symplectic form on E corresponding to λ. There exists an isomorphism [Bal05, Examples 1.1.21, 1.1.22]
$$\left(E\oplus E^\vee, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\cong\left(E\oplus E, \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}\right),$$
so we have
$$b_*(H(E)) = b_*(E,\phi)b_*(E,-\phi) = (1+b_1(E,\phi)t)(1+b_1(E,-\phi)t) = (1+e(E,\lambda)t)(1+e(E,-\lambda)t).$$
By Lemma 5 we have $e(E,-\lambda)=\epsilon e(E,\lambda)$, thus
$$b_*(H(E)) = (1+e(\mathcal{T})t)(1+\epsilon e(\mathcal{T})t) = 1+(1+\epsilon)e(\mathcal{T})t+\epsilon e(\mathcal{T})^2t^2.$$
In order to obtain the formula for the total Pontryagin class one should drop the middle term and change the sign in front of $b_2(H(E))=\epsilon e(\mathcal{T})^2$. □

Lemma 7. *Let $\mathcal{T}$ be a rank 2 special linear bundle over a smooth variety X. Then $\mathcal{T}\cong\mathcal{T}^\vee$ and $e(\mathcal{T})=e(\mathcal{T}^\vee)$.*

Proof. Set $\mathcal{T} = (E, \lambda_E)$. The trivialization $\lambda_E : \Lambda^2 E \xrightarrow{\simeq} \mathcal{O}_X$ defines a symplectic form on E and an isomorphism $\phi : E \xrightarrow{\simeq} E^\vee$, thus it is sufficient to check that

$$\lambda_{E^\vee} \circ \det \phi = \lambda_E.$$

This could be checked locally, so we can suppose that $E \cong \mathcal{O}_X^2$ and, in view of Lemma 1, $(E, \lambda_E) \cong (\mathcal{O}_X^2, 1)$. Fixing a basis $\{e_1, e_2\}$ such that $e_1 \wedge e_2 = 1$ and taking the dual basis $\{e_1^\vee, e_2^\vee\}$ for $(\mathcal{O}_X^2)^\vee$, we have

$$\phi(e_1) = (e_1 \wedge -) = e_2^\vee, \quad \phi(e_2) = (e_2 \wedge -) = -e_1^\vee.$$

Thus we obtain

$$\det \phi(e_1 \wedge e_2) = e_2^\vee \wedge (-e_1^\vee) = e_1^\vee \wedge e_2^\vee$$

and

$$\lambda_{E^\vee} \det \phi(e_1 \wedge e_2) = \lambda_{E^\vee}(e_1^\vee \wedge e_2^\vee) = 1. \qquad \square$$

DEFINITION 8. For a vector bundle E we denote by E^0 the complement to the zero section. For a special linear bundle $\mathcal{T} = (E, \lambda)$ we put $\mathcal{T}^0 = E^0$.

DEFINITION 9. Let $\mathcal{T}$ be a rank n special linear bundle over a smooth variety X. The *Gysin sequence* is a long exact sequence of $A^{*,*}(X)$-modules

$$\cdots \xrightarrow{\partial} A^{*-2n,*-n}(X) \xrightarrow{\cup e(\mathcal{T})} A^{*,*}(X) \to A^{*,*}(\mathcal{T}^0) \xrightarrow{\partial} A^{*-2n+1,*-n}(X) \to \cdots$$

obtained from the localization sequence for the zero section $X \to \mathcal{T}$ via the homotopy invariance and the Thom isomorphism.

LEMMA 8. *Let (E, λ_E) be a special linear bundle over a smooth variety X:*

(i) *let λ'_E be any other trivialization of* $\det E$. *Then*

$$A^{0,0}(X) \cup e(E, \lambda_E) = A^{0,0}(X) \cup e(E, \lambda'_E);$$

(ii) *for the dual special linear bundle $(E^\vee, \lambda_{E^\vee})$,*

$$A^{0,0}(X) \cup e(E, \lambda_E) = A^{0,0}(X) \cup e(E^\vee, \lambda_{E^\vee}).$$

Proof. Set $n = \operatorname{rank} E$ and denote the projections $E^0 \to X$ and $E^{\vee 0} \to X$ by p and p', respectively.

(i) Consider the Gysin sequences corresponding to the trivializations λ_E and λ'_E:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{0,0}(X) \arrow[r, "\cup e(E,\lambda_E)"] & A^{2n,n}(X) \arrow[r, "p^A"] \arrow[d, "="] & A^{2n,n}(E^0) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & A^{0,0}(X) \arrow[r, "\cup e(E,\lambda'_E)"] & A^{2n,n}(X) \arrow[r, "p^A"] & A^{2n,n}(E^0) \arrow[r] & \cdots
\end{tikzcd}$$

We have

$$A^{0,0}(X) \cup e(E, \lambda_E) = \ker p^A = A^{0,0}(X) \cup e(E, \lambda'_E).$$

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(ii) Consider
$$Y = \{(v, f) \in E \times_X E^\vee \mid f(v) = 1\}.$$
Projections $p_1 : Y \to E^0$ and $p_2 : Y \to E^{\vee 0}$ have fibres isomorphic to $\mathbb{A}^{n-1}$, thus
$$A^{*,*}(E^0) \cong A^{*,*}(Y) \cong A^{*,*}(E^{\vee 0})$$
and we have a canonical isomorphism $A^{*,*}(E^0) \cong A^{*,*}(E^{\vee 0})$ over $A^{*,*}(X)$. Now proceed as in the first part and consider the Gysin sequences

$$\begin{tikzcd} \cdots \arrow[r] & A^{0,0}(X) \arrow[r, "\cup e(E,\lambda_E)"] & A^{2n,n}(X) \arrow[r, "p^A"] \arrow[d, "="] & A^{2n,n}(E^0) \arrow[r] \arrow[d, "\cong"] & \cdots \\ \cdots \arrow[r] & A^{0,0}(X) \arrow[r, "\cup e(E^\vee,\lambda_{E^\vee})"] & A^{2n,n}(X) \arrow[r, "p'^A"] & A^{2n,n}(E^{\vee 0}) \arrow[r] & \cdots \end{tikzcd}$$

We have
$$A^{0,0}(X) \cup e(E, \lambda_E) = \ker p^A = \ker p'^A = A^{0,0}(X) \cup e(E^\vee, \lambda_{E^\vee}). \qquad \square$$

Lemma 9. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X such that there exists a nowhere vanishing section $s : X \to \mathcal{T}$. Then $e(\mathcal{T}) = 0$.*

Proof. Set $\operatorname{rank} \mathcal{T} = n$ and consider the Gysin sequence
$$\cdots \to A^{0,0}(X) \xrightarrow{\cup e(\mathcal{T})} A^{2n,n}(X) \xrightarrow{j^A} A^{2n,n}(\mathcal{T}^0) \to \cdots$$
The section s induces the splitting s^A for j^A, thus j^A is injective and
$$e(\mathcal{T}) = 1 \cup e(\mathcal{T}) \in \ker j^A = \{0\}. \qquad \square$$

4. Pushforwards along closed embeddings

In this section we give a construction of pushforwards along the closed embeddings with special linear normal bundles for an *SL*-oriented cohomology theory. This is quite similar to the construction of such pushforwards for oriented [PS03, Nen06] or symplectically oriented [PW11a] cohomology theories and twisted Witt groups [Nen07], so we follow [PW11a] and [Nen07] adapting them to a special linear context.

Definition 10. Let $i : Z \to X$ be a closed embedding of smooth varieties. The *deformation space* $D(Z, X)$ is obtained as follows:

(i) consider $X \times \mathbb{A}^1$;
(ii) blow it up along $Z \times 0$;
(iii) remove the blow-up of $X \times 0$ along $Z \times 0$.

This construction produces a smooth variety $D(Z, X)$ over $\mathbb{A}^1$. The fibre over 0 is canonically isomorphic to the normal bundle to Z in X which we denote by N_i. The fibre over 1 is isomorphic to X. We have the corresponding closed embeddings $i_0 : N_i \to D(Z, X)$ and $i_1 : X \to D(Z, X)$. There is a closed embedding $z : Z \times \mathbb{A}^1 \to D(Z, X)$ such that over 0 it coincides with the zero section $s : Z \to N_i$ of the normal bundle and over 1 it coincides with the closed embedding $i : Z \to X$. Finally, we have a projection $p : D(Z, X) \to X$.

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Thus we have homomorphisms of $A^{*,*}(X)$-modules (via p^A),

$$A^{*,*}(\mathrm{Th}(N_i)) \xleftarrow{i_0^A} A^{*,*}(\mathrm{Th}(z)) \xrightarrow{i_1^A} A^{*,*}(\mathrm{Th}(i)).$$

These homomorphisms are isomorphisms since in the homotopy category $H_\bullet(k)$ we have isomorphisms $i_0 : \mathrm{Th}(N_i) \cong \mathrm{Th}(z)$ and $i_1 : \mathrm{Th}(i) \cong \mathrm{Th}(z)$ [MV99, Theorem 2.23]. We set

$$d_i^A = i_1^A \circ (i_0^A)^{-1} : A^{*,*}(\mathrm{Th}(N_i)) \to A^{*,*}(\mathrm{Th}(i))$$

to be the *deformation to the normal bundle isomorphism*. The functoriality of the deformation space $D(Z,X)$ (see [Nen07, Proposition 3.4]) makes the deformation to the normal bundle isomorphism functorial.

Definition 11. For a closed embedding $i : Z \to X$ of smooth varieties a *special linear normal bundle* is a pair (N_i, λ) with N_i the normal bundle to Z in X and $\lambda : \det N_i \xrightarrow{\simeq} \mathcal{O}_Z$ an isomorphism of line bundles.

Definition 12. Let $i : Z \to X$ be a closed embedding of smooth varieties with a rank n special linear normal bundle (N_i, λ). Denote by $\tilde{\imath}_A$ the composition of the Thom and deformation to the normal bundle isomorphisms,

$$\tilde{\imath}_A = d_i^A \circ (-\cup th(N_i,\lambda)) : A^{*,*}(Z) \xrightarrow{\simeq} A^{*+2n,*+n}(\mathrm{Th}(i)).$$

For the quotient map $z : X \to \mathrm{Th}(i)$ the composition

$$i_A = z^A \circ \tilde{\imath}_A : A^{*,*}(Z) \to A^{*+2n,*+n}(X)$$

is the *pushforward map*. Note that in general i_A depends on the trivialization of $\det N_i$.

Remark 5. We have an analogous definition of the pushforward map for a closed embedding $i : Z \to X$ in every cohomology theory possessing a Thom class for the normal bundle N_i. In particular, we have pushforwards in the stable cohomotopy groups for closed embeddings with a trivialized normal bundle (N_i, θ), where $\theta : N_i \xrightarrow{\simeq} \mathcal{O}_Z^n$ is an isomorphism of vector bundles, since there is a Thom class $th(\mathcal{O}_Z^n) = \Sigma_T^n 1$ and suspension isomorphism

$$(-\cup \Sigma_T^n 1) : \pi^{*,*}(Z) \xrightarrow{\simeq} \pi^{*+2n,*+n}(\mathrm{Th}(\mathcal{O}_Z^n)).$$

Definition 13. Let $i : Z \to X$ be a closed embedding of smooth varieties with a rank n special linear normal bundle. Then using the notation of pushforward maps the localization sequence boils down to the following long exact sequence of $A^{*,*}(X)$-modules:

$$\cdots \xrightarrow{\partial} A^{*-2n,*-n}(Z) \xrightarrow{i_A} A^{*,*}(X) \xrightarrow{j^A} A^{*,*}(X-Z) \xrightarrow{\partial} A^{*-2n+1,*-n}(Z) \xrightarrow{i_A} \cdots$$

We refer to this sequence as *the Gysin sequence*, similar to Definition 9.

In the rest of this section we sketch some properties of the pushforward maps. The next lemma is similar to [PW11a, Proposition 7.4].

Lemma 10. *Consider the following cartesian diagram with all the varieties involved being smooth:*

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "i'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Let i, i' be closed embeddings with special linear normal bundles (N_i, λ) and $(N_{i'}, \lambda')$. Suppose, moreover, that g' induces an isomorphism $((g')^* N_i, (g')^*\lambda) \cong (N_{i'}, \lambda')$. Then we have $g^A \tilde{\imath}_A = \tilde{\imath}'_A g'^A$.

Proof. This follows from functoriality of the deformation to the normal bundle and functoriality of Thom classes. ☐

The next proposition is an analogue of [PW11a, Proposition 7.6].

Proposition 1. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X with a section $s : X \to \mathcal{T}$ meeting the zero section r transversally in Y. Then for the inclusion $i : Y \to X$ and every $b \in A^{*,*}(X)$ we have*

$$i_A i^A(b) = b \cup e(\mathcal{T}).$$

Proof. Let $z^A : A^{*,*}(\mathrm{Th}(i)) \to A^{*,*}(X)$ and $\overline{z}^A : A^{*,*}(\mathrm{Th}(\mathcal{T})) \to A^{*,*}(\mathcal{T})$ be the extension of support maps and let $p : \mathcal{T} \to X$ be the structure map for the bundle. Consider the diagram

$$\begin{tikzcd}
A^{*,*}(X) \arrow[r, "\tilde{r}_A", "\cup th(\mathcal{T})"'] \arrow[d, "i^A"] & A^{*,*}(\mathrm{Th}(\mathcal{T})) \arrow[r, "\overline{z}^A"] \arrow[d, "s^A"] & A^{*,*}(\mathcal{T}) \arrow[d, shift right, "r^A"'] \arrow[d, shift left, "s^A"] \\
A^{*,*}(Y) \arrow[r, "\tilde{\imath}_A"] & A^{*,*}(\mathrm{Th}(i)) \arrow[r, "z^A"] & A^{*,*}(X) \arrow[u, bend right, "p^A"']
\end{tikzcd}$$

The pullbacks along the two sections of p are inverses of the same isomorphism p^A, so $s^A = r^A$. The right-hand square consists of pullbacks, thus it is commutative. The left-hand square commutes by Lemma 10. Hence we have

$$i_A i^A(b) = z^A \tilde{\imath}_A i^A(b) = r^A \overline{z}^A(b \cup th(\mathcal{T})) = b \cup e(\mathcal{T}).$$ ☐

Pushforward maps are compatible with compositions of closed embeddings. The following proposition is similar to [Nen07, Proposition 5.1] and the same reasoning applies, so we omit the proof.

Proposition 2. *Let $Z \xrightarrow{i} Y \xrightarrow{j} X$ be closed embeddings of smooth varieties with special linear normal bundles $(N_{ji}, \lambda_{ji}), (N_i, \lambda_i), (i^* N_{ji}/N_i, \lambda_j)$ such that $\lambda_i \otimes \lambda_j = \lambda_{ji}$. Then $j_A i_A = (ji)_A$.*

5. Preliminary computations in the stable cohomotopy groups and the stable Hopf map

We will carry out preliminary computations involving $\pi^{*,*}$ and various motivic spheres. The main result of this section is Proposition 3, proved by a rather lengthy computation. We track down all the canonical isomorphisms involved, so the formulas are somewhat messy.

Throughout this section we use $X = \mathbb{A}^{n+1} - \{0\}$ for a punctured affine space with $n \geqslant 1$. Let $x = (1, 1, 0, \ldots, 0)$ be a point on X. First of all, recall the following well-known isomorphisms [MV99, Lemma 2.15, Example 2.20].

