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A note on the ruled surface

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The generators and their orthogonal trajectories form, perhaps, the most useful set of parametric curves for the study of the local geometry of a ruled surface[1]. It is not generally realised, however, that the fundamental quantities of the surface can be expressed quite simply in terms of the geodesic curvature, the geodesic torsion and the normal curvature of the directrix, that particular orthogonal trajectory which is chosen as base curve. Certain of the results are similar in form to those arising in the special case of a surface which is generated by the principal normals to a given curve[2], except that the curvature and torsion are geodetic. In addition it is possible to obtain in an elegant form the differential equation of the curved asymptotic lines and the expression for the mean curvature.

Let M, (ξ, η, ζ), be the point in which the directrix meets the generator with direction cosines (l, m, n). Then any point on the surface has coordinates of the form $(\xi + lu, \eta + mu, \zeta + nu)$, where u is the distance of the point from the directrix, measured along the generator, and $\xi, \eta, \zeta, l, m, n$ are functions of the arc-parameter v of the directrix. Orthodox calculations lead to the following results, in which κ_g, τ_g denote the geodesic curvature and torsion and κ_n the normal curvature of the directrix[3] at M, and dashes denote differentiation with regard to v.

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[1] L. P. Eisenhart, *Differential Geometry* (1909), 247-9.

[2] C. E. Weatherburn, *Differential Geometry of Three Dimensions* (1931), 142.

[3] In the definition of these quantities, as in the choice of axial directions for the trihedral, we follow Eisenhart.

The linear element of the surface takes the form

$$ds^2 = du^2 + \{(1 - u\kappa_g)^2 + u^2 \tau_g^2\} dv^2.$$

The parameter of distribution, β, has the value

$$\beta = \tau_g/(\kappa_g^2 + \tau_g^2).$$

Since β is an invariant for the generator and is therefore unaltered by change of directrix, it follows that $\tau_g/(\kappa_g^2 + \tau_g^2)$ has the same value for all orthogonal trajectories at the points at which they cross a given generator.

The equation of the line of striction becomes

$$(\kappa_g^2 + \tau_g^2)\, u = \kappa_g.$$

The total curvature, K, at the point (u, v) is given by

$$K = -\tau_g^2/\{(1 - u\kappa_g)^2 + u^2 \tau_g^2\}^2,$$

which at M becomes $-\tau_g^2$ and at the central point $-(\kappa_g^2 + \tau_g^2)^2/\tau_g^2$.

The mean curvature at any point (u, v) is given by

$$[\kappa_n - u(2\kappa_n \kappa_g + \tau'_g) + u^2\{\kappa_n(\kappa_g^2 + \tau_g^2) + (\kappa_g \tau'_g - \tau_g \kappa'_g)\}]/\{(1 - u\kappa_g)^2 + u^2 \tau_g^2\}^{3/2},$$

which at M becomes κ_n and at the central point becomes

$$\{\kappa_n \tau_g^2 (\kappa_g^2 + \tau_g^2) - \kappa_g \tau_g (\kappa_g \kappa'_g + \tau_g \tau'_g)\}/(\kappa_g^2 + \tau_g^2)^2.$$

The differential equation of the curved asymptotic lines is

$$2\tau_g \frac{du}{dv} = \kappa_n - u(2\kappa_n \kappa_g + \tau'_g) + u^2\{\kappa_n(\kappa_g^2 + \tau_g^2) + (\kappa_g \tau'_g - \tau_g \kappa'_g)\},$$

while the differential equation of the lines of curvature takes the form

$$\tau_g\{(1 - u\kappa_g)^2 + u^2 \tau_g^2\} dv^2 - \tau_g du^2$$
$$+ [\kappa_n - u(2\kappa_n \kappa_g + \tau'_g) + u^2\{\kappa_n(\kappa_g^2 + \tau_g^2) + (\kappa_g \tau'_g - \tau_g \kappa'_g)\}]\, du\, dv = 0.$$

Two special cases of interest are surfaces generated by the binormals to a given curve and surfaces generated by the normals to a given curve. If this curve be taken as directrix and if κ and τ denote the curvature and torsion at the point v, then in the former case $\kappa_n = \kappa$, $\kappa_g = 0$, $\tau_g = \tau$, the directrix being the line of striction, while in the latter case $\kappa_n = 0$, $\kappa_g = \kappa$, $\tau_g = \tau$. The above formulae are correspondingly simplified.