GENERALIZATION OF A THEOREM OF P. D. FINCH'S ON INTEGRATION OF SET-FUNCTIONS

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Let $\mathscr{M}$ be a σ-field of subsets of a space $\mathscr{X}$. A partition of $\mathscr{X}$ means a countable partition Π of $\mathscr{X}$ into sets belonging to $\mathscr{M}$; the set of partitions is directed by refinement. A. Kolmogoroff in 1930 [1] discussed an integral

$$(1) \qquad I_F(S) = (K)\int_S dF = \lim \sum_{M\in\Pi} F(M \cap S)$$

(Moore-Smith limit as Π gets finer) for set-functions F defined on $\mathscr{M}$. When it exists, I_F is σ-additive, and if by chance F is already σ-additive, then $I_F = F$.

In [2], P. D. Finch studies the same integral for the case that $F = f\mu$, μ being a positive σ-finite measure and f an arbitrary set-function on $\mathscr{M}$, whose values on sets of μ-measure 0 are evidently irrelevant. This case is not really special, since any Kolmogoroff-integrable F can be so written, modulo sets of μ-measure 0, by using for μ the total variation of I_F, but the giving of μ enriches the situation with new questions such as those treated below. Let us call a function F on $\mathscr{M}$ *K-integrable* if I_F as defined in (1) exists, and call f *F-integrable* for μ if $f\mu$ is K-integrable.

An important example of an F-integrable function f is the quotient ν/μ of a signed measure by a measure. Here the singular part of ν with respect to μ has no effect on the integral of $f\mu$, because it can be isolated in a set of μ-measure 0. In fact it is easy to see that $I_{f\mu} = \nu_a$, the absolutely continuous part of ν. If we let θ be the Radon-Nikodym derivative of ν_a with respect to μ, then this equation can be written

$$(2) \qquad (K)\int d\, f\mu = \int \theta\, d\mu.$$

More generally, as remarked by Finch, any F-integrable f satisfies (2) if for θ we use the derivative of $I_{f\mu}$. Thus the generalized integration can be regarded as a way of assigning to certain set-functions specific random variables of which they are to be regarded as generalized averages. The questions I propose to answer are these:

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1. When do two F-integrable set-functions have the same integral, and hence the same θ in (2)?

2. Given f F-integrable, how badly can the values of f be shaken up without destroying integrability? That is, for what real functions g will $g(f)$ still be integrable?

3. How does replacing f by $g(f)$ change θ?

The first question was answered by Kolmogoroff, but has to be re-answered in Finch's terms for application to the others. The theorem referred to in the title ([3] Theorem 1) is a partial answer to questions 2 and 3 for the case that f has the form ν/μ and g is of bounded variation.

As in other integration theories, questions of F-integrability have two sides, one having to do with regularity of the integrand, the other with its size. I address myself first to the regularity side, which is to say the proofs are written for bounded f. The results hold in reality more generally, but their application to unbounded f requires fuller explanation, which I save for the last part of the paper. There is no loss of generality in supposing μ to be finite throughout, rather than only σ-finite.

If $S \in \mathscr{M}$ and Π is a partition, I shall write $S < \Pi$ to mean that $S \subset M$ for some $M \in \Pi$. Adapting Kolmogoroff's phrase, I call set-functions f_1 and f_2 *differentially equivalent* for μ when for every $\delta > 0$ there exists a partition Π such that

$$|f_1(S)-f_2(S)| < \delta \text{ for all } S < \Pi,\ \mu(S) > 0. \tag{3}$$

THEOREM 1. *Suppose that f_1 (or f_2) is F-integrable for μ. Then a necessary and sufficient condition for $I_{f_1\mu} = I_{f_2\mu}$ is that f_1 and f_2 be differentially equivalent for μ.*

PROOF. *Sufficiency.* Let $S \in \mathscr{M}$, $\mu(S) > 0$. Given $\varepsilon > 0$, choose a partition Π (of S) to satisfy (3) with $\delta = \varepsilon/2\mu(S)$, and so fine that for any Π' finer than Π

$$\Big|I_{f_1\mu}(S) - \sum_{\Pi'} f_1\mu\Big| < \tfrac{1}{2}\varepsilon$$

Then we have

$$\Big|I_{f_1\mu}(S) - \sum_{\Pi'} f_2\mu\Big| < \tfrac{1}{2}\varepsilon + \sum_{\Pi'} |f_1 - f_2|\mu < \varepsilon$$

whence $I_{f_1\mu}(S) = I_{f_2\mu}(S)$ by definition.