Definition 14. Set $\sigma = \sigma_2^{-1}\sigma_1 : (X, x) \xrightarrow{\simeq} (\mathbb{G}_m, 1) \wedge T^{\wedge n}$ for the canonical isomorphism in the pointed homotopy category. It is defined via

$$(X, x) \xrightarrow{\sigma_1} X/((\mathbb{A}^1 \times (\mathbb{A}^n - \{0\})) \cup (\{1\} \times \mathbb{A}^n)) \xleftarrow{\sigma_2} (\mathbb{G}_m, 1) \wedge T^{\wedge n},$$

where σ_1 is induced by the identity map on X and σ_2 is induced by the natural embedding $\mathbb{G}_m \times \mathbb{A}^n \subset X$. Recall that σ_1 is an isomorphism since $(\mathbb{A}^1 \times (\mathbb{A}^n - \{0\})) \cup (\{1\} \times \mathbb{A}^n)$

is $\mathbb{A}^1$-contractible: one can contract it in the following two steps: one first projects it onto $\{1\}\times\mathbb{A}^n$, contracting $\mathbb{A}^1\times(\mathbb{A}^n-\{0\})$ to $\{1\}\times(\mathbb{A}^n-\{0\})$, and then contracts the obtained affine space. The second morphism σ_2 is induced by the excision isomorphism $(\mathbb{G}_{m+},+)\wedge T^{\wedge n}\cong X/(X-(\mathbb{A}^1\times\{(0,0,\ldots,0)\}))$, so it is an isomorphism as well.

We write $s=s_2^{-1}s_1:(\mathbb{A}^2-\{0\},(1,1))\xrightarrow{\simeq}(\mathbb{G}_m,1)\wedge T$ for this isomorphism in the particular case of $n=1$.

Another isomorphism that we need can be easily expressed via the cone construction.

Definition 15. Let $i:Y\to Z$ be a morphism of pointed motivic spaces. The space $\operatorname{Cone}(i)$ defined via the cocartesian square

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "in_1"] & Z \arrow[d] \\ Y\wedge\Delta^1 \arrow[r] & \operatorname{Cone}(i) \end{tikzcd}$$

is called the *cone of the morphism* i.

Definition 16. Set $\rho=\rho_2\circ\rho_1^{-1}:T\xrightarrow{\simeq}(\mathbb{G}_m,1)\wedge S^1_s=S^{2,1}$ for the canonical isomorphism in the homotopy category defined via

$$T\xleftarrow{\rho_1}\operatorname{Cone}(i_\rho)\xrightarrow{\rho_2}(\mathbb{G}_m,1)\wedge S^1_s,$$

where i_ρ stands for the natural embedding $(\mathbb{G}_m,1)\to(\mathbb{A}^1,1)$ and the isomorphisms ρ_1 and ρ_2 are induced by the maps $\Delta^1\to pt$ and $\mathbb{A}^1\to pt$, respectively.

Definition 17. For every pointed motivic space Y put

$$\Sigma_T=(\mathrm{id}_Y\wedge\rho)^\pi\Sigma^{2,1}:\pi^{*,*}(Y)\to\pi^{*+2,*+1}(Y\wedge T)$$

and set $\Sigma_T^n=\Sigma_T\circ\Sigma_T\circ\cdots\circ\Sigma_T$ for the n-fold composition.

Consider the localization sequence for the embedding $\{0\}\to\mathbb{A}^{n+1}$,

$$\cdots\to\pi^{p,q}(T^{\wedge n+1})\to\pi^{p,q}(\mathbb{A}^{n+1})\to\pi^{p,q}(X)\xrightarrow{\partial}\pi^{p+1,q}(T^{\wedge n+1})\to\cdots$$

Here we identify the Thom space $\mathbb{A}^{n+1}/X=\mathbb{A}^{n+1}/(\mathbb{A}^{n+1}-\{0\})$ with $T^{\wedge n+1}=(\mathbb{A}^1/\mathbb{G}_m)^{\wedge n+1}$ via the canonical choice of coordinates on $\mathbb{A}^{n+1}$ (see [MV99, § 3, Proposition 2.17(2)]). Canonical isomorphisms described above together with the choice of the point x on X provide a splitting for the connecting homomorphism ∂. We discuss this in the next lemma. Put

$$\tau:T^{\wedge n}\wedge(\mathbb{G}_m,1)\to(\mathbb{G}_m,1)\wedge T^{\wedge n},\quad \tau_c:T^{\wedge n}\wedge T\to T\wedge T^{\wedge n}$$

for the twisting isomorphisms defined via $(x_0,x_1,\ldots,x_n)\mapsto(x_n,x_0,\ldots,x_{n-1})$.

Lemma 11. *For the canonical morphism* $r:(X_+,+)\to(X,x)$ *we have*

$$\partial r^\pi=(\tau_c^\pi)^{-1}(\mathrm{id}\wedge\rho)^\pi\Sigma^{1,0}\tau^\pi(\sigma^\pi)^{-1}.$$

Proof. On the right-hand side we have

$$(\tau_c^\pi)^{-1}(\mathrm{id}\wedge\rho)^\pi\Sigma^{1,0}\tau^\pi(\sigma^\pi)^{-1}=(\tau_c^\pi)^{-1}(\mathrm{id}\wedge\rho)^\pi(\sigma^{-1}\tau\wedge\mathrm{id})^\pi\Sigma^{1,0}=((\sigma^{-1}\tau\wedge\mathrm{id})(\mathrm{id}\wedge\rho)\tau_c^{-1})^\pi\Sigma^{1,0}.$$

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Put $Y = (\mathbb{A}^1 \times (\mathbb{A}^n - \{0\})) \cup (\{1\} \times \mathbb{A}^n)$. Let $i_Y : X/Y \to \mathbb{A}^{n+1}/Y$, $i_G : T^{\wedge n} \wedge (\mathbb{G}_m, 1) \to T^{\wedge n} \wedge (\mathbb{A}^1, 1)$ and $i_+ : (X_+, +) \to (\mathbb{A}^{n+1}_+, +)$ be the natural embeddings and let $j_1 : (\mathbb{A}^{n+1}_+, +) \to \mathrm{Cone}(i_+)$ and $j_2 : \mathrm{Cone}(i_+) \to \mathrm{Cone}(j_1)$ be the canonical maps for the cone construction.

Consider the diagram

$$\begin{tikzcd}
T^{\wedge n} \wedge (\mathbb{G}_m, 1) \wedge S^1_s \arrow[d, "\tau\wedge\mathrm{id}"', "\simeq"] & T^{\wedge n} \wedge \mathrm{Cone}(i_\rho) \arrow[l, "\mathrm{id}\wedge\rho_2"', "\simeq"] \arrow[r, "\mathrm{id}\wedge\rho_1", "\simeq"'] \arrow[d, "w"', "\simeq"] & T^{\wedge(n+1)} \arrow[r, "\tau_c", "\simeq"'] & T^{\wedge(n+1)} \\
(\mathbb{G}_m, 1) \wedge T^{\wedge n} \wedge S^1_s \arrow[d, "\sigma_2\wedge\mathrm{id}"', "\simeq"] & \mathrm{Cone}(i_G) \arrow[d, "t"', "\simeq"] & & \\
(X/Y) \wedge S^1_s & \mathrm{Cone}(i_Y) \arrow[l, "u", "\simeq"'] \arrow[rruu, "\psi_3", "\simeq"'] & & \mathrm{Cone}(i_+) \arrow[ll, "v", "\simeq"'] \arrow[uu, "\simeq", "\psi_1"'] \arrow[d, "j_2"] \\
(X, x) \wedge S^1_s \arrow[u, "\sigma_1\wedge\mathrm{id}"', "\simeq"] & & (X_+, +) \wedge S^1_s \arrow[ll, "r\wedge\mathrm{id}"'] & \mathrm{Cone}(j_1) \arrow[l, "\psi_2"', "\simeq"]
\end{tikzcd}$$

Here ψ_1, ψ_2 and ψ_3 are induced by $\Delta^1 \to pt$, v is induced by $\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}/Y$ and $X_+ \to X/Y$, u is induced by $\mathbb{A}^{n+1}/Y \to pt$, w is an obvious isomorphism $T^{\wedge n} \wedge \mathrm{Cone}(i_\rho) \cong \mathrm{Cone}(\mathrm{id} \wedge i_\rho)$ and t is induced by the commutative square

$$\begin{tikzcd}
T^{\wedge n} \wedge (\mathbb{G}_m, 1) \arrow[r] \arrow[d, "i_G"] & X/Y \arrow[d, "i_Y"] \\
T^{\wedge n} \wedge (\mathbb{A}^1, 1) \arrow[r, "\tau'_c"] & \mathbb{A}^{n+1}/Y
\end{tikzcd}$$

where $\tau'_c(x_0, x_1, \ldots, x_n) = (x_n, x_0, \ldots, x_{n-1})$.

One can easily verify that the large diagram is commutative. By the very definition we have

$$\partial r^\pi = (\psi_2 j_2 \psi_1^{-1})^\pi \Sigma^{1,0} r^\pi = ((r \wedge \mathrm{id})\psi_2 j_2 \psi_1^{-1})^\pi \Sigma^{1,0},$$

thus it is sufficient to show that

$$(r \wedge \mathrm{id})\psi_2 j_2 \psi_1^{-1} = (\sigma^{-1}\tau \wedge \mathrm{id})(\mathrm{id} \wedge \rho)\tau_c^{-1}$$

which follows from the commutativity of the above diagram. □

Definition 18. The *Hopf map* is the morphism of pointed motivic spaces

$$H : (\mathbb{A}^2 - \{0\}, (1,1)) \to (\mathbb{P}^1, [1:1])$$

given by $H(x, y) = [x : y]$. Let $\vartheta = \vartheta_2^{-1}\vartheta_1$ be the composition

$$\vartheta : (\mathbb{P}^1, [1:1]) \xrightarrow{\vartheta_1} \mathbb{P}^1/\mathbb{A}^1 \xleftarrow{\vartheta_2} T,$$

where ϑ_1 is induced by the identity map on $\mathbb{P}^1$ and ϑ_2 is the excision isomorphism given by $\vartheta_2(x) = [x : 1]$. Then the *stable Hopf map* is the unique element $\eta \in \pi^{-1,-1}(pt)$ such that $s^\pi \Sigma_T \Sigma^{1,1} \eta = \Sigma_T^\infty(\rho\vartheta H)$, i.e. η is the stabilization of H moved to $\pi^{-1,-1}(pt)$ via the canonical isomorphisms.

LEMMA 12. *Let $\widetilde{H} : (\mathbb{A}^2 - \{0\}, (1,1)) \to (\mathbb{P}^1, [1:1])$ be the morphism of pointed motivic spaces given by $\widetilde{H}(x,y) = [y:x]$ and let $\widetilde{\eta} \in \pi^{-1,-1}(pt)$ be the unique element such that $s^\pi \Sigma_T \Sigma^{1,1} \widetilde{\eta} = \Sigma_T^\infty(\rho\vartheta\widetilde{H})$. Then $\widetilde{\eta} = \epsilon \cup \eta$.*

Proof. Let $\phi : (\mathbb{A}^2 - \{0\}, (1,1)) \to (\mathbb{A}^2 - \{0\}, (1,1))$ be the reflection given by $\phi(x,y) = (y,x)$. Put $Y = (\mathbb{A}^1 \times (\mathbb{A}^2 - \{0\})) \cup (\{1\} \times \mathbb{A}^2)$ and consider the commutative diagram

$$\begin{tikzcd}
(\mathbb{A}^2 - \{0\}, (1,1)) \wedge T \arrow[r, "\phi\wedge -\mathrm{id}_T"] \arrow[d, "\psi_2"', "\simeq"] & (\mathbb{A}^2 - \{0\}, (1,1)) \wedge T \arrow[d, "\psi_2"', "\simeq"] \\
(\mathbb{A}^3 - \{0\})/Y \arrow[r, "\psi_3", "\simeq"'] & (\mathbb{A}^3 - \{0\})/Y \\
(\mathbb{G}_m, 1) \wedge T \wedge T \arrow[r, "\mathrm{id}\wedge(-\mathrm{id}_{T\wedge T})"] \arrow[u, "\psi_1", "\simeq"'] & (\mathbb{G}_m, 1) \wedge T \wedge T \arrow[u, "\psi_1", "\simeq"']
\end{tikzcd}$$

Here ψ_1 is induced by the inclusion $\mathbb{G}_m \times \mathbb{A}^2 \to \mathbb{A}^3 - \{0\}$, ψ_2 is given by $\psi_2(x,y,z) = ((x+y)/2, (x-y)/2, z)$ and $\psi_3(x,y,z) = (x,-y,-z)$. All the morphisms ψ_i are isomorphisms: ψ_1 is an excision isomorphism, ψ_3 is an involution and ψ_2 can be decomposed in an obvious way,

$$(\mathbb{A}^2 - \{0\}, (1,1)) \wedge T \xrightarrow{\psi_2'} (\mathbb{A}^2 - \{0\})/((\mathbb{A}^1 \times \mathbb{G}_m) \cup (\{1\} \times \mathbb{A}^1)) \wedge T \to (\mathbb{A}^3 - \{0\})/Y,$$

with the first map $\psi_2'(x,y,z) = ((x+y)/2, (x-y)/2, z)$ being an isomorphism since $((\mathbb{A}^1 \times \mathbb{G}_m) \cup (\{1\} \times \mathbb{A}^1))$ is $\mathbb{A}^1$-contractible and the second map being an excision isomorphism. It is well known that $-\mathrm{id}_{T\wedge T} = \mathrm{id}_{T\wedge T}$ in the homotopy category: one has

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so there is an $\mathbb{A}^1$-homotopy between $\mathrm{id}_{T\wedge T}$ and $-\mathrm{id}_{T\wedge T}$ given by the matrix

$$h(t) = \begin{pmatrix} 1 & 0 \\ -2t & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2t & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Thus we obtain $\phi \wedge -\mathrm{id}_T = \psi_2^{-1}\psi_1(\mathrm{id} \wedge \mathrm{id}_{T\wedge T})\psi_1^{-1}\psi_2 = \mathrm{id} \wedge \mathrm{id}_T$, yielding

$$(\rho\vartheta\widetilde{H}) \wedge \mathrm{id}_T = ((\rho\vartheta\widetilde{H}) \wedge \mathrm{id}_T)(\phi \wedge -\mathrm{id}_T) = (\rho\vartheta H) \wedge -\mathrm{id}_T.$$

Taking the Σ_T^∞-suspension and using the suspension isomorphism Σ_T^{-1}, we get

$$\Sigma_T^\infty(\rho\vartheta H) \cup \epsilon = \Sigma_T^\infty(\rho\vartheta\widetilde{H}).$$

The suspension isomorphisms as well as ρ^π and s^π are homomorphisms of $\pi^{0,0}(pt)$-modules, and ϵ is central, thus

$$s^\pi \Sigma_T \Sigma^{1,1}(\epsilon \cup \eta) = \Sigma_T^\infty(\rho\vartheta H) \cup \epsilon = \Sigma_T^\infty(\rho\vartheta\widetilde{H}) = s^\pi \Sigma_T \Sigma^{1,1}(\widetilde{\eta}).$$

The claim follows via taking $(s^\pi)^{-1}$ and desuspending. $\square$

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Recall that for stable cohomotopy groups we have canonical Thom classes for trivialized vector bundles $th(\mathcal{O}_X^n) = \Sigma_T^n 1$ and pushforwards i_π for the closed embeddings with a trivialized normal bundle (N_i, θ).

We fix the following notation. Let $i : \mathbb{G}_m \to X$ be a closed embedding to the zeroth coordinate given by $i(t) = (t, 0, \ldots, 0)$. Identify the normal bundle

$$N_i \cong U = \mathbb{G}_m \times \mathbb{A}^n \subset X$$

with the Zariski neighbourhood U of $\mathbb{G}_m$ and define the trivialization $\theta : U \xrightarrow{\simeq} \mathbb{G}_m \times \mathbb{A}^n$ via

$$\theta(t, x_1, \ldots, x_n) = (t, x_1/t, x_2, \ldots, x_n).$$

This particular trivialization will arise naturally in Lemma 15. There is a pushforward map

$$i_\pi : \pi^{0,0}(\mathbb{G}_m) \to \pi^{2n,n}(X)$$

induced by the trivialization θ. In Lemma 15 we will need to know the image of the unit under this pushforward, or, more precisely, $\partial i_\pi(1)$.

