CONCERNING METRIZABILITY OF POINTWISE PARACOMPACT MOORE SPACES

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Although it is known that there exists a pointwise paracompact Moore space which is not metrizable **(1)**, very little seems to be known about the metrizability of pointwise paracompact Moore spaces. This paper is devoted to determining some of the conditions under which a pointwise paracompact Moore space is metrizable.

The statement that S is a Moore space means that there exists a sequence of collections of regions in S satisfying Axiom 0 and the first three parts of Axiom 1 of **(2)**. A Moore space is complete if and only if it satisfies all of Axiom 1 of **(2)**.

The statement that S is pointwise paracompact means that if H is an open covering of S, there exists a refinement H' of H covering S such that no point of S belongs to infinitely many elements of H'.

The statement that S is a locally peripherally separable space means that if P is a point and D a domain containing P, then there is a domain D' containing P such that D' is a subset of D and the boundary of D' is separable.

The statement that the collection G of point sets is discrete means that the closures of the sets of G are mutually exclusive and any subcollection of G has a closed sum.

The statement that S is strongly screenable means that if H is an open covering of S, there exists a sequence $H_1, H_2, \ldots$ such that each H_i is a discrete collection of domains, H_i is a refinement of H, and $\sum H_i$ covers S.

If G is a collection of point sets, then G^* denotes the point set to which x belongs if and only if x is a point of some element of G.

If D is a domain and E is the boundary of D, then the statement that B is accessible means that if P is a point of B and R is a region containing P, then there exist points Q and Q' such that Q is in D, Q' is in $R \cdot B$, and there is an arc with end points Q and Q' which, except for Q', lies wholly in D.

THEOREM 1. *A locally separable Moore space is metrizable if and only if it is pointwise paracompact.*

Proof. Suppose that S is locally separable and H is an open covering of S. If S is pointwise paracompact, there is a refinement H' of H such that no point of S belongs to infinitely many elements of H' and if h is an element of H', then h is separable. If h is an element of H', there is a countable subset K

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which is dense in h. Each element of H' which intersects h contains some point of K. But no point of S belongs to infinitely many elements of H', so only countably many elements of H' intersect h.

For each element h of H', denote by h, $H_{h,1}$, $H_{h,2}$, ... a sequence of collections of elements of H' such that an element g of H' belongs to $H_{h,1}$ if and only if g intersects h and if n is a positive integer greater than 1, an element g of H' belongs to $H_{h,n}$ if and only if g intersects $H^*_{h,n-1}$ but does not intersect h if $n = 2$ or $(h + H^*_{h,1} + \ldots + H^*_{h,n-2})$ if n is greater than 2. It is clear that for each positive integer n, the point set $(h + H^*_{h,1} + \ldots + H^*_{h,n})$ is separable. By definition, each $H_{h,n}$ is a countable subcollection of H' and the sequence h, $H_{h,1}$, $H_{h,2}$, ... is countable.

If h is an element of H', let M^*_h denote the point set

$$(h + H^*_{h,1} + H^*_{h,2} + \ldots).$$

Then M^*_h has no boundary. For suppose P is a boundary point of M^*_h. Some element g of H' contains P and also contains a point of some $H^*_{h,i}$. But then g belongs to $H_{h,i+1}$ and P is an interior point of M^*_h.

Denote by w a well-ordering of the elements of H' and by w' the maximal sub-sequence of w whose first term is the first term of w and such that if w'' is any initial segment of w', then the first term of w' following each term of w'' in w' is the first term t of w such that if h is a term of w' which precedes t in w, then M^*_h does not intersect M^*_t. It follows that if P is a point of S, there is an h of w' such that M^*_h contains P. For if there is no h such that M^*_h contains P, then P is a boundary point of $\sum_{h \in w'} M^*_h$; otherwise w' is not maximal. But some element g of H' contains P and thus intersects some M^*_h where h is in w'. Then g is a subset of M^*_h and P is not a boundary point of $\sum_{h \in w'} M^*_h$.