Necessity. Suppose both integrals exist and f_1 and f_2 are not differentially equivalent for μ. Then there exists $\delta > 0$ such that for any partition Π of $\mathscr{X}$ there is some $S < \Pi$ with $\mu(S) > 0$ and $|f_1(S)-f_2(S)| \geqq \delta$. Apply this to a partition Π so fine that both f_1 and f_2 have variations $\leqq \frac{1}{4}\delta$ on every $M \in \Pi$ with $\mu(M) > 0$ ([2] Theorem 3.3). Then

$$|\mu(S)^{-1}I_{f_i\mu}(S) - f_i(S)| < \tfrac{1}{4}\delta \qquad (i = 1, 2).$$

Since $|f_1(S)-f_2(S)| \geqq \delta$, this implies $I_{f_1\mu}(S) \neq I_{f_2\mu}(S)$, finishing the proof.

If θ is a measurable real-valued function on $\mathscr{X}$, write $\theta(\mu)$ for the distribution of θ as a random variable, that is the measure defined for Borel sets E of the real line by $\theta(\mu)(E) = \mu(\theta^{-1}(E))$.

THEOREM 2. *Suppose θ is μ-summable, and let $\nu = \int \theta \, d\mu$. The following three conditions on a bounded real function g of a real variable are equivalent*:

(2.1) *$g(\nu/\mu)$ is F-integrable for μ.*

(2.2) *The set of discontinuities of g has measure* 0 *for the continuous (non-atomic) part of $\theta(\mu)$.*

(2.3) $(K) \int_S d\, g(\nu/\mu)\mu = \int_S g(\theta) d\mu \quad (S \in \mathscr{M})$.

PROOF. $\theta(\mu)$ has at most countably many atoms situated at real points $x_1, x_2, \cdots$. Let $M_n = \theta^{-1}(x_n) \in \mathscr{M}$, and set $M_0 = \mathscr{X} - \bigcup M_n$. It suffices to prove the theorem separately for each of the subspaces M_n. For $n \neq 0$ this is trivial, because then θ is constant on M_n, while on M_0 μ has a continuous distribution. We can therefore assume that $\theta(\mu)$ is continuous.

LEMMA. *Let a = ess inf θ and b = ess sup θ. Then $\nu(S)/\mu(S)$ for $\mu(S) > 0$ takes every value in the interval (a, b) (which may be infinite).*

PROOF. For any $c > a$, $S_c = \theta^{-1}(a, c)$ has positive measure, and $a < \nu(S_c)/\mu(S_c) < c$. Similarly ν/μ has values arbitrarily close to b. Now (μ, ν) is an atom-free vector valued measure having values on rays from the origin into the right half-plane with slopes approximating a and b. Since by Lyapounov's theorem the range of (μ, ν) is convex, (μ, ν) has values also on all rays with intermediate slopes.

Proof that (2.1) *implies* (2.2). Suppose the set of discontinuities of g has positive $\theta(\mu)$-measure. Then for some $\varepsilon > 0$ the set E_ε of all points where g has saltus $\geqq \varepsilon$ has positive measure. For any partition Π of $\mathscr{X}$ there is some $M \in \Pi$ which intersects $\theta^{-1}(E_\varepsilon)$ in a set of positive measure. Applying the lemma to the subspace M gives an open interval containing points of E_ε, every point of the interval being $\nu(S)/\mu(S)$ for some $S \subset M$ with $\mu(S) > 0$. But then $g(\nu/\mu)$ has variation $\geqq \varepsilon$ on M, which by [2] Theorem 3.3 rules out F-integrability for $g(\nu/\mu)$.