Proposition 3. *In the above notation we have* $\partial i_\pi(1) = (-1)^n \epsilon \cup \Sigma_T^{n+1} \eta$.

Proof. From the construction of the pushforward map we have

$$i_\pi(1) = z^\pi d_i^\pi(th(U, \theta))$$

with $z^\pi : \pi^{*,*}(\mathrm{Th}(i)) \to \pi^{*,*}(X)$ being the support extension and d_i^π the deformation to the normal bundle isomorphism. Represent i as the composition

$$i : \mathbb{G}_m \xrightarrow{i_1} U \xrightarrow{i_2} X$$

and let $w : \mathrm{Th}(i_1) \xrightarrow{\simeq} \mathrm{Th}(i)$ be the induced isomorphism in the homotopy category. Recall that for U there is a natural isomorphism [Nen07, proof of Proposition 3.1] $D(\mathbb{G}_m, U) \cong U \times \mathbb{A}^1$ and $d_{i_1}^\pi = \mathrm{id}$. By the functoriality of the deformation construction we have $d_i^\pi = (w^\pi)^{-1}$, so we need to compute

$$\partial z^\pi (w^\pi)^{-1}(th(U, \theta)).$$

Decomposing z as

$$z : (X_+, +) \xrightarrow{r} (X, x) \xrightarrow{z_1} \mathrm{Th}(i)$$

and using Lemma 11, we obtain

$$\begin{aligned}\partial z^\pi (w^\pi)^{-1}(th(U, \theta)) &= \partial r^\pi z_1^\pi (w^\pi)^{-1}(th(U, \theta)) \\ &= (\tau_c^\pi)^{-1}(\mathrm{id} \wedge \rho)^\pi \Sigma^{1,0}(\tau^\pi (\sigma^\pi)^{-1} z_1^\pi (w^\pi)^{-1}(th(U, \theta))).\end{aligned}$$

We can represent the Thom class $th(U, \theta) \in \pi^{2n,n}(\mathrm{Th}(i_1))$ by Σ_T^∞-suspension of the composition

$$\mathrm{Th}(i_1) \xrightarrow{\widetilde{H}_2} T^{\wedge n} \xrightarrow{\rho^{\wedge n}} (S^{2,1})^{\wedge n} \xrightarrow{\Xi_n} S^{2n,n},$$

where $\widetilde{H}_2$ is given by $\widetilde{H}_2(t, x_1, x_2, \ldots, x_n) = (x_1/t, x_2, \ldots, x_n)$, and Ξ_n is the canonical shuffling isomorphism.

Identifying the first copy of T with $\mathbb{P}^1/\mathbb{A}^1$ via $\theta_2(x) = [x : 1]$, we rewrite $\widetilde{H}_2$ as $\widetilde{H}_2 = (\theta_2^{-1} \wedge \mathrm{id})\widetilde{H}_1$ with $\widetilde{H}_1$ given by

$$\widetilde{H}_1(t, x_1, x_2, \ldots, x_n) = ([x_1 : t], x_2, \ldots, x_n).$$

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Put $Y = (\mathbb{A}^1 \times (\mathbb{A}^n - \{0\})) \cup (\{1\} \times \mathbb{A}^n)$ and consider the diagram

$$\begin{tikzcd}
(\mathbb{P}^1/\mathbb{A}^1) \wedge T^{\wedge n-1} & T \wedge T^{\wedge n-1} \arrow[l, "\vartheta_2 \wedge \mathrm{id}", "\simeq"'] \arrow[r, "\Xi_n \rho^{\wedge n}", "\simeq"'] & S^{2n,n} \\
\mathrm{Th}(i_1) \arrow[u, "\widetilde{H}_1"] \arrow[r, "w_1", "\simeq"'] & ((\mathbb{A}^2 - \{0\})/\mathbb{A}^1 \times \mathbb{G}_m) \wedge T^{\wedge n-1} \arrow[ul, "\widetilde{H}_3 \wedge \mathrm{id}"'] \arrow[r, "w_2", "\simeq"'] & \mathrm{Th}(i) \arrow[dl, "j'"'] \\
(\mathbb{A}^2 - \{0\}, (1,1)) \wedge T^{\wedge n-1} \arrow[ur, "j \wedge \mathrm{id}"] \arrow[r, "\psi_1", "\simeq"'] & X/Y & (X, x) \arrow[l, "\sigma_1"', "\simeq"] \arrow[u, "z_1"']
\end{tikzcd}$$

Here $\widetilde{H}_3(x,y) = [y : x]$ and all the other maps are given by the tautological inclusions, i.e. w_1 is induced by the inclusion $U = \mathbb{G}_m \times \mathbb{A}^n \subset (\mathbb{A}^2 - \{0\}) \times \mathbb{A}^{n-1}$, w_2 and ψ_1 are induced by $(\mathbb{A}^2 - \{0\}) \times \mathbb{A}^{n-1} \subset X$, j' is given by the identity map on X and j is given by identity map on $\mathbb{A}^2 - \{0\}$. Morphisms w_1 and w_2 are excision isomorphisms and ψ_1 is the composition of the isomorphism $s_1 \wedge \mathrm{id}$ and an excision isomorphism (see the next diagram), so it is an isomorphism as well. One can easily check that this diagram is commutative. Hence

$$\begin{aligned}
z_1^{\pi}(w^{\pi})^{-1}(th(U,\theta)) &= z_1^{\pi}((w_2 w_1)^{\pi})^{-1}(th(U,\theta)) \\
&= \Sigma_T^{\infty}(\Xi_n \rho^{\wedge n}(\vartheta_2^{-1}\widetilde{H}_3 j \wedge \mathrm{id})\psi_1^{-1}\sigma_1) = (\psi_1^{-1}\sigma_1)^{\pi}\Sigma_T^{n-1}(\Sigma_T^{\infty}(\rho\vartheta\widetilde{H})),
\end{aligned}$$

with $\widetilde{H} = \widetilde{H}_3 j$. There is the following commutative diagram consisting of isomorphisms:

$$\begin{tikzcd}
(\mathbb{A}^2 - \{0\}, (1,1)) \wedge T^{\wedge n-1} \arrow[rr, "\psi_1", "\simeq"'] \arrow[d, "s_1 \wedge \mathrm{id}"', "\simeq"] & & X/Y \\
(\mathbb{A}^2 - \{0\})/((\mathbb{A}^1 \times \mathbb{G}_m) \cup (\{1\} \times \mathbb{A}^1)) \wedge T^{\wedge n-1} \arrow[urr, "\simeq"'] & & (\mathbb{G}_m, 1) \wedge T \wedge T^{\wedge n-1} \arrow[ll, "s_2 \wedge \mathrm{id}"', "\simeq"] \arrow[u, "\sigma_2", "\simeq"']
\end{tikzcd}$$

All the maps in the diagram are induced by the tautological inclusions, s_2 is induced by $\mathbb{G}_m \times \mathbb{A}^1 \subset \mathbb{A}^2 - \{0\}$ and s_1 is given by the identity map on $\mathbb{A}^2 - \{0\}$. Morphisms σ_2, s_2 and the diagonal morphism are excision isomorphisms and s_1 is an isomorphism via the usual contraction argument.

Thus we have

$$\begin{aligned}
\Sigma^{1,0}((\psi_1^{-1}\sigma_1\sigma^{-1}\tau)^{\pi}\Sigma_T^{n-1}(\Sigma_T^{\infty}(\rho\vartheta\widetilde{H}))) &= \Sigma^{1,0}(((s_1^{-1}s_2 \wedge \mathrm{id})\tau)^{\pi}\Sigma_T^{n-1}(\Sigma_T^{\infty}(\rho\vartheta\widetilde{H}))) \\
&= \Sigma^{1,0}(\tau^{\pi}\Sigma_T^{n-1}((s_1^{-1}s_2)^{\pi}\Sigma_T^{\infty}(\rho\vartheta\widetilde{H}))).
\end{aligned}$$

To sum up, the above considerations together with Lemma 12 yield

$$\begin{aligned}
\partial i_{\pi}(1) &= ((\mathrm{id} \wedge \rho)\tau_c^{-1})^{\pi}\Sigma^{1,0}(\tau^{\pi}\Sigma_T^{n-1}((s_1^{-1}s_2)^{\pi}\Sigma_T^{\infty}(\rho\vartheta\widetilde{H}))) \\
&= \epsilon \cup ((\mathrm{id} \wedge \rho)\tau_c^{-1})^{\pi}\Sigma^{1,0}(\tau^{\pi}\Sigma_T^{n}\Sigma^{1,1}\eta) = \epsilon \cup ((\tau \wedge \mathrm{id})(\mathrm{id} \wedge \rho)\tau_c^{-1})^{\pi}\Sigma^{1,0}\Sigma_T^{n}\Sigma^{1,1}\eta.
\end{aligned}$$

Now we examine the homomorphism

$$\Theta = ((\tau \wedge \mathrm{id})(\mathrm{id} \wedge \rho)\tau_c^{-1})^{\pi}\Sigma^{1,0}\Sigma_T^{n}\Sigma^{1,1} : \pi^{-1,-1}(pt) \to \pi^{2n+1,n}(T^{\wedge n+1}).$$

Unravelling the notation, this homomorphism can be represented as an external product with a Σ_T^{∞}-suspension of the map $T^{\wedge n+1} \to S^{2n+2,n+1}$ corresponding to the following picture consisting of copies of the morphism ρ and identity maps:

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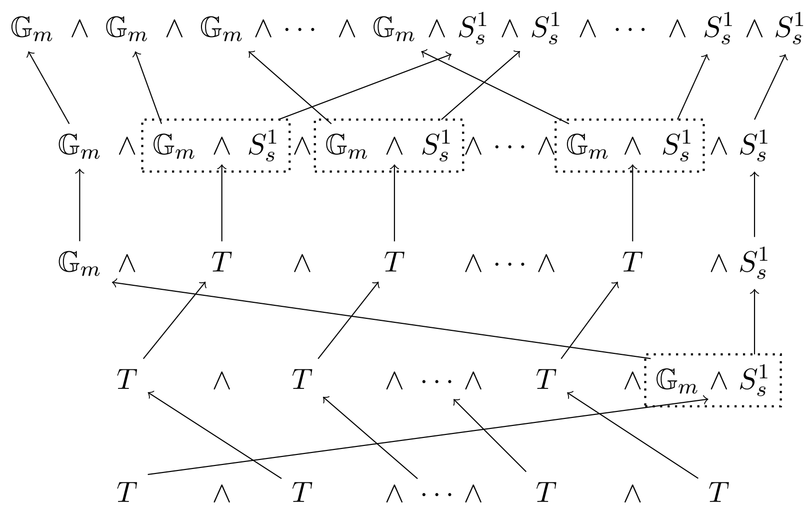
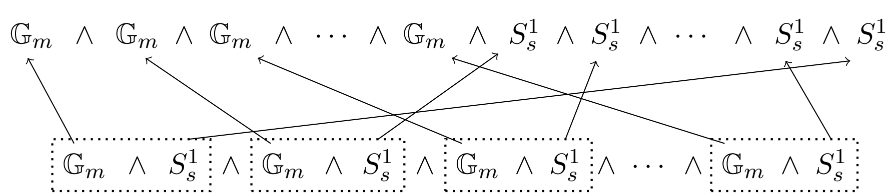
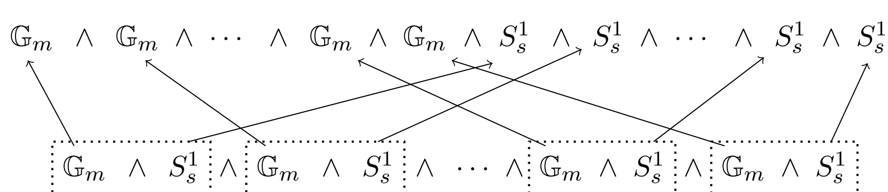


Here the top row of arrows is the canonical shuffling isomorphism, the top row and second row combined correspond to $\Sigma^{1,0}\Sigma_T^n\Sigma^{1,1}$, the third row is $\tau \wedge \mathrm{id}$ and the bottom row is $(\mathrm{id} \wedge \rho)\tau_c^{-1}$. Taking the composition, we obtain (writing $\mathbb{G}_m \wedge S_s^1$ instead of T)



The corresponding picture for Σ_T^{n+1} is



These pictures coincide up to a cyclic permutation of S_s^1-s. This permutation automorphism equals $(-1)^n$ in the homotopy category, thus $\Theta = (-1)^n\Sigma_T^{n+1}$. $\square$

6. Inverting the stable Hopf map

Let $A^{*,*}(-)$ be the bigraded ring cohomology theory represented by a commutative monoid $A \in \mathcal{SH}(k)$. Inverting $\eta \in A^{-1,-1}(pt)$, we obtain a new cohomology theory with $(2i, i)$ groups isomorphic to $(2i+n, i+n)$ ones by means of the cup product with η^{-n}. Put

$$A^n(Y) = (A_\eta^{*,*}(Y))^{n,0} = (A^{*,*}(Y) \otimes_{A^{*,*}(pt)} A^{*,*}(pt)[\eta^{-1}])^{n,0},$$
$$A^*(Y) = (A_\eta^{*,*}(Y))^{*,0} = \bigoplus_{n\in\mathbb{Z}} A^n(Y).$$

One can easily see that this is a cohomology theory. For the algebraic K-theory represented by BGL [PPR09b] this construction gives $\mathrm{BGL}^*(-) = 0$ since we have $\eta \in \mathrm{BGL}^{-1,-1}(pt) = K_{-1}(pt) = 0$ and $\mathrm{BGL}^{*,*}_\eta(-) = 0$. As we will see in Corollary 1, it is always the case that an oriented cohomology theory degenerates to a trivial cohomology theory. Thus we are interested in cohomology theories with a special linear orientation but without a general one. Our running example is hermitian K-theory represented by the spectrum BO that transforms to Witt groups, i.e. for every smooth variety X there is a natural isomorphism $BO^i_\eta(X) \cong W^i(X)$ (see [Aan13]).

For stable cohomotopy groups there is the following result by Morel.

Theorem 1. *There exists a canonical isomorphism* $(\pi^{*,*}_\eta(pt))^{0,0} \xrightarrow{\simeq} W^0(pt)$.

Proof. This follows from [Mor12, Corollary 6.43]. □

Definition 19. From now on $A^*(-)$ denotes a graded ring cohomology theory obtained via the above construction, i.e.

$$A^*(Y) = (A^{*,*}_\eta(Y))^{*,0}$$

for a bigraded SL-oriented ring cohomology theory $A^{*,*}(-)$ represented by a commutative monoid $A \in \mathcal{SH}(k)$. We have Thom and Euler classes and all the machinery of SL-oriented theories, including the Gysin sequences and pushforwards. In order to stay in the chosen grading we need to modify the Thom and Euler classes as follows:

$$th'(\mathcal{T}) = (-1)^{n(n-1)/2} th(\mathcal{T}) \cup \eta^n, \quad e'(\mathcal{T}) = (-1)^{n(n-1)/2} e(\mathcal{T}) \cup \eta^n,$$

for a special linear bundle $\mathcal{T}$ of rank n. The sign is introduced for the sake of multiplicativity of the characteristic classes. For ease of notation, we will omit the primes and write just $th(\mathcal{T})$ and $e(\mathcal{T})$ and refer to them as Thom and Euler classes. These classes are of degree n.

Remark 6. Note that from ϵ-commutativity we have $\eta \cup \eta = -\epsilon \cup (\eta \cup \eta)$, thus, inverting η, we obtain $\epsilon = -1$ in $A^*(pt)$.