Each M_h is a collection of only countably many domains. For each h of w' denote by w_h a simply infinite well-ordering of the elements of M_h. For each positive integer n, denote by H_n the collection to which g belongs if and only if there is an element h of w' such that g is the nth term of w_h. It follows immediately that each H_n is a discrete collection of domains which refines H and $\sum H^*_i$ is S. Thus S is strongly screenable, and Bing has proved **(1)** that each such Moore space is metrizable.

It is well known **(3)** that each metrizable Moore space is paracompact, so each metrizable Moore space is pointwise paracompact and the proof is complete.

COROLLARY. *A complete, pointwise paracompact Moore space is metrizable provided that each point is contained in a region which does not contain uncountably many mutually exclusive domains.*

Proof. Suppose that P is a point of the space and R is a region containing P such that R does not contain uncountably many mutually exclusive domains. Moore **(2)** has proved that each Axiom 1 space in which there do not exist uncountably many mutually exclusive domains is separable. Then R, treated as a space, is separable and the original space is locally separable. By Theorem 1, each locally separable, pointwise paracompact Moore space is metrizable.

THEOREM 2. *A Moore space is metrizable if and only if the boundary of each domain is strongly screenable.*

Note. The proof of this theorem, which is simpler than my original proof, is due to E. E. Grace.

Proof. Suppose that H is an open covering of S. Denote by G a collection of domains whose closures are mutually exclusive such that each domain of G is a subset of some domain of H and each point of S is a point of G^* or a limit point of G^*. Then $\overline{G^*}$ is S and $\overline{G^*} - G^*$ is strongly screenable since it is the boundary of G^*. Thus there exists a sequence $H_1, H_2, \ldots$ satisfying the notion of strong screenability with respect to H and the boundary of G^*.

Denote by G' the collection to which the domain d belongs if and only if there is a domain g of G such that d is $g - (g \cdot \overline{\sum H^*_i})$. Then each point in $\overline{G'^*}$ must belong to the closure of some element of G' since no point of $\overline{G^*} - G^*$ is a limit point of G'^*. Thus G' is a discrete collection of domains.

Suppose that M is the boundary of $\sum H^*_i$. Then each point of S belongs to either G'^*, M, or $\sum H^*_i$. But M is strongly screenable since it is the boundary of a domain. Thus there exists a sequence $H'_1, H'_2, \ldots$ satisfying the notion of strong screenability with respect to H and M. Clearly, the sequence $G', H_1, H'_1, H_2, H'_2, \ldots$ satisfies the notion of strong screenability with respect to H and S. Bing has proved **(1)** that each strongly screenable Moore space is metrizable, so the condition has been proved sufficient.

Since each metrizable space is strongly screenable, it is obvious that the condition of the theorem is necessary.

THEOREM 3. *If S is a locally peripherally separable Moore space such that the boundary of each domain is accessible, then S is metrizable if and only if it is pointwise paracompact.*

Proof. To prove that the condition is sufficient, suppose that H is an open covering of S such that each element of H has a separable boundary, D is a domain in S, B is the boundary of D, and P is a point of B. There is a refinement H' of H such that no point of S belongs to infinitely many elements of H'.

Denote by h_p an element of H containing P and by $H'_{p,1}$ the subcollection of H' to which g belongs if and only if g contains a point in the boundary of h_p. Since the boundary of h_p is separable, $H'_{p,1}$ is at most countable. Denote by $H_{p,1}$ one particular subcollection of H such that each element of $H'_{p,1}$ belongs to some element of $H_{p,1}$, each element of $H_{p,1}$ contains an element of $H'_{p,1}$, and $H_{p,1}$ is countable. Suppose that B_1 is the boundary of $(h_p + H^*_{p,1})$, Q is a point of B_1, and g is an element of H' containing Q. Suppose that R is a region containing Q such that R is a subset of g. Since B_1 is accessible, there is a point x in $(h_p + H^*_{p,1})$ and there is a point y in $R \cdot B_1$