Proof that (2.2) *implies* (2.3) Let $f_1 = g(\nu/\mu)$ and $f_2 = \int g(\theta) d\mu/\mu$. I shall prove that f_1 and f_2 are differentially equivalent for μ. Then it will follow from Theorem 1 that $I_{f_1\mu} = I_{f_2\mu} = f_2\mu$, and this is (2.3). To this end let $\varepsilon > 0$ be given. The set $E_{\frac{1}{2}\varepsilon} = E$ where g has saltus $\geqq \frac{1}{2}\varepsilon$ being closed, its complement is a countable disjoint union of intervals. Partition each of these intervals into countably many disjoint subintervals on each of which g has oscillation $< \varepsilon$, and let Π_R be the partition of the line consisting of E and all the intervals so formed. Let $\Pi = \theta^{-1}(\Pi_R)$ be the corresponding partition of $\mathscr{X}$. This is the partition required to show differential equivalence. Indeed, suppose $S < \Pi$, $\mu(S) > 0$. Since $\mu(\theta^{-1}(E)) = 0$, S is contained in $\theta^{-1}(I)$ for some interval I of Π_R, and $g(I)$ is contained in some interval J of length ε. Since $\theta(S) \subset I$, $\nu(S) = \int_S \theta \, d\mu \in \mu(S)I$. (This estimation works only because I is an interval!) Therefore $f_1(S) = g(\nu(S)/\mu(S)) \in J$. But also $g(\theta(S)) \subset J$, whence also $f_2(S) \in J$. Thus $|f_1(S) - f_2(S)| < \varepsilon$.

Since (2.3) obviously implies (2.1), Theorem 2 is proved.

REMARK. The bounded functions g which satisfy the three conditions of the theorem for all μ and θ are just those having at most countably many discontinuities. This function class includes the functions of bounded variation, so that Theorem 2 generalizes Theorem 1 of [3].

The next theorem generalizes Theorem 2 as far as possible to arbitrary integrable functions f in place of ν/μ.

THEOREM 3. *Let f and f_1 be set-functions F-integrable and differentially equivalent for μ. Let g be a bounded real function satisfying* (2.2) *with θ the derivative with respect to μ of $\nu = I_{f\mu} = I_{f_1\mu}$. Then $g(f)$ and $g(f_1)$ are F-integrable and differentiably equivalent for μ.*

PROOF. Nothing is lost by taking $f_1 = \nu/\mu$, because of the transitivity of differential equivalence. Given $\varepsilon > 0$, partition the real line as in the proof of Theorem 2 into E and countably many disjoint intervals on each of which g has oscillation $< \varepsilon$. Refine this partition by separating from each interval its end points, if any, so that aside from E each cell is either a point or an open interval. It suffices to consider the problem on each subspace $M = \theta^{-1}(I)$ for I in this partition, and this is trivial except when I is one of the open intervals. For this case form a partition Π of M as follows. Let $I = (a, b)$, and let $\{\delta_n\}$ be a decreasing sequence of numbers tending to 0 as limit, with $\delta_1 < \frac{1}{2}(b-a)$. Let $I_n = (a+\delta_n, b-\delta_n)$, and let $M_n = \theta^{-1}(I_n - I_{n-1})$ for $n = 2, 3, \ldots$ and $M_1 = \theta^{-1}(I_1)$. Using the differential equivalence of f and f_1 find for each n a partition Π_n of M_n such that $S \subset M_n$, $S < \Pi_n$, $\mu(S) > 0$ implies $|f(S) - f_1(S)| < \delta_n$. The desired partition Π of M is formed by combining all the Π_n to refine the partition $\{M_n\}$. To see that this works, suppose $S < \Pi$, $\mu(S) > 0$. Then $S \subset M_n$ for some n. Since $\theta(S) \subset I_n$, we have $f_1(S) = \nu(S)/\mu(S) \in I_n$, and since $|f(S) - f_1(S)| < \delta_n$, we have also $f(S) \in I$. Finally, g has oscillation $< \varepsilon$ on I, so that $|g(f(S)) - g(f_1(S))| < \varepsilon$.