We conclude this section with the following simplification of Lemma 6.

Lemma 13. *Let $A^*(-)$ be a graded ring cohomology theory described in Definition 19 and let $\mathcal{T} = (E, \lambda)$ be a rank 2 special linear bundle. Then*

$$b_*(H(E)) = 1 - e(\mathcal{T})^2 t^2, \quad p_*(\mathcal{T}) = 1 + e(\mathcal{T})^2 t^2.$$

Proof. This follows from the above remark and Lemma 6. □

7. The complement to the zero section

In this section we compute the cohomology of the complement to the zero section of a special linear vector bundle. It turns out that there is a good answer in terms of characteristic classes only in the case of odd rank.

Recall that for a special linear bundle $\mathcal{T}$ we denote by $\mathcal{T}^0$ the complement to the zero section. We start with the following lemma concerning the case of a special linear bundle possessing a nonvanishing section.

Definition 20. We denote an operator of the $\cup$-product with an element by the symbol of the element, writing α for $- \cup \alpha$.

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Lemma 14. *Let $\mathcal{T}$ be a rank k special linear bundle over a smooth variety X with a nowhere vanishing section $s : X \to \mathcal{T}$. Then for some $\alpha \in A^{k-1}(\mathcal{T}^0)$ we have an isomorphism*

$$(1, \alpha) : A^*(X) \oplus A^{*+1-k}(X) \to A^*(\mathcal{T}^0).$$

Proof. Consider the Gysin sequence

$$\cdots \to A^{*-k}(X) \xrightarrow{0} A^*(X) \xrightarrow{j^A} A^*(\mathcal{T}^0) \xrightarrow{\partial_A} A^{*-k+1}(X) \xrightarrow{0} \cdots$$

The section s induces the splitting s^A for j^A, hence gives a splitting r for ∂_A. We have the claim for $\alpha = r(1)$, since all the maps involved are homomorphisms of $A^*(X)$-modules. □

We want to obtain an isomorphism which does not depend on the choice of a section, so we act as one acts in the projective bundle theorem for oriented cohomology theories: take a certain special linear bundle over $\mathcal{T}^0$ and compute its Euler class.

Definition 21. Let $p : E \to X$ be a vector bundle over a smooth variety X. Restricting the 'diagonal' section $\mathcal{O}_E \to p^*E$ to E^0, we get an exact sequence

$$0 \to \mathcal{O}_{E^0} \to p^*E|_{E^0} \to \mathcal{T}_E \to 0.$$

If E is a special linear bundle, then, by Lemma 2, so is $\mathcal{T}_E$.

For the derived Witt groups there is a computation of $W^*(\mathbb{A}^n - \{0\})$ due to Balmer and Gille. In the case of the odd dimension it can be formulated as follows.

Theorem 2. *Let $(E, \lambda) = (\mathcal{O}_{pt}^{2n+1}, 1)$ be a trivialized special linear bundle of odd rank over a point with $n \geqslant 1$. Then we have an isomorphism of graded $W^*(pt)$-algebras*

$$W^*(pt)[e]\Big/(e^2) \xrightarrow{\simeq} W^*(\mathbb{A}^{2n+1} - \{0\})$$

induced by $e \mapsto e(\mathcal{T}_E) \in W^{2n}(\mathbb{A}^{2n+1} - \{0\})$.

Proof. See [BG05, Theorem 8.13] for an explicit basis of $W^*(\mathbb{A}^{2n+1} - \{0\})$ over $W^*(pt)$ in terms of a certain Koszul complex. One can identify this Koszul complex with the Euler class $e(\mathcal{T}_E)$ (see [Nen07, § 2.5] for the definition of Thom classes in derived Witt groups). □

We can derive an analogous result for $A^*(-)$ from our computation in stable cohomotopy groups.

Lemma 15. *Let $(E, \lambda) = (\mathcal{O}_{pt}^{2n+1}, 1)$, $n \geqslant 1$, be a trivialized special linear bundle over a point. Then we have an isomorphism of graded $A^*(pt)$-algebras*

$$A^*(pt)[e]\Big/(e^2) \xrightarrow{\simeq} A^*(E^0).$$

induced by $e \mapsto e(\mathcal{T}_E) \in A^{2n}(E^0)$.

Proof. Consider the Gysin sequence

$$\cdots \to A^{*-2n-1}(pt) \xrightarrow{0} A^*(pt) \to A^*(E^0) \xrightarrow{\partial_A} A^{*-2n}(pt) \xrightarrow{0} \cdots$$

The bundle E is trivial, hence $e(E, \lambda) = 0$ and the Gysin sequence consists of short exact sequences.

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Consider the dual special linear bundle $\mathcal{T}_E^\vee$. Taking the dual trivialization of $E^\vee$, we obtain

$$\mathcal{T}_E^\vee = \{(x_0, \ldots, x_{2n}, y_0, \ldots, y_{2n}) \in E^0 \times E^\vee \mid x_0 y_0 + \cdots + x_{2n} y_{2n} = 0\}.$$

There is a section $s : E^0 \to \mathcal{T}_E^\vee$ with

$$s(x_0, x_1, x_2, \ldots, x_{2n-1}, x_{2n}) = (x_0, x_1, \ldots, x_{2n}, 0, x_2, -x_1, \ldots, x_{2n}, -x_{2n-1}).$$

This section meets the zero section in $\mathbb{G}_m \cong \{(t, 0, \ldots, 0) \mid t \neq 0\}$. Proposition 1 states that $e(\mathcal{T}_E^\vee) = i_A(1)$ for the inclusion $i : \mathbb{G}_m \to \mathbb{A}^{2n+1} - \{0\}$ with the trivialization of $\det N_i$ arising from the trivialization of $\det \mathcal{T}_E^\vee$. Identify $N_i \cong \mathcal{T}_E^\vee|_{\mathbb{G}_m}$ with $U = \mathbb{G}_m \times \mathbb{A}^{2n} \subset E^0$ via

$$(t, 0, \ldots, 0, 0, y_1, \ldots, y_{2n}) \mapsto (t, y_1, \ldots, y_{2n}).$$

The isomorphism $\lambda_{\mathcal{T}_E^\vee} : \det \mathcal{T}_E^\vee|_{\mathbb{G}_m} \xrightarrow{\simeq} \mathcal{O}_{\mathbb{G}_m}$ arises from the canonical trivialization of $E^\vee|_{\mathbb{G}_m}$ and morphism $\phi : E^\vee|_{\mathbb{G}_m} \to \mathcal{O}_{\mathbb{G}_m}$ with

$$\phi(t, y_0, y_1, \ldots, y_{2n}) = (t, t y_0).$$

Thus over t for $\mathbf{y}^i = (y_1^i, y_2^i, \ldots, y_{2n}^i)$ we have

$$\lambda_{\mathcal{T}_E^\vee}(\mathbf{y}^1 \wedge \mathbf{y}^2 \wedge \cdots \wedge \mathbf{y}^{2n}) = \det \begin{pmatrix} 1/t & 0 & 0 & \cdots & 0 \\ 0 & y_1^1 & y_1^2 & \cdots & y_1^{2n} \\ 0 & y_2^1 & y_2^2 & \cdots & y_2^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & y_{2n}^1 & y_{2n}^2 & \cdots & y_{2n}^{2n} \end{pmatrix}$$

and $\theta : (U, \lambda_{\mathcal{T}_E^\vee}) \xrightarrow{\simeq} (\mathcal{O}_{\mathbb{G}_m}^{2n}, 1)$ with $\theta(t, y_1, y_2, \ldots, y_{2n}) = (t, y_1/t, y_2, \ldots, y_{2n})$ is an isomorphism of special linear bundles.

Consider the following diagram with i_π being the pushforward in stable cohomotopy groups for the closed embedding i with the trivialization θ of the normal bundle:

$$\begin{tikzcd} A^0(\mathbb{G}_m) \arrow[r, "i_A"] & A^{2n}(E^0) \arrow[r, "\partial_A"] & A^0(pt) \\ \pi^0(\mathbb{G}_m) \arrow[u] \arrow[r, "i_\pi"] & \pi^{2n}(E^0) \arrow[u] \arrow[r, "\partial_\pi"] & \pi^0(pt) \arrow[u] \end{tikzcd}$$

The left-hand side commutes since θ is an isomorphism of special linear bundles. The right-hand side consists of the structure morphisms for A^* and the boundary maps for the Gysin sequences of the inclusion $\{0\} \to E$, hence commutes as well. Proposition 3 states that $\partial_\pi i_\pi(1) = -1$, thus

$$\partial_A(e(\mathcal{T}_E^\vee)) = \partial_A i_A(1) = -1.$$

Hence, examining the short exact sequence

$$0 \to A^*(pt) \to A^*(E^0) \xrightarrow{\partial_A} A^{*-2n}(pt) \to 0$$

given by the Gysin sequence, we obtain that $\{1, e(\mathcal{T}_E^\vee)\}$ is a basis of $A^*(E^0)$ over $A^*(pt)$.

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There is a nowhere vanishing section of $\mathcal{T}_E^\vee \oplus \mathcal{T}_E^\vee$ constructed analogously to s defined above, so

$$e(\mathcal{T}_E^\vee)^2 = e(\mathcal{T}_E^\vee \oplus \mathcal{T}_E^\vee) = 0.$$

Lemma 8 yields that for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in A^*(pt)$ we have

$$e(\mathcal{T}_E) = (\alpha_1 + \beta_1 \cup e(\mathcal{T}_E^\vee)) \cup e(\mathcal{T}_E^\vee) = \alpha_1 \cup e(\mathcal{T}_E^\vee),$$

$$e(\mathcal{T}_E^\vee) = (\alpha_2 + \beta_2 \cup e(\mathcal{T}_E^\vee)) \cup e = \alpha_2 \cup \alpha_1 \cup e(\mathcal{T}_E^\vee).$$

We already know that $\{1, e(\mathcal{T}_E^\vee)\}$ is a basis, thus $\alpha_2 \cup \alpha_1 = 1$ and α_1 is invertible. Hence $\{1, \alpha_1 \cup e(\mathcal{T}_E^\vee)\} = \{1, e(\mathcal{T}_E)\}$ is a basis as well and $e(\mathcal{T}_E)^2 = \alpha_1^2 \cup e(\mathcal{T}_E^\vee)^2 = 0$, so the claim follows. □

Corollary 1. *Let $A^{*,*}(-)$ be an oriented cohomology theory represented by a commutative monoid $A \in \mathcal{SH}(k)$. Then $A^*(pt) = 0$.*

Proof. There is a natural special linear orientation on $A^{*,*}(-)$ obtained by setting $th(E, \lambda) = th(E)$, with the latter Thom class arising from the orientation of $A^{*,*}(-)$. Hence for a rank n special linear bundle we have $e(E, \lambda) = c_n(E)$. By the above lemma, for $E = \mathcal{O}_{pt}^3$ there is an isomorphism

$$(1, c_2(\mathcal{T}_E)) : A^*(pt) \oplus A^{*-2}(pt) \xrightarrow{\simeq} A^*(E^0).$$

Multiplicativity of total Chern classes yields $c_*(\mathcal{O}_{E^0})c_*(\mathcal{T}_E) = c_*(\mathcal{O}_{E^0}^3)$, hence $c_2(\mathcal{T}_E) = 0$. The above isomorphism yields $A^*(pt) = 0$. □

Having a canonical basis for $A^*(\mathcal{O}_{pt}^{2n+1} - \{0\})$ over $A^*(pt)$ as in Lemma 15, we can glue it and obtain a basis for the cohomology of the complement to the zero section of an arbitrary special linear bundle of odd rank.

Theorem 3. *Let (E, λ) be a special linear bundle of rank $2n + 1, n \geqslant 1$, over a smooth variety X. Then for $e = e(\mathcal{T}_E)$ we have an isomorphism*

$$(1, e) : A^*(X) \oplus A^{*-2n}(X) \xrightarrow{\simeq} A^*(E^0).$$

Proof. The general case is reduced to the case of the trivial vector bundle E via the usual Mayer–Vietoris arguments. In the latter case we have a commutative diagram of the Gysin sequences

$$\begin{tikzcd}
0 \arrow[r] & A^*(X) \arrow[r] & A^*(E^0) \arrow[r, "\partial_A"] & A^{*-2n}(X) \arrow[r] & 0 \\
0 \arrow[r] & A^*(pt) \arrow[r] \arrow[u] & A^*(E'^0) \arrow[r, "\partial_A"] \arrow[u] & A^{*-2n}(pt) \arrow[r] \arrow[u, "p^A"] & 0
\end{tikzcd}$$

with $E' = \mathcal{O}_{pt}^{2n+1}$. By Lemma 15 the element $\partial_A(e(\mathcal{T}_{E'}))$ generates $A^{*-2n}(pt)$ as a module over $A^*(pt)$, thus for a certain $\alpha \in A^*(pt)$ we have $\alpha \cup \partial_A(e(\mathcal{T}_{E'})) = 1$. Using $E = p^*E'$, we obtain

$$\alpha \cup \partial_A(e(\mathcal{T}_E)) = \alpha \cup p^A \partial_A(e(\mathcal{T}_{E'})) = 1,$$

so $\partial_A(e(\mathcal{T}_E))$ generates $A^{*-2n}(X)$ over $A^*(X)$. Hence $(1, e)$ is an isomorphism. □

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Remark 7. In the case of $\operatorname{rank} E = 1$ one still has an isomorphism: a special linear bundle of rank one is a trivialized line bundle, hence there is an isomorphism

$$A^*(X) \oplus A^*(X) \cong A^*(E^0) = A^*(X \times \mathbb{G}_m)$$

induced by the isomorphism $A^*(pt) \oplus A^*(pt) \cong A^*(\mathbb{G}_m)$.

Corollary 2. *Let $\mathcal{T}$ be a special linear bundle of odd rank over a smooth variety X. Then $e(\mathcal{T}) = 0$.*

Proof. Set $\operatorname{rank} \mathcal{T} = 2n+1$ and $e = e(\mathcal{T})$. Consider the Gysin sequence

$$\cdots \to A^0(X) \xrightarrow{e} A^{2n+1}(X) \xrightarrow{j^A} A^{2n+1}(\mathcal{T}^0) \to A^1(X) \to \cdots$$

The above calculations show that j^A is injective, hence $e = 0$. □

8. Special linear projective bundle theorem

In this section we obtain a special linear version of the projective bundle theorem. First of all, we introduce the varieties that behave in a special linear context similarly to projective spaces and Grassmannians in the context of general orientation.

Definition 22. For $k < n$ consider the group

$$P'_k = \begin{pmatrix} SL_k & * \\ 0 & SL_{n-k} \end{pmatrix}.$$

The quotient variety $\mathrm{SGr}(k,n) = SL_n/P'_k$ is called a *special linear Grassmann variety*. Put $\mathrm{SGr}_X(k,n) = X \times \mathrm{SGr}(k,n)$. We denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the *tautological special linear bundles* over $\mathrm{SGr}_X(k,n)$ with $\operatorname{rank} \mathcal{T}_1 = k$ and $\operatorname{rank} \mathcal{T}_2 = n-k$.

Remark 8. Consider the parabolic subgroup

$$P_k = SL_n \cap \begin{pmatrix} GL_k & * \\ 0 & GL_{n-k} \end{pmatrix}.$$

We have a projection $SL_n/P'_k \to SL_n/P_k$ identifying the special linear Grassmann variety with the complement to the zero section of the determinant of the tautological vector bundle over the ordinary Grassmann variety $\mathrm{Gr}(k,n)$. This yields the following geometrical description of $\mathrm{SGr}(k,n)$. Fix a vector space V of dimension n. Then

$$\mathrm{SGr}(k,n) = \{(U \leqslant V, \lambda \in (\Lambda^k U)^0) \mid \dim U = k\}.$$

In particular, we have $\mathrm{SGr}(1,n) \cong \mathbb{A}^n - \{0\}$, since the above description says that it is the variety of nonzero vectors in one-dimensional subspaces of V, i.e. all the nonzero vectors of V. On the other hand, one can see that $\mathrm{SGr}(1,n) \cong \mathbb{A}^n - \{0\}$ from the definition: the canonical left action of SL_n on $\mathbb{A}^n - \{0\}$ is transitive and the stabilizer of the point $(1,0,\ldots,0)$ equals P'_1.