such that some arc has x and y as end points and lies, except for y, wholly in $(h_p + H^*_{p,1})$. Then g must contain a boundary point of some element of $H_{p,1}$. For if y is not a boundary point of some element of $H_{p,1}$, denote by x', y a subinterval of the arc x, y such that each point of x', y belongs to g. But some element of $H_{p,1}$ contains x' and so the arc x', y contains a boundary point of that element. Thus, if g is an element of H' which contains a point of $\overline{(h_p + H^*_{p,1})} - (h_p + H^*_{p,1})$, then g must contain a boundary point of some element of $H_{p,1}$. Since each element of $H_{p,1}$ has a separable boundary, it follows that at most countably many elements of H' intersect B_1. If $H'_{p,2}$ is the subcollection of H' to which g belongs if and only if g contains a point of B_1, let $H_{p,2}$ denote one particular countable subcollection of H which covers $H'^*_{p,2}$. The preceding argument establishes that the boundary of $(h_p + H^*_{p,1} + H^*_{p,2})$ is intersected by at most countably many elements of H'. This process continued indefinitely defines a sequence h_p, $H_{p,1}$, $H_{p,2}$, . . . such that h_p contains P, $H_{p,1}$ is a countable subcollection of H which covers the boundary of h_p, and $H_{p,n+1}$ is a countable subcollection of H which covers the boundary of $H_{p,n}$ for each positive integer n. An argument similar to the one used above establishes that the point set $(h_p + H^*_{p,1} + H^*_{p,2} + \ldots)$ has no boundary.

Let w denote a well-ordering of the points of B. If P_1 is the first term of w, suppose that h_{P_1}, $H_{P_1,1}$, $H_{P_1,2}$, . . . is such a sequence as that defined above and $M^*_{P_1}$ is the point set to which x belongs if and only if x is a point of h_{P_1} or, for some positive integer n, x is a point of $H^*_{P_1,n}$. If P_2 is the first term of w such that P_2 is not a point of M_{P_1}, denote by h_{P_2}, $H_{P_2,1}$, $H_{P_2,2}$, . . . a sequence based on P_2 in the subspace $S - M^*_{P_1}$. This process continued indefinitely defines a maximal sub-sequence w' of w which is perhaps uncountable such that the first term of w is the first term of w' and if w'' is an initial segment of w',then the first term of w' following each term of w'' in w' is the first term Q of w such that if M is the point set to which x belongs if and only if there is a term P of w'' such that x is a point of M^*_P, then Q is not a point of M.

An argument similar to that used earlier in this proof establishes that if G is the collection to which the point set N belongs if and only if there is a term P of w' such that N is M^*_P, then G is a discrete collection of closed point sets. It has already been established that each point set of G has no boundary. To see that G is discrete, suppose that G' is a subcollection of G such that G'^* has a boundary. If Q is a point of that boundary, some element g of H' contains Q. Some region R contains Q and is a subset of g. Since the boundary of G'^* is accessible, there is a point x in $g \cdot G'^*$ and there is a point y in the intersection of R and the boundary of G'^* such that some arc has end points x and y and, except for y, lies wholly in $g \cdot G'^*$. But x is in some $H^*_{P,n}$ for some P of w' and some positive integer n. Then some domain of $H_{P,n}$ contains x and the arc with end points x and y must contain a boundary point of that domain. But this means that g is a subset of $H^*_{P,n+1}$ and Q is not a boundary point of G'^*.

Now if P is a term of w', it has been noted that M_P is the sum of at most countably many domains. For each term P of w', let w_p be a simply infinite

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sequence which is a well-ordering of the elements of M_P. For each positive integer n, let H_n be the collection to which the domain h belongs if and only if there is a term P of w' such that h is the nth term of w_p. Since G is a discrete collection, it follows that each H_n is a discrete collection of domains. Thus the boundary of D is strongly screenable and by Theorem 2, S is metrizable.

That the condition of the theorem is necessary follows as in Theorem 1.

THEOREM 4. *A locally peripherally separable, locally arcwise connected Moore space is metrizable if and only if it is pointwise paracompact.*

Proof. In a locally arcwise connected Moore space, the boundary of each domain is accessible. It follows from Theorem 3 that the condition is sufficient.

That the condition is necessary has already been established.

THEOREM 5 *A complete, connected "im kleinen," locally peripherally separable Moore space is metrizable if and only if it is pointwise paracompact.*

Proof. It is established in **(2)** that each complete, connected "im kleinen" Moore space is locally arcwise connected. Theorem 4 applies to prove the condition sufficient.

It is clear that the condition is necessary.

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