COROLLARY. (2) *and* (2.2) *together imply*

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$$(K)\int dg(f)\mu = \int g(\theta)d\mu.$$

The following simple example shows that, in contrast to Theorem 2, (2.2) is not necessary for the F-integrability of $g(f)$, and that F-integrability of both $g(f)$ and $g(f_1)$ is not sufficient for equality of their integrals. Let $(\mathscr{X}, \mathscr{M}, \mu)$ be the Borel unit interval with Lebesgue measure. Let $\theta(x) = x$, and $\nu = \int \theta\, d\mu$. Define for $\mu(M) > 0$

$$f(M) = \begin{cases} \nu(M)/\mu(M) \text{ when this is rational} \\ r, \text{ where } r \text{ is rational and } |r-\nu(M)/\mu(M)| < \mu(M) \text{ otherwise;} \end{cases}$$

$f_1(M)$ similarly, interchanging rational and irrational.

Let g be the characteristic function of the rationals. One verifies easily that f, f_1, and ν/μ are all differentially equivalent for μ. Moreover $g(f) \equiv 1$ and $g(f_1) \equiv 0$ are trivially F-integrable for μ, with unequal integrals. Yet g is discontinuous on a set of measure 1, and $\theta(\mu)$ is continuous.

For set-functions f which are not essentially bounded [2] the above results are in doubt because they depend on the necessity of the condition for F-integrability proved by Finch only for essentially bounded functions. I shall call f bounded on M if $\{f(S) \mid S \subset M, \mu(M) > 0\}$ is bounded, and I shall call f *locally bounded* if there exists a partition Π such that f is bounded on M for every $M \in \Pi$. This is weaker than essential boundedness. I shall prove that if the integral of f for μ as defined by (1) exists, then f is locally bounded. Finch's theorem on integrability then applies to each cell of the resulting partition, removing the above objection. Note that for Theorems 2 and 3 to apply also to unbounded functions g, (2.2) should be strengthened by requiring g to be summable for $\theta(\mu)$.

Assuming that $(K)\int d\, f\mu$ exists, there exists a partition such that $\sum_\Pi f\mu$ is convergent and bounded on all finer partitions Π. By restricting our attention to one cell of this partition at a time we can assume at the outset that $|\sum_\Pi f\mu| \leqq 1$ for all Π. It is necessary to prove first that $f\mu$ is locally bounded. For this it suffices to show the existence of a set $M \in \mathscr{M}$ with $\mu(M) > 0$ and f bounded on M, for then a maximal disjoint family of such M gives the required partition. If no such M exists we can build a decreasing sequence of sets $\{M_n\}$ with $M_0 = \mathscr{X}$, $\mu(M_n) > 0$ and $|f(M_n)| > |f(\mathscr{X}-M_{n-1})|+2$. Then since

$$|f(M_n)+f(M_{n-1}-M_n)+f(\mathscr{X}-M_{n-1})| \leqq 1$$

we have $|f(M_{n-1}-M_n)| \geqq 1$. But then setting $M_\infty = \bigcap M_n$ and $\Pi = \{M_n \mid 0 \leqq n \leqq \infty\}$ we have $\sum_\Pi f\mu$ divergent.

We can now construct a partition to show f itself is locally bounded. First choose a maximal disjoint sequence $\{M_1^m\}$ from $\mathscr{M}$ with $\mu(M_1^m) > 0$

and $f(M_1^m) > 1$. (If no such sets exist, all the better!) Let $M_1 = \mathscr{X} - \bigcup_m M_1^m$. Then f is bounded above on M_1. Next choose, disjoint from M_1 a maximal disjoint sequence $\{M_2^m\}$ with $f(M_2^m) > 2$, and set $M_2 = M_1 - \bigcup_m M_2^m$. Then f is bounded above on M_2. Continue thus indefinitely, and let $M_\infty = \mathscr{X} - \bigcup_n M_n$. Clearly $\{M_n \mid 1 \leqq n \leqq \infty\}$ is the required partition, provided we show that $\mu(M_\infty) = 0$. But for any n the partition Π consisting of the set $\bigcup_{i=1}^n M_i$ and the sets M_n^m gives a sum $\sum_\Pi f\mu > n\mu(M_\infty) - k$, where k is a bound for $f\mu$ on $\mathscr{X}$. If $\mu(M_\infty) > 0$ this gives unbounded sums. For the partition constructed f is bounded above on each cell. Repeat the argument in each cell to bound f below.

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