Theorem 4. *For a smooth variety X we have the isomorphisms:*

$$(1, e_1, \ldots, e_1^{2n-2}, e_2) : \bigoplus_{i=0}^{2n-2} A^{*-2i}(X) \oplus A^{*-2n+2}(X) \xrightarrow{\simeq} A^*(\mathrm{SGr}_X(2,2n)),$$

$$(1, e_1, e_1^2, \ldots, e_1^{2n-1}) : \bigoplus_{i=0}^{2n-1} A^{*-2i}(X) \xrightarrow{\simeq} A^*(\mathrm{SGr}_X(2,2n+1)),$$

with $e_1 = e(\mathcal{T}_1)$, $e_2 = e(\mathcal{T}_2)$.

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Proof. We will deal with several special linear Grassmann varieties at once, so we will use $\mathcal{T}_i(r,k)$ for $\mathcal{T}_i$ over $\mathrm{SGr}_X(r,k)$ and abbreviate $e(\mathcal{T}_i(r,k))$ to $e_i(r,k)$ and $e(\mathcal{T}_i(r,k)^\vee)$ to $e_i^\vee(r,k)$. The proof is by induction on the dimension of the Grassmannian.

The base case. We have $\mathrm{SGr}_X(2,3) \cong \mathrm{SGr}_X(1,3) \cong \mathbb{A}^3_X - \{0\}$ and under these isomorphisms the bundle $\mathcal{T}_1(2,3)^\vee$ goes to $\mathcal{T}_2(1,3)$ which goes to $\mathcal{T}_{\mathcal{O}^3_X}$ in the notation of Definition 21. Note that $\operatorname{rank} \mathcal{T}_1(2,3) = 2$, thus $\mathcal{T}_1(2,3) \cong \mathcal{T}_1(2,3)^\vee$ and $e(\mathcal{T}_1(2,3)) = e(\mathcal{T}_1(2,3)^\vee)$. Hence Theorem 3 gives the claim for $\mathrm{SGr}_X(2,3)$.

Basic geometry. Fix a vector space V of dimension $k+1$, a subspace $W \leqslant V$ of codimension one and forms $\mu_1 \in (\Lambda^{k+1}V)^0, \mu_2 \in (\Lambda^k W)^0$. Then we have the following diagram constructed in the same vein as in the case of ordinary Grassmannians:

$$\begin{tikzcd}
\mathrm{SGr}(2,k) \arrow[r, "i"] & \mathrm{SGr}(2,k+1) & Y \arrow[l, "j"'] \arrow[d, "p"] \\
 & & \mathrm{SGr}(1,k)
\end{tikzcd}$$

where the inclusion i corresponds to the pairs $(U, \mu \in (\Lambda^2 U)^0)$ with $U \leqslant W$, $\dim U = 2$; the open complement Y consists of the pairs $(U, \mu \in (\Lambda^2 U)^0)$ with $\dim U = 2, \dim U \cap W = 1$; the projection p is given by $p(U,\mu) = (U \cap W, \mu')$ where μ' is given by the isomorphisms $\Lambda^k W \otimes V/W \cong \Lambda^{k+1} V$ and $\Lambda^{k-1}(U \cap W) \otimes V/W \cong \Lambda^2 U$ and forms μ_1, μ_2 and μ. Here i is a closed embedding, j is an open embedding and p is an $\mathbb{A}^k$-bundle. Take an arbitrary $f \in V^\vee$ such that $\ker f = W$. This gives rise to a constant section of the trivial bundle $(\mathcal{O}^{k+1}_{\mathrm{SGr}(2,k+1)})^\vee$, hence a section of $\mathcal{T}_1(2,k+1)^\vee$. The latter section vanishes exactly over $i(\mathrm{SGr}(2,k))$. Note that we have $\operatorname{rank} \mathcal{T}_1(2,k+1) = 2$, hence $e_1^\vee(2,k+1) = e_1(2,k+1)$.

The case $k = 2n-1$. Identify $A^*(X \times Y) \cong A^*(\mathrm{SGr}_X(1,2n-1))$ via p^A and consider the localization sequence

$$\cdots \to A^{*-2}(\mathrm{SGr}_X(2,2n-1)) \xrightarrow{i_A} A^*(\mathrm{SGr}_X(2,2n)) \xrightarrow{j^A} A^*(\mathrm{SGr}_X(1,2n-1)) \to \cdots.$$

Theorem 3 states that $\{1, e_2(1,2n-1)\}$ is a basis of $A^*(\mathrm{SGr}_X(1,2n-1))$ over $A^*(X)$. We have $j^*\mathcal{T}_2(2,2n) \cong p^*\mathcal{T}_2(1,2n-1)$ and

$$j^A(e_2(2,2n)) = e_2(1,2n-1),$$

hence j^A is a split surjection (over $A^*(X)$) with the splitting defined by

$$1 \mapsto 1,\ e_2(1,2n-1) \mapsto e_2(2,2n).$$

Then i_A is injective. Hence to obtain a basis of $A^*(\mathrm{SGr}_X(2,2n))$ it is sufficient to calculate the pushforward for a basis of $A^{*-2}(\mathrm{SGr}_X(2,2n-1))$ and combine it with $\{1, e_2(2,2n)\}$. Using the induction, we know that

$$\{1, e_1(2,2n-1), \ldots, e_1(2,2n-1)^{2n-3}\}$$

is a basis of $A^*(\mathrm{SGr}_X(2,2n-1))$. We have $i^*(\mathcal{T}_1(2,2n)) \cong \mathcal{T}_1(2,2n-1)$, hence

$$e_1(2,2n-1) = i^A(e_1(2,2n)).$$

By Proposition 1 we have
$$i_A(e_1(2,2n-1)^l) = e_1(2,2n)^{l+1},$$
obtaining the desired basis
$$\{e_1(2,2n), e_1(2,2n)^2, \ldots, e_1(2,2n)^{2n-2}, 1, e_2(2,2n)\}$$
of $A^*(\mathrm{SGr}_X(2,2n))$ over $A^*(X)$.

The case $k = 2n$. Again identify $A^*(X \times Y) \cong A^*(\mathrm{SGr}_X(1,2n))$ via p^A and consider the localization sequence
$$\cdots \xrightarrow{\partial_A} A^{*-2}(\mathrm{SGr}_X(2,2n)) \xrightarrow{i_A} A^*(\mathrm{SGr}_X(2,2n+1)) \xrightarrow{j^A} A^*(\mathrm{SGr}_X(1,2n)) \xrightarrow{\partial_A} \cdots.$$
Using the induction, we know a basis of $A^*(\mathrm{SGr}_X(2,2n))$, namely
$$\{1, e_1(2,2n), e_1(2,2n)^2, \ldots, e_1(2,2n)^{2n-2}, e_2(2,2n)\}$$
and Lemma 14 gives us a noncanonical basis $\{1, \alpha\}$ for $A^*(\mathrm{SGr}_X(1,2n))$. Examine $i_A(e_2(2,2n))$. It cannot be computed using Proposition 1 since it seems that $e_2(2,2n)$ cannot be pulled back from $A^*(\mathrm{SGr}_X(2,2n+1))$, so we use the following argument. Consider a nonzero vector $w \in W$. This induces constant sections of $\mathcal{O}^{2n}_{\mathrm{SGr}(2,2n)}$ and $\mathcal{O}^{2n+1}_{\mathrm{SGr}(2,2n+1)}$ and sections of $\mathcal{T}_2(2,2n)$ and $\mathcal{T}_2(2,2n+1)$. The latter sections vanish over $\mathrm{SGr}_X(1,2n-1)$ and $\mathrm{SGr}_X(1,2n)$, respectively. Here $\mathrm{SGr}_X(1,2n-1)$ corresponds to vectors in $W/\langle w\rangle$ and $\mathrm{SGr}_X(1,2n)$ corresponds to vectors in $V/\langle w\rangle$. Hence we have the following commutative diagram consisting of closed embeddings:
$$\begin{tikzcd}
\mathrm{SGr}_X(1,2n-1) \arrow[r, "r'"] \arrow[d, "i'"] & \mathrm{SGr}_X(2,2n) \arrow[d, "i"] \\
\mathrm{SGr}_X(1,2n) \arrow[r, "r"] & \mathrm{SGr}_X(2,2n+1)
\end{tikzcd}$$
By Proposition 1 we have $e_2(2,2n) = r'_A(1)$, so, using Proposition 2, we obtain $i_A(e_2(2,2n)) = r_A i'_A(1)$. Notice that $N_{i'}$ is a trivial bundle of rank one. In fact, there is a section of trivial bundle $\mathcal{T}_1(1,2n)^\vee$ over $\mathrm{SGr}_X(1,2n)$ constructed using the same element f such that $\ker f = W$ and this section meets the zero section exactly at $\mathrm{SGr}_X(1,2n-1)$. So we have
$$i_A(e_2(2,2n)) = r_A i'_A(1) = r_A(e_1^\vee(1,2n)) = r_A(0) = 0.$$

We claim that $\ker i_A = A^*(X) \cup e_2(2,2n)$ and $\operatorname{Im} j^A = A^*(X) \cup 1$. We have $j^A(1) = 1$, hence $\partial_A(1) = 0$ and
$$\ker i_A = \operatorname{Im} \partial_A = A^*(X) \cup \partial_A(\alpha).$$
The localization sequence is exact, so we have
$$e_2(2,2n) = \partial_A(y \cup \alpha) = y \cup \partial_A(\alpha)$$
for some $y \in A^*(X)$, and since $e_2(2,2n)$ is an element of the basis, y is not a zero divisor. Consider the presentation of $\partial_A(\alpha)$ with respect to the chosen basis:
$$\partial_A(\alpha) = x_0 \cup 1 + x_1 \cup e_1(2,2n) + \cdots + x_{2n-2} \cup e_1(2,2n)^{2n-2} + z \cup e_2(2,2n).$$

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We have $y \cup \partial_A(\alpha) = e_2(2,2n)$, hence $y \cup z = 1$ and every $y \cup x_i = 0$, hence $x_i = 0$. Then $\partial_A(\alpha) = z \cup e_2(2,2n)$ and

$$\ker i_A = \operatorname{Im} \partial_A = A^*(X) \cup \partial_A(\alpha) = A^*(X) \cup e_2(2,2n).$$

We have

$$\partial_A(x_0 \cup 1 + x_1 \cup \alpha) = x_1 \cup \partial_A(\alpha) = x_1 \cup z \cup e_2(2,2n),$$

hence $\operatorname{Im} j^A = \ker \partial_A = A^*(X) \cup 1$.

There is an obvious splitting for $A^*(\mathrm{SGr}_X(2,2n+1)) \xrightarrow{j^A} \operatorname{Im} j^A$, $1 \mapsto 1$. Then calculating by the same vein as in the odd-dimensional case the pushforwards for the basis of $\operatorname{Coker} \partial_A$, $\{e_1(2,2n)^l\}$, and adding $\{1\}$ to them, we obtain the desired basis of $\mathrm{SGr}_X(2,2n+1)$,

$$\{e_1(2,2n+1),\ldots,e_1(2,2n+1)^{2n-1},1\}. \qquad \square$$

Definition 23. Let $\mathcal{T}$ be a rank n special linear bundle over a smooth variety X. We define the *relative special linear Grassmann variety* $\mathrm{SGr}(k,\mathcal{T})$ twisting $\mathrm{SGr}_X(k,n)$ with the cocycle $\gamma \in H^1(X, SL_n)$ given by $\mathcal{T}$. This variety is an $\mathrm{SGr}(k,n)$-bundle over X. Note that in the particular case of $k=1$ we have $\mathrm{SGr}(k,\mathcal{T}) = \mathcal{T}^0$ since we twist $X \times (\mathbb{A}^n - \{0\})$ with the cocycle defining γ. Similarly to the above, we denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the tautological special linear bundles over $\mathrm{SGr}(k,\mathcal{T})$.

Theorem 5. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X.*

(i) *If* $\operatorname{rank}\mathcal{T} = 2n$ *then there is an isomorphism*

$$(1,e_1,\ldots,e_1^{2n-2},e_2) : \bigoplus_{i=0}^{2n-2} A^{*-2i}(X) \oplus A^{*-2n+2}(X) \xrightarrow{\simeq} A^*(\mathrm{SGr}(2,\mathcal{T}))$$

with $e_1 = e(\mathcal{T}_1)$, $e_2 = e(\mathcal{T}_2)$.

(ii) *If* $\operatorname{rank}\mathcal{T} = 2n+1$ *then there is an isomorphism*

$$(1,e,e^2,\ldots,e^{2n-1}) : \bigoplus_{i=0}^{2n-1} A^{*-2i}(X) \xrightarrow{\simeq} A^*(\mathrm{SGr}(2,\mathcal{T}))$$

with $e = e(\mathcal{T}_1)$.

Proof. The general case is reduced to the case of the trivial bundle $\mathcal{T}$ via the usual Mayer–Vietoris arguments. The latter case follows from Theorem 4. $\square$

9. A splitting principle

In this section we assert a splitting principle for *SL*-oriented cohomology theories with inverted stable Hopf map. The principle states that from the viewpoint of such cohomology theory every special linear bundle is a direct sum of rank 2 special linear bundles and at most one trivial linear bundle.

Definition 24. For $k_1 < k_2 < \cdots < k_m$ consider the group

$$P'_{k_1,\ldots,k_{m-1}} = \begin{pmatrix} SL_{k_1} & * & \cdots & * \\ 0 & SL_{k_2-k_1} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & SL_{k_m-k_{m-1}} \end{pmatrix}$$

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and define a *special linear flag variety* as the quotient

$$\mathcal{SF}(k_1,\dots,k_m) = SL_{k_m}/P'_{k_1,\dots,k_{m-1}}.$$

In particular, we are interested in the varieties

$$\mathcal{SF}(2n) = \mathcal{SF}(2,4,\dots,2n), \quad \mathcal{SF}(2n+1) = \mathcal{SF}(2,4,\dots,2n,2n+1).$$

These varieties are called *maximal SL_2 flag varieties*. Similar to the case of the special linear Grassmannians, we denote by $\mathcal{T}_i$ the tautological special linear bundles over $\mathcal{SF}(k_1,k_2,\dots,k_m)$ with rank $\mathcal{T}_i = k_i - k_{i-1}$.

Remark 9. For $k_1 < k_2 < \cdots < k_m$ consider the parabolic group

$$P_{k_1,\dots,k_{m-1}} = SL_{k_m} \cap \begin{pmatrix} GL_{k_1} & * & \cdots & * \\ 0 & GL_{k_2-k_1} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & GL_{k_m-k_{m-1}} \end{pmatrix}.$$

The projection

$$\mathcal{SF}(k_1,k_2,\dots,k_m) = SL_n/P'_{k_1,\dots,k_m} \to SL_n/P_{k_1,\dots,k_m} = \mathcal{F}(k_1,k_2,\dots,k_m)$$

yields the following geometrical description of the special linear flag varieties. Consider a vector space V of dimension k_m. Then have

$$\mathcal{SF}(k_1,k_2,\dots,k_m) = \{(V_1 \leqslant \cdots \leqslant V_{m-1} \leqslant V, \lambda_1,\dots,\lambda_{m-1}) \mid \dim V_j = k_j, \lambda_j \in (\Lambda^{k_j} V_j)^0\}.$$

Definition 25. Let $\mathcal{T}$ be a rank n special linear bundle over a smooth variety X. We define the *relative special linear flag variety* $\mathcal{SF}(k_1,k_2,\dots,k_{m-1},\mathcal{T})$ twisting $X \times \mathcal{SF}(k_1,k_2,\dots,k_{m-1},n)$ with the cocycle given by $\mathcal{T}$. This variety is an $\mathcal{SF}(k_1,k_2,\dots,k_{m-1},n)$-bundle over X. We denote the relative version of the maximal SL_2 flag variety by $\mathcal{SF}(\mathcal{T})$.

Theorem 6. *Let $\mathcal{T}$ be a special linear bundle of rank k over a smooth variety X. Then $A^*(\mathcal{SF}(2,4,\dots,2n,\mathcal{T}))$ is a free module over $A^*(X)$ with the following basis: if k is odd,*

$$\{e_1^{m_1} e_2^{m_2} \cdots e_n^{m_n} \mid 0 \leqslant m_i \leqslant k-2i\};$$

if k is even,

$$\left\{ u_1 u_2 \cdots u_n \;\middle|\; u_i = \left[\begin{array}{l} e_i^{m_i},\ 0 \leqslant m_i \leqslant k-2i \\ e_{i+1} e_{i+2} \cdots e_{n+1} \end{array} \right. \right\},$$

where $e_i = e(\mathcal{T}_i, \lambda_{\mathcal{T}_i})$.

Proof. Proceed by induction on n. For $n=1$ the claim follows from Theorem 5.

Consider the projection

$$p : Y = \mathcal{SF}(2,4,\dots,2n,\mathcal{T}) \to \mathcal{SF}(2,4,\dots,2n-2,\mathcal{T}) = Y_1$$

that forgets about the last subspace. Denote the tautological bundles over Y by $\mathcal{T}_i$ and the tautological bundles over Y_1 by $\mathcal{T}'_i$.

Suppose that k is odd. Using an isomorphism $Y \cong \mathrm{SGr}(2,\mathcal{T}'_n)$ and Theorem 5, we obtain that $A^*(Y)$ is a free module over $A^*(Y_1)$ with the basis

$$\mathcal{B} = \{1, e_n, \dots, e_n^{k-2n}\}.$$

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Using the induction, we have the following basis for $A^*(Y_1)$:

$$\mathcal{B}_1 = \{e_1'^{m_1} e_2'^{m_2} \cdots e_{n-1}'^{m_{n-1}} \mid 0 \leqslant m_i \leqslant k-2i\},$$

with $e_i' = e(\mathcal{T}_i')$. One has $p^*(\mathcal{T}_i') \cong \mathcal{T}_i$ and $p^A(e_i') = e_i$ for $i \leqslant n-1$. Computing the pullback for $\mathcal{B}_1$ and multiplying it with $\mathcal{B}$, we obtain the desired basis.

Now let k be even. This case is completely analogous to that of odd k. We have an isomorphism $Y \cong \mathrm{SGr}(2, \mathcal{T}_n')$. By Theorem 5 we know that $A^*(Y)$ is a free module over $A^*(Y_1)$ with the basis

$$\mathcal{B} = \left\{ u_n \,\middle|\, u_n = \left[\begin{array}{l} e_n^{m_n},\ 0 \leqslant m_n \leqslant k-2n \\ e_{n+1} \end{array} \right. \right\}.$$

Using the induction we have the following basis for $A^*(Y_1)$:

$$\mathcal{B}_1 = \left\{ u_1 u_2 \cdots u_{n-1} \,\middle|\, u_i = \left[\begin{array}{l} e_i'^{m_i},\ 0 \leqslant m_i \leqslant k-2i \\ e_{i+1}' e_{i+2}' \cdots e_n' \end{array} \right. \right\},$$

with $e_i' = e(\mathcal{T}_i')$. Note that $p^*(\mathcal{T}_i') \cong \mathcal{T}_i$ and $p^A(e_i') = e_i$ for $i \leqslant n-1$. Multiplicativity of the Euler classes yields $p^A(e_n') = e_n e_{n+1}$. Computing the pullback for $\mathcal{B}_1$ and multiplying it with $\mathcal{B}$, we obtain the desired basis of Y. □

A straightforward consequence of the splitting principle is the following corollary relating top characteristic classes of special linear bundles.

Corollary 3. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X and put $n = [\frac{1}{2} \operatorname{rank} \mathcal{T}]$. Then we have:*

(i) $e(\mathcal{T}) = e(\mathcal{T}^\vee)$;
(ii) $b_{2i+1}(H(\mathcal{T})) = 0$ *for all* i;
(iii) $p_i(\mathcal{T}) = 0$ *for* $i > n$;
(v) *if* $\operatorname{rank} \mathcal{T} = 2n$ *then* $p_n(\mathcal{T}) = e(\mathcal{T})^2$.

Proof. Consider $r = r_2 r_1 : Y \xrightarrow{r_1} \mathcal{SF}(\mathcal{T}) \xrightarrow{r_2} X$ with r_1 being an $\mathbb{A}^s$-bundle splitting the $r^*\mathcal{T}$ into a sum of special linear vector bundles isomorphic to $r_1^*\mathcal{T}_i$. One can construct Y in a similar way to Lemma 3. From Theorem 6 we know that r^A is an injection. We have $r^*\mathcal{T} \cong \bigoplus_i r_1^*\mathcal{T}_i$ and $r^*\mathcal{T}^\vee \cong \bigoplus_i r_1^*\mathcal{T}_i^\vee$. Note that $\operatorname{rank} r_1^*\mathcal{T}_i \leqslant 2$, hence $r_1^*\mathcal{T}_i \cong r_1^*\mathcal{T}_i^\vee$ and we obtain $r^*\mathcal{T} \cong r^*\mathcal{T}^\vee$, so $r^A e(\mathcal{T}) = r^A e(\mathcal{T}^\vee)$ and $e(\mathcal{T}) = e(\mathcal{T}^\vee)$.

By Lemma 13 and multiplicativity of total Borel classes we have

$$b_*(r^* H(\mathcal{T})) = b_*(H(r^*\mathcal{T})) = \prod_{i=1}^{n} (1 - e(r_1^*\mathcal{T}_i)^2 t^2),$$

thus the odd Borel classes vanish and $p_i = 0$ for $i > n$. Moreover, for a special linear bundle of even rank this equality yields

$$r^A p_n(\mathcal{T}) = (-1)^n r^A b_{2n}(\mathcal{T}) = (-1)^{2n} \prod_{i=1}^{n} e(r_1^*\mathcal{T}_i)^2 = (r^A e(\mathcal{T}))^2. \qquad \square$$

Another consequence of the splitting principle is multiplicativity of total Pontryagin classes.

Lemma 16. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X and let $\mathcal{T}_1 \leqslant \mathcal{T}$ be a special linear subbundle. Then $p_*(\mathcal{T}) = p_*(\mathcal{T}_1) p_*(\mathcal{T}/\mathcal{T}_1)$.*

Proof. Considering the $\mathbb{A}^r$-bundle $p : Y \to X$ described in Lemma 3, one may assume that $\mathcal{T} \cong \mathcal{T}_1 \oplus \mathcal{T}/\mathcal{T}_1$. The claim of the lemma follows from the second item of the above corollary and multiplicativity of total Borel classes. □

We finish this section with the theorem claiming that every special linear bundle of even rank is cohomologically symplectic in a precise sense.

THEOREM 7. *Let $\mathcal{T} = (E, \lambda)$ be a special linear bundle of even rank over a smooth variety X. Then there exists a morphism of smooth varieties $p : Y \to X$ such that $A^*(Y)$ is a free $A^*(X)$-module (via p^A) and p^*E has a canonical symplectic form ϕ compatible with trivialization $p^*\lambda$.*

Proof. Consider the same morphism $p : Y \to X$ as we used in the proof of Corollary 3, i.e. Y is an $\mathbb{A}^r$-bundle over $\mathcal{SF}(\mathcal{T})$ such that

$$p^*\mathcal{T} \cong \bigoplus \mathcal{T}_i,$$

where $\mathcal{T}_i$ are special linear bundles of rank two. Theorem 6 yields that $A^*(Y)$ is a free $A^*(X)$-module. The special linear bundles $\mathcal{T}_i = (E, \lambda_i)$ have canonical symplectic forms ϕ_i induced by trivializations λ_i. Hence $E \cong \bigoplus E_i$ has a symplectic form $\phi = \phi_1 \perp \phi_2 \perp \cdots \perp \phi_n$ which is compatible with $\lambda = \lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n$. □

10. Cohomology of the partial flags

We now turn to the computation of the relations satisfied by Pontryagin and Euler classes of the tautological bundles.

THEOREM 8. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X. Put $e = e(\mathcal{T}), p_i = p_i(\mathcal{T})$. Then we have the following isomorphisms of $A^*(X)$-algebras:*

(i) *for* $\operatorname{rank} \mathcal{T} = 2n$,

$$\phi_1 : A^*(X)[e_1, e_2] \Big/ R_{2,2n} \to A^*(\mathrm{SGr}(2, \mathcal{T})),$$

where

$$R_{2,2n} = \left(e_1 e_2 - e, e_2^2 + \sum_{i=0}^{n-1} (-1)^{i-1} p_{n-i-1} e_1^{2i} \right)$$

and the homomorphism is induced by $\phi_1(e_1) = e(\mathcal{T}_1), \phi_1(e_2) = e(\mathcal{T}_2)$;

(ii) *for* $\operatorname{rank} \mathcal{T} = 2n + 1$,

$$\phi_2 : A^*(X)[e_1] \Bigg/ \left(\sum_{i=0}^{n} (-1)^i p_{n-i} e_1^{2i} \right) \to A^*(\mathrm{SGr}(2, \mathcal{T})),$$

the homomorphism is induced by $\phi_2(e_1) = e(\mathcal{T}_1)$.

Proof. In view of Theorem 5 it is sufficient to show that the claimed relations hold, since ϕ_1 and ϕ_2 are supposed to map the bases to the bases. We have an isomorphism $p^*\mathcal{T}/\mathcal{T}_1 \cong \mathcal{T}_2$ for the natural projection $p : \mathrm{SGr}(2, \mathcal{T}) \to X$, so the relation $e(\mathcal{T}_1)e(\mathcal{T}_2) = e$ follows from the multiplicativity of the Euler class. In order to obtain the other relation, compute the total Pontryagin class

$$p_*(p^A\mathcal{T}) = p_*(\mathcal{T}_1)p_*(\mathcal{T}_2).$$

Dividing by $p_*(\mathcal{T}_1)=1+p_1(\mathcal{T}_1)t^2$ in $A^*(\mathrm{SGr}(2,\mathcal{T}))[[t]]$, we get

$$1+(p_1-p_1(\mathcal{T}_1))t^2+\cdots+\left(\sum_{j=0}^{i}(-1)^j p_{i-j}p_1(\mathcal{T}_1)^j\right)t^{2i}+\cdots=\sum p_i(\mathcal{T}_2)t^{2i}.$$

For $\operatorname{rank}\mathcal{T}=2n$, recall that by Corollary 3,

$$p_1(\mathcal{T}_1)=e(\mathcal{T}_1)^2,\quad p_{n-1}(\mathcal{T}_2)=e(\mathcal{T}_2)^2$$

and compare the coefficients at t^{2n-2}. In the other case, by the same Corollary 3, we have $p_1(\mathcal{T}_1)=e(\mathcal{T}_1)^2$ and $p_n(\mathcal{T}_2)=0$, so the claim follows from the comparison of the coefficients of t^{2n}. ☐

Corollary 4. *The homomorphisms of $A^*(pt)$-algebras,*

(i) $\phi_1: A^*(pt)[e_1,e_2]\big/(e_1e_2,\ e_1^{2n-2}+(-1)^n e_2^2)\xrightarrow{\simeq}A^*(\mathrm{SGr}(2,2n))$,

(ii) $\phi_2: A^*(pt)[e_1]\big/(e_1^{2n})\xrightarrow{\simeq}A^*(\mathrm{SGr}(2,2n+1))$,

induced by $\phi_1(e_1)=e(\mathcal{T}_1)$, $\phi_1(e_2)=e(\mathcal{T}_2)$ and $\phi_2(e_1)=e(\mathcal{T}_1)$ are isomorphisms.

Proof. The characteristic classes of the trivial bundle vanish, so the claim follows from the above theorem. ☐

In order to write down the relations for the cohomology of the special linear flag varieties we need to carry out certain computations involving symmetric polynomials. Put $h_i(x_1,x_2,\ldots,x_n)$ for the ith complete symmetric polynomial in n variables.

Lemma 17. *Let $\mathcal{O}_X^n$ be the trivialized special linear bundle over a smooth variety X. Suppose that there is an isomorphism of special linear bundles $(\mathcal{O}_X^n,1)\cong(\bigoplus_{i=1}^k\mathcal{T}_i)\oplus\mathcal{T}'$ for the special linear bundles $\mathcal{T}_i$ of rank 2. Then*

$$p_i(\mathcal{T}')=(-1)^i h_i(e(\mathcal{T}_1)^2,e(\mathcal{T}_2)^2,\ldots,e(\mathcal{T}_k)^2).$$

Proof. Using multiplicativity of total Pontryagin classes and Lemma 13, we obtain

$$\left(\prod_{i=1}^k(1+e(\mathcal{T}_i)^2t^2)\right)p_*(\mathcal{T}')=p_*(\mathcal{O}_X^n)=1.$$

The claim follows from the comparison of the coefficients of t^{2i} in the series obtained inverting $(1+e(\mathcal{T}_i)^2t^2)$ in $A^*(X)[[t]]$:

$$\begin{aligned}1+p_1(\mathcal{T}')t^2+p_2(\mathcal{T}')t^4+\cdots&=\prod_{i=1}^k(1-e(\mathcal{T}_i)^2t^2+e(\mathcal{T}_i)^4t^4-\cdots)\\&=1-h_1(e(\mathcal{T}_1)^2,\ldots,e(\mathcal{T}_k)^2)t^2+h_2(e(\mathcal{T}_1)^2,\ldots,e(\mathcal{T}_k)^2)t^4-\cdots.\end{aligned}$$ ☐

Proposition 4. *We have the following isomorphisms of $A^*(pt)$-algebras:*

(i) $\phi_1: A^*(pt)[e_1,\ldots,e_m,e'_m]\big/I_{2m,2n}\xrightarrow{\simeq}A^*(\mathcal{SF}(2,4,\ldots,2m,2n))$, *where*

$$\begin{aligned}I_{2m,2n}=(&e_1e_2\cdots e_me'_m,(-1)^n e_2^2\cdots e_m^2e'^2_m+h_{n-1}(e_1^2),\\&(-1)^{n-1}e_3^2\cdots e_m^2e'^2_m+h_{n-2}(e_1^2,e_2^2),\ldots,\\&(-1)^{n-m+1}e'^2_m+h_{n-m}(e_1^2,e_2^2,\ldots,e_m^2))\end{aligned}$$

and the isomorphism is induced by $\phi_1(e_i)=e(\mathcal{T}_i)$, $\phi_1(e'_m)=e(\mathcal{T}_{m+1})$;

(ii) $\phi_2 : A^*(pt)[e_1, \ldots, e_m] \big/ I_{2m,2n+1} \xrightarrow{\simeq} A^*(\mathcal{SF}(2, 4, \ldots, 2m, 2n+1))$, *where*
$$I_{2m,2n+1} = (h_n(e_1^2), h_{n-1}(e_1^2, e_2^2), \ldots, h_{n-m+1}(e_1^2, e_2^2, \ldots, e_m^2))$$
and the isomorphism is induced by $\phi_2(e_i) = e(\mathcal{T}_i)$.

Proof. The detailed proof would be quite messy, so we present the reasoning only for part (ii). The even case is quite the same, but the formulas are a little more complicated.

The special linear flag varieties considered are iterated $\mathrm{SGr}(2,k)$-bundles, so one may proceed by induction on m. The case of $m = 1$ was dealt with in Corollary 4. Put $\mathcal{SF}_m = \mathcal{SF}(2, 4, \ldots, 2m, 2n+1)$ and write $\mathcal{T}_{i,m}$ for the ith tautological special linear bundle over $\mathcal{SF}_m$. As usual, taking an $\mathbb{A}^r$-bundle $Y \to \mathcal{SF}_m$ allows us to assume that $\mathcal{O}^{2n+1}_{\mathcal{SF}_m}$ splits into a sum $(\bigoplus_{i=1}^m \mathcal{T}_{i,m}) \oplus \mathcal{T}_{m+1,m}$. Put
$$\mathcal{T}'_{j,m} = \bigoplus_{i=j+1}^{m+1} \mathcal{T}_{i,m}$$
and for ease of notation for the Euler classes write $e_{i,m} = e(\mathcal{T}_{i,m}), e'_{i,m} = e(\mathcal{T}'_{i,m})$.

There is a natural isomorphism $\mathcal{SF}_{m+1} \cong \mathrm{SGr}(2, \mathcal{T}_{m+1,m})$. By Theorem 8 there is an isomorphism
$$A^*(\mathcal{SF}_m)[e_{m+1,m+1}] \Big/ \left(\sum_{i=0}^{n-m} (-1)^i p_{n-m-i}(\mathcal{T}'_{m,m}) e^{2i}_{m+1,m+1}\right) \cong A^*(\mathcal{SF}_{m+1}).$$
The induction assumption provides the description for the coefficients of this polynomial algebra, i.e.
$$A^*(\mathcal{SF}_m) \cong A^*(pt)[e_{1,m}, \ldots, e_{m,m}] \big/ I_{2m,2n+1}.$$
Note that $e_{i,m} = e_{i,m+1}$ for $i \leqslant m$. Thus it is sufficient to show that
$$\sum_{i=0}^{n-m} (-1) p^i_{n-m-i}(\mathcal{T}'_{m,m}) e^{2i}_{m+1,m+1} = (-1)^{n-m} h_{n-m}(e^2_{1,m+1}, e^2_{2,m+1}, \ldots, e^2_{m+1,m+1}).$$
Applying Lemma 17, we obtain
$$p_{n-m-i}(\mathcal{T}'_{m,m}) = (-1)^{n-m-i} h_{n-m-i}(e^2_{1,m}, e^2_{2,m}, \ldots, e^2_{m,m}).$$
The claim follows from the well-known identity
$$h_k(x_1, \ldots, x_l) = \sum_{i=0}^{k} h_i(x_1, \ldots, x_{l-1}) x_l^{k-i}. \qquad \square$$

Remark 10. The relations in the proposition arise from the comparison of the different descriptions for the top Pontryagin class of $\mathcal{T}'_{i,m}$:
$$(e_{i+1} \cdots e_{m+1})^2 = e(\mathcal{T}'_{i,m})^2 = p_{n-i}(\mathcal{T}'_i) = (-1)^{n-i} h_{n-i}(e_1^2, \ldots, e_i^2).$$
In the even case there is another relation expressing triviality of the Euler class of the bundle $(\bigoplus_{i=1}^m \mathcal{T}_i) \oplus \mathcal{T}_{m+1}$.

There is another description for the ideals $I_{2m,2n}$ and $I_{2m,2n+1}$.

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Lemma 18. *For the above ideals $I_{2m,2n}$ and $I_{2m,2n+1}$ we have*

$$
\begin{aligned}
I_{2m,2n} &= (e_1e_2\cdots e_me'_m, (-1)^{n-m+1}e'^2_m + h_{n-m}(e_1^2, e_2^2, \ldots, e_m^2), h_{n-m+1}(e_1^2, \ldots, e_m^2),\\
&\qquad h_{n-m+2}(e_1^2, \ldots, e_m^2), \ldots, h_{n-1}(e_1^2, \ldots, e_m^2)),\\
I_{2m,2n+1} &= (h_{n-m+1}(e_1^2, \ldots, e_m^2), h_{n-m+2}(e_1^2, \ldots, e_m^2), \ldots, h_n(e_1^2, \ldots, e_m^2)).
\end{aligned}
$$

Proof. These equalities follow from the obvious identity

$$h_i(x_1, x_2, \ldots, x_n) = h_i(x_1, x_2, \ldots, x_{n-1}) + x_nh_{i-1}(x_1, x_2, \ldots, x_n). \qquad \square$$

Remark 11. The vanishing of the polynomials $h_i(e_1^2, \ldots, e_m^2)$ corresponds to the vanishing of the Pontryagin classes $p_i(\mathcal{T}_{m+1}), i > n-m$, for the tautological bundle $\mathcal{T}_{m+1}$ over $\mathcal{SF}(2, 4, \ldots, 2m, 2n+1)$.

Remark 12. Consider the case of the maximal SL_2 flag variety, i.e. $\mathcal{SF}(2n)$ and $\mathcal{SF}(2n+1)$. Investigation of the above relations yields that the cohomology of these varieties coincide with the algebras of coinvariants for the Weyl groups $W(D_n)$ and $W(B_n)$, respectively. In other words, there are isomorphisms

$$
\begin{aligned}
A^*(\mathcal{SF}(2n)) &\cong A^*(pt)[e_1, e_2, \ldots, e_n]\Big/(s_1, s_2, \ldots, s_{n-1}, t),\\
A^*(\mathcal{SF}(2n+1)) &\cong A^*(pt)[e_1, e_2, \ldots, e_n]\Big/(s_1, s_2, \ldots, s_n),
\end{aligned}
$$

where $s_i = \sigma_i(e_1^2, e_2^2, \ldots, e_n^2)$ for the elementary symmetric polynomials σ_i and $t = e_1e_2\cdots e_n$.

11. Symmetric polynomials

In this section we deal with the polynomials invariant under the action of the Weyl group $W(B_n)$ or $W(D_n)$ and obtain certain spanning sets for the polynomial rings. Our method is an adaptation of that used in [Ful97, § 10, Proposition 3].

Consider $\mathbb{Z}^n$ and fix the usual basis $\{e_1, \ldots, e_n\}$. Let

$$W(B_n) = \{\phi \in \mathrm{Aut}(\mathbb{Z}^n) \mid \phi(e_i) = \pm e_j\}$$

be the Weyl group of the root system B_n and let

$$W(D_n) = \{\phi \in \mathrm{Aut}(\mathbb{Z}^n) \mid \phi(e_i) = (-1)^{k_i}e_j,\ (-1)^{\sum k_i} = 1\}$$

be the Weyl group of the root system D_n. Identifying $R = \mathbb{Z}[e_1, \ldots, e_n]$ with the symmetric algebra $\mathrm{Sym}^*((\mathbb{Z}^n)^\vee)$ in the usual way, we obtain the actions of these Weyl groups on R. Let $R_B = R^{W(B_n)}$ and $R_D = R^{W(D_n)}$ be the algebras of invariants.

For the elementary polynomials $\sigma_i \in \mathbb{Z}[x_1, \ldots, x_n]$ consider

$$s_i = \sigma_i(e_1^2, \ldots, e_n^2), \quad t = \sigma_n(e_1, \ldots, e_n).$$

One can easily check that $R_B = \mathbb{Z}[s_1, \ldots, s_n]$ and $R_D = \mathbb{Z}[s_1, \ldots, s_{n-1}, t]$.

In order to compute spanning sets for R over R_B and R_D we need 'decreasing degree' equalities provided by the following lemma.

LEMMA 19. *There exist polynomials $g_i, h_i \in R$ such that*

$$e_1^{2n} = \sum_{i=1}^{n} g_i s_i, \quad e_1^{2n-1} = \sum_{i=1}^{n-1} h_i s_i + h_n t.$$

One may assume that g_i and h_i are homogeneous of appropriate degrees, i.e. $\deg g_i = 2n - 2i$, $\deg h_i = 2n - 2i - 1$ and $\deg h_n = n - 1$.

Proof. Let $I_B = (s_1, \ldots, s_n)$ and $I_D = (s_1, \ldots, s_{n-1}, t)$ be the ideals generated by the homogeneous invariant polynomials of positive degree. We need to show that $e_1^{2n} \in I_B$ and $e_1^{2n-1} \in I_D$. Set $S_B = R/I_B$, $S_D = R/I_D$.

Consider $S_B[[x]]$. Since all the s_i belong to I_B we have

$$(1 - \bar{e}_1^2 x)(1 - \bar{e}_2^2 x) \cdots (1 - \bar{e}_n^2 x) = 1,$$

hence

$$(1 - \bar{e}_2^2 x)(1 - \bar{e}_3^2 x) \cdots (1 - \bar{e}_n^2 x) = 1 + \bar{e}_1^2 x + \bar{e}_1^4 x^2 + \cdots.$$

Comparing the coefficients at x^n, we obtain $\bar{e}_1^{2n} = 0$, thus $e_1^{2n} \in I_B$.

Consider $S_D[[x]]$. As above, we have

$$(1 - \bar{e}_1^2 x^2)(1 - \bar{e}_2^2 x^2) \cdots (1 - \bar{e}_n^2 x^2) = 1,$$

hence

$$(1 + \bar{e}_1 x)(1 - \bar{e}_2^2 x^2)(1 - \bar{e}_3^2 x^2) \cdots (1 - \bar{e}_n^2 x^2) = 1 + \bar{e}_1 x + \bar{e}_1^2 x^2 + \cdots.$$

Comparing the coefficients at x^{2n-1}, we obtain

$$\bar{e}_1^{2n-1} = (-1)^{n-1} \bar{e}_1 \bar{e}_2^2 \cdots \bar{e}_n^2 = (-1)^{n-1} \bar{t} \bar{e}_2 \bar{e}_3 \cdots \bar{e}_n = 0,$$

thus $e_1^{2n-1} \in I_D$. □

PROPOSITION 5. *In the above notation we have the following spanning sets:*

(i) $\mathcal{B}_1 = \{e_1^{m_1} e_2^{m_2} \cdots e_n^{m_n} \mid 0 \leqslant m_i \leqslant 2n - 2i + 1\}$ *spans R over R_B;*

(ii) $\mathcal{B}_2 = \left\{u_1 u_2 \cdots u_{n-1} \middle| u_i = \left[\begin{smallmatrix} e_i^{m_i}, 0 \leqslant m_i \leqslant 2n-2i \\ e_{i+1} e_{i+2} \cdots e_n \end{smallmatrix}\right.\right\}$ *spans R over R_D.*

Proof. In both cases proceed by induction on n. The base case of $n = 1$ is clear. Denote by $\mathcal{B}'_1$ and $\mathcal{B}'_2$ the spanning sets in $R' = \mathbb{Z}[e_2, \ldots, e_n]$ and let $s'_i, t' \in R'$ be the corresponding invariant polynomials. Note that $s_i = e_1^2 s'_{i-1} + s'_i$ and $t = e_1 t'$.

Suppose that one cannot express some monomial as a linear combination of $\mathcal{B}_1$ (or $\mathcal{B}_2$) with R_B-coefficients (or R_D-coefficients). Consider among these monomials the ones of minimal degree and choose among them a monomial with the largest degree at e_1, denoting it by $f = e_1^{k_1} e_2^{k_2} \cdots e_n^{k_n} \in R$.

(1) For $k_1 \geqslant 2n$ we can use Lemma 19 and substitute $\sum g_i s_i$ for e_1^{2n}, obtaining

$$f = \sum s_i g_i e_1^{k_1 - 2n} e_2^{k_2} \cdots e_n^{k_n}$$

with $\deg g_i e_1^{k_1-2n} e_2^{k_2} \cdots e_n^{k_n} < \deg f$, so the right-hand side is an R_B-linear combination of $\mathcal{B}_1$. Now suppose that $k_1 < 2n$. By the induction we have

$$e_2^{k_2} e_3^{k_3} \cdots e_n^{k_n} = \sum \alpha_j(s'_1, \ldots, s'_{n-1}) b'_j$$

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for some $b'_j \in \mathcal{B}'_1$ and $\alpha_j \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$. We can assume that all the summands on the right-hand side are homogeneous of total degree $k_2 + \cdots + k_n$. Since $s_i = e_1^2 s'_{i-1} + s'_i$ one has

$$\alpha_j(s_1, \ldots, s_{n-1}) = \alpha_j(s'_1, \ldots, s'_{n-1}) + \sum_{l>0} e_1^l \beta_{jl}$$

for some $\beta_{jl} \in R'_B$. Thus we obtain

$$f = \sum_j e_1^{k_1} \alpha_j(s_1, \ldots, s_{n-1}) b'_j - \sum_{j,l} e_1^{k_1+l} \beta_{jl} b'_j.$$

Note that $e_1^{k_1} b'_j \in \mathcal{B}_1$, so the first sum is an R_B-linear combination of the monomials from the spanning set. The second sum consists of monomials of degree $\deg f$, and degrees at e_1 of these monomials are greater than k_1, so by the choice of f it is an R_B-linear combination of the monomials from the spanning set. Hence f is an R_B-linear combination of $\mathcal{B}_1$, contradicting the assumption.

(2) As above, for $k_1 \geqslant 2n-1$ we can use Lemma 19 and lower the total degree, so suppose that $k_1 < 2n-1$. By induction we have

$$e_2^{k_2} e_3^{k_3} \cdots e_n^{k_n} = \sum \alpha_j(s'_1, \ldots, s'_{n-2}, t') b'_j$$

for some $b'_j \in \mathcal{B}'_2$ and $\alpha_j \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$. We may assume that all the summands on the right-hand side are homogeneous of degree $k_2 + k_3 + \cdots + k_n$. One has $t'^2 = s'_{n-1}$, hence

$$\alpha_j(s'_1, \ldots, s'_{n-2}, t) = \widetilde{\alpha}_j(s'_1, \ldots, s'_{n-2}, s'_{n-1}) + t' \widehat{\alpha}_j(s'_1, \ldots, s'_{n-2}, s'_{n-1}).$$

As above, we can substitute s_i into $\widetilde{\alpha}_j$ and $\widehat{\alpha}_j$ and obtain some $\widetilde{\beta}_{jl}, \widehat{\beta}_{jl} \in R'_D$. Thus we have

$$\begin{aligned} f = &\sum_j e_1^{k_1} \widetilde{\alpha}_j(s_1, \ldots, s_{n-1}) b'_j + \sum_j e_1^{k_1} \widehat{\alpha}_j(s_1, \ldots, s_{n-1}) t' b'_j \\ &- \sum_{j,l} e_1^{k_1+l} \widetilde{\beta}_{jl} b'_j - \sum_{j,l} e_1^{k_1+l} \widehat{\beta}_{jl} t' b'_j. \end{aligned}$$

In the first sum we have $e_1^{k_1} b'_j \in \mathcal{B}_2$. One has $t' b'_j \in \mathcal{B}_2$, so in the case of $k_1 = 0$ the second sum is a linear combination of the elements from the spanning set, otherwise, if $k_1 \geqslant 1$, one has $t = e_1 t'$ and $\deg(e_1^{k_1-1} \widehat{\alpha}_j(s_1, \ldots, s_{n-1}) b'_j) < \deg f$, so in both cases the second sum is an R_D-linear combination of $\mathcal{B}_2$. The third and fourth sums are dealt with like the second one in (1); the monomials have the same degree as f but the degrees at e_1 are greater. Thus we obtain that f is an R_D-linear combination of $\mathcal{B}_2$, contradicting the assumption. □

12. Cohomology of the special linear Grassmannians and BSL_n

We are now ready to compute the cohomology of the special linear Grassmannians. Recall that $h_i(x_1, x_2, \ldots, x_n) = g_i(\sigma_1, \sigma_2, \ldots, \sigma_n)$ for a certain polynomial $g_i \in \mathbb{Z}[y_1, y_2, \ldots, y_n]$.

Theorem 9. *For the special linear Grassmannians we have the following isomorphisms of $A^*(pt)$-algebras:*

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(i) $\phi_1 : A^*(pt)[p_1, p_2, \ldots, p_m, e, e'] \big/ J_{2m,2n} \xrightarrow{\cong} A^*(\mathrm{SGr}(2m, 2n))$, *where*

$$J_{2m,2n} = (ee', e^2 - p_m, (-1)^{n-m+1} e'^2 + g_{n-m}(p_1, p_2, \ldots, p_m), g_{n-m+1}(p_1, p_2, \ldots, p_m), \\ g_{n-m+2}(p_1, p_2, \ldots, p_m), \ldots, g_{n-1}(p_1, p_2, \ldots, p_m))$$

and the isomorphism is induced by $\phi_1(p_i) = p_i(\mathcal{T}_1)$, $\phi_1(e) = e(\mathcal{T}_1)$ *and* $\phi_1(e') = e(\mathcal{T}_2)$;

(ii) $\phi_2 : A^*(pt)[p_1, p_2, \ldots, p_m, e] \big/ J_{2m,2n+1} \xrightarrow{\cong} A^*(\mathrm{SGr}(2m, 2n+1))$, *where*

$$J_{2m,2n+1} = (e^2 - p_m, g_{n-m+1}(p_1, p_2, \ldots, p_m), g_{n-m+2}(p_1, p_2, \ldots, p_m), \ldots, g_n(p_1, p_2, \ldots, p_m))$$

and the isomorphism is induced by $\phi_2(p_i) = p_i(\mathcal{T}_1)$ *and* $\phi_2(e) = e(\mathcal{T}_1)$.

Proof. (i) Consider the special linear flag variety $p : \mathcal{SF}(\mathcal{T}_1) \to \mathrm{SGr}(2m, 2n)$. The homomorphism p^A is injective by Theorem 6. There is an isomorphism

$$\mathcal{SF}(\mathcal{T}_1) \cong \mathcal{SF}(2, 4, \ldots, 2m, 2n).$$

Denote this variety by $\mathcal{SF}$. Proposition 4 yields that there is an injection

$$p^A : A^*(\mathrm{SGr}(2m, 2n)) \to A^*(pt)[e_1, \ldots, e_m, e'_m] \big/ I_{2m,2n}.$$

We have $p^A(e(\mathcal{T}_1)) = e_1 e_2 \cdots e_m, p^A(e(\mathcal{T}_2)) = e'_m$ and, by Lemma 13 and multiplicativity of total Pontryagin classes, $p^A(p_i(\mathcal{T}_1)) = \sigma_i(e_1^2, e_2^2, \ldots, e_m^2)$. From now on we omit p^A and regard $A^*(\mathrm{SGr}(2m, 2n))$ as a subalgebra of $A^*(\mathcal{SF})$. Lemma 18 shows that

$$J_{2m,2n} \subset I_{2m,2n} = J_{2m,2n} A^*(pt)[e_1, \ldots, e_m, e'_m];$$

moreover,

$$J_{2m,2n} = I_{2m,2n} \cap A^*(pt)[p_1, \ldots, p_m, e, e'],$$

since by Proposition 5 $A^*(pt)[e_1, \ldots, e_m, e'_m]$ is a free module over $A^*(pt)[p_1, \ldots, p_m, e, e']$.

Hence there exists the claimed map

$$\phi_1 : A^*(pt)[p_1, p_2, \ldots, p_m, e, e'] \big/ J_{2m,2n} \to A^*(\mathrm{SGr}(2m, 2n)) \subset A^*(\mathcal{SF})$$

with $\phi_1(p_i) = p_i(\mathcal{T}_1) = \sigma_i(e_1^2, e_2^2, \ldots, e_m^2)$, $\phi_1(e) = e(\mathcal{T}_1) = e_1 e_2 \cdots e_m$ and $\phi_1(e') = e(\mathcal{T}_2) = e'_m$, and it is injective.

Applying Theorem 6, we obtain that the set

$$\mathcal{B} = \left\{ u_1 u_2 \cdots u_{m-1} \;\middle|\; u_i = \left[\begin{array}{l} e_i^{m_i},\ 0 \leqslant m_i \leqslant 2n - 2i \\ e_{i+1} e_{i+2} \cdots e_m \end{array} \right. \right\},$$

forms a basis of $A^*(\mathcal{SF})$ over $A^*(\mathrm{SGr}(2m, 2n))$. Note that by the same theorem $A^*(\mathcal{SF})$ is generated as an $A^*(pt)$-algebra by $e_1, e_2, \ldots, e_m, e'_m$, thus by Proposition 5 we know that $\mathcal{B}$ spans $A^*(\mathcal{SF})$ over the algebra

$$\mathrm{Im}(\phi_1) = A^*(pt)[\phi_1(p_1), \phi_1(p_2), \ldots, \phi_1(p_{m-1}), \phi_1(e), \phi_1(e')],$$

hence $\mathrm{Im}(\phi_1) = A^*(\mathrm{SGr}(2m, 2n))$ and ϕ_1 is surjective.

(ii) This can be obtained via similar reasoning. $\square$

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Remark 13. It seems that there is no good description for $A^*(\mathrm{SGr}(2m+1,2n))$ in terms of the Euler and Pontryagin characteristic classes. For instance, consider the simplest example $\mathrm{SGr}(1,2)\cong\mathbb{A}^2-\{0\}$. It is isomorphic to the unpointed motivic sphere $S^{3,2}$, so $A^*(\mathrm{SGr}(1,2))\cong A^*(pt)\oplus A^{*-1}(pt)$, but we do not have an appropriate nontrivial special linear bundle over $A^*(\mathbb{A}^2-\{0\})$ to take the characteristic class. Another complication comes from the fact that the grading shift is odd whereas our characteristic classes lie in the even degrees.

We now turn to the computation of the cohomology rings of the classifying spaces

$$\mathrm{BSL}_n=\varinjlim_{m\in\mathbb{N}}\mathrm{SGr}(n,m).$$

The case of BSL_{2n} easily follows from Theorem 9. In order to compute the cohomology of BSL_{2n+1} we will use a certain Gysin sequence relating $A^*(\mathrm{BSL}_{2n+1})$ to $A^*(\mathrm{BSL}_{2n})$.

Recall that A^* is constructed from a representable cohomology theory. In this setting we have the following proposition relating the cohomology groups of a limit space to the limit of the cohomology groups [PPR09b, Lemma A.5.10].

Proposition 6. *For any sequence of motivic spaces $X_1\xrightarrow{i_1}X_2\xrightarrow{i_2}X_3\xrightarrow{i_3}\cdots$ and any p we have an exact sequence of abelian groups*

$$0\to\varprojlim{}^1 A^{p-1}(X_k)\to A^p(\varinjlim X_k)\to\varprojlim A^p(X_k)\to 0.$$

As usual, the $\lim^1$ term vanishes whenever the Mittag-Leffler condition is satisfied, i.e. if for every i there exists some k such that for every $j\geqslant k$ one has $\mathrm{Im}(A^*(X_j)\to A^*(X_i))=\mathrm{Im}(A^*(X_k)\to A^*(X_i))$.

Consider the sequence of embeddings

$$\cdots\to\mathrm{SGr}(2n,2m+1)\xrightarrow{i_{2m+1}}\mathrm{SGr}(2n,2m+3)\to\cdots$$

By Theorem 9 we know that i^A_{2m+1} is surjective, hence

$$A^p(\mathrm{BSL}_{2n})\cong\varprojlim A^p(\mathrm{SGr}(2n,2m+1))\cong\varprojlim A^p(\mathrm{SGr}(2n,m)).$$

The sequence of the tautological special linear bundles $\mathcal{T}_1$ over $\mathrm{SGr}(2n,m)$ gives rise to a bundle $\mathcal{T}$ over BSL_{2n}. We have a sequence of embeddings of the Thom spaces

$$\cdots\to\mathrm{Th}(\mathcal{T}_1(2n,2m+1))\xrightarrow{j_{2m+1}}\mathrm{Th}(\mathcal{T}_1(2n,2m+3))\to\cdots$$

where $\mathcal{T}_1(i,j)$ is the first tautological special linear bundle over $\mathrm{SGr}(i,j)$. Since all the morphisms $\mathcal{T}_1(2n,k)\to\mathcal{T}_1(2n,l)$ considered are inclusions there is a canonical isomorphism $\mathcal{T}/(\mathcal{T}-\mathrm{BSL}_{2n})=\mathrm{Th}(\mathcal{T})\cong\varinjlim\mathcal{T}_1(2n,m)$. For every k we have an isomorphism

$$A^{*-2n}(\mathrm{SGr}(2n,k))\xrightarrow{\cup th(\mathcal{T}_1(2n,k))}A^*(\mathrm{Th}(\mathcal{T}_1(2n,k))),$$

so j^A_{2m+1} are surjective as well as i_{2m+1} and

$$A^p(\mathrm{Th}(\mathcal{T}))\cong\varprojlim A^p(\mathcal{T}_1(2n,m)).$$

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Definition 26. Let $\mathcal{T}$ be the tautological bundle over BSL_{2n}. Denote by $b_i(\mathcal{T}), e(\mathcal{T}) \in A^*(\mathrm{BSL}_{2n})$ and $th(\mathcal{T}) \in A^*(\mathrm{Th}(\mathcal{T}))$ the elements corresponding to the sequences of the classes of the tautological bundles,

$$p_i(\mathcal{T}) = (\dots, p_i(\mathcal{T}_1(2n,m)), p_i(\mathcal{T}_1(2n,m+1)), \dots),$$
$$e(\mathcal{T}) = (\dots, e(\mathcal{T}_1(2n,m)), e(\mathcal{T}_1(2n,m+1)), \dots),$$
$$th(\mathcal{T}) = (\dots, th(\mathcal{T}_1(2n,m)), th(\mathcal{T}_1(2n,m+1)), \dots),$$

with $\mathcal{T}_1(2n,m)$ being the tautological special linear bundle over $\mathrm{SGr}(2n,m)$.

The above considerations show that we have a Gysin sequence for the tautological bundle over the classifying space BSL_{2n}.

Lemma 20. *Let $\mathcal{T}$ be the tautological bundle over BSL_{2n}. Then there exists a long exact sequence*

$$\cdots \to A^{*-2n}(\mathrm{BSL}_{2n}) \xrightarrow{\cup e(\mathcal{T})} A^*(\mathrm{BSL}_{2n}) \xrightarrow{j^A} A^*(\mathrm{BSL}_{2n-1}) \xrightarrow{\partial} \cdots$$

Proof. For the zero section inclusion of motivic spaces $\mathrm{BSL}_{2n} \to \mathcal{T}$ we have the long exact sequence

$$\cdots \to A^*(\mathrm{Th}(\mathcal{T})) \to A^*(\mathcal{T}) \to A^*(\mathcal{T}^0) \xrightarrow{\partial} \cdots$$

The isomorphisms

$$A^{*-2n}(\mathrm{SGr}(2n,k)) \xrightarrow{\cup th(\mathcal{T}_1(2n,k))} \mathrm{Th}(\mathcal{T}_1(2n,k))$$

induce an isomorphism $A^{*-2n}(\mathrm{BSL}_{2n}) \xrightarrow{\cup th(\mathcal{T})} A^*(\mathrm{Th}(\mathcal{T}))$, so we can substitute $A^{*-2n}(\mathrm{BSL}_{2n})$ for the first term in the above sequence. Using homotopy invariance, we exchange $\mathcal{T}$ for BSL_{2n}. By the definition of $e(\mathcal{T})$ the first arrow represents the cup product $\cup e(\mathcal{T})$.

We have isomorphisms

$$\mathcal{T}^0 \cong \varinjlim \mathrm{SGr}(1, 2n-1, m) \cong \varinjlim \mathrm{SGr}(2n-1, 1, m).$$

The sequence of projections

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{SGr}(2n-1,1,m) \arrow[r] \arrow[d] & \mathrm{SGr}(2n-1,1,m+1) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{SGr}(2n-1,m+1) \arrow[r] & \mathrm{SGr}(2n-1,m+2) \arrow[r] & \cdots
\end{tikzcd}$$

induces a morphism $\mathcal{T}^0 \xrightarrow{r} \mathrm{BSL}_{2n-1}$. Note that

$$\mathrm{SGr}(2n-1,1,m) \cong \mathcal{T}_2(2n-1,m+1)^0,$$

and $\mathcal{T}^0$ is an $(\mathbb{A}^\infty - \{0\})$-bundle over BSL_{2n-1}, so by [MV99, § 4, Proposition 2.3] r is an isomorphism in the homotopy category and we can substitute $A^*(\mathrm{BSL}_{2n-1})$ for the third term in the long exact sequence. □

Definition 27. For a graded ring R^* let $R^*[[t]]_h$ be the homogeneous power series ring, i.e. a graded ring with

$$R^*[[t]]_h^k = \left\{ \sum a_i t^i \;\middle|\; \deg a_i + i \deg t = k \right\}.$$

Note that $R^*[[t]]_h = \varprojlim R^*[t]/t^n$, where the limit is taken in the category of graded algebras. We set $R^*[[t_1, \dots, t_n]]_h = R^*[[t_1, \dots, t_{n-1}]]_h[[t_n]]_h$.

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THEOREM 10. *For* $\deg e = 2n, \deg p_i = 4i$ *we have isomorphisms*

$$A^*(pt)[[p_1, \dots, p_{n-1}, e]]_h \xrightarrow{\cong} A^*(\mathrm{BSL}_{2n}),$$
$$A^*(pt)[[p_1, \dots, p_n]]_h \xrightarrow{\cong} A^*(\mathrm{BSL}_{2n+1}).$$

Proof. The case of BSL_{2n} follows from Theorem 9 and Proposition 6, since for the sequence

$$\cdots \to \mathrm{SGr}(2n, 2m+1) \xrightarrow{i_{2m+1}} \mathrm{SGr}(2n, 2m+3) \to \cdots$$

the pullbacks i_{2m+1}^A are surjective and $\lim^1$ vanishes, yielding

$$A^*(\mathrm{BSL}_2) \cong \varprojlim A^*(\mathrm{SGr}(2n, 2m+1)) = \varprojlim A^*(pt)[p_1, p_2, \dots, p_n, e]/J_{2n,2m+1}$$
$$= A^*(pt)[[p_1, \dots, p_{n-1}, e]]_h.$$

For the odd case, consider the long exact sequence from Lemma 20 for BSL_{2n+2}. By the above calculations $e(\mathcal{T})$ is not a zero divisor, so the the map $\cup e(\mathcal{T})$ is injective and we have a short exact sequence

$$0 \to A^{*-2n-2}(\mathrm{BSL}_{2n+2}) \xrightarrow{\cup e(\mathcal{T})} A^*(\mathrm{BSL}_{2n+2}) \to A^*(\mathrm{BSL}_{2n+1}) \to 0.$$

Identifying $A^*(\mathrm{BSL}_{2n+2})$ with the homogeneous power series and killing e, we obtain the desired result. □

Remark 14. Another way to compute $A^*(\mathrm{BSL}_{2n+1})$ is to use the calculation for $A^*(\mathrm{SGr}(2n+1, 2m+1)) \cong A^*(\mathrm{SGr}(2m-2n, 2m+1))$. The Euler classes are unstable, so the image

$$\mathrm{Im}(A^*(\mathrm{SGr}(2n+1, 2m+3)) \to A^*(\mathrm{SGr}(2n+1, 2m+1)))$$

is generated by the Pontryagin classes $p_i(\mathcal{T}_2)$ and $\lim^1$ vanishes. One could express $p_i(\mathcal{T}_2)$ in terms of $p_i(\mathcal{T}_1)$, obtaining the desired result.

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