ON A DUAL INTEGRAL EQUATION WITH A TRIGONOMETRIC KERNEL

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(Received 31 March, 1976)

1. In this note we formally solve the following dual integral equations:

$$\int_0^\infty \psi(u)\left[1+\frac{u+e^{-u}\sinh u}{\sinh^2 u}\right]\cos uz\,du = h \quad (0<z<a), \tag{1}$$

$$\int_0^\infty \frac{\psi(u)}{u}\cos uz\,du = 0 \quad (a<z<\infty), \tag{2}$$

where h is a constant and the Fourier cosine transform of $u^{-1}\psi(u)$ is assumed to exist. These dual equations arise in a crack problem in elasticity theory.

2. Solution of equations. We follow one of the standard procedures for solving dual integral equations (see [**2**]) and seek a solution in the form of an integral that satisfies (2) identically.

Define a function ϕ by the equation

$$\psi(u) = \int_0^a \phi(y)\sin uy\,dy. \tag{3}$$

When we substitute for $\psi(u)$ from (3) into (2) and interchange the order of integration we find that (2) is identically satisfied. By substituting (3) into (1) and interchanging the order of integration in the resulting double integral we obtain

$$\int_0^a \phi(y)\,dy\int_0^\infty [\coth u + u\,\mathrm{cosech}^2 u]\sin uy\cos uz\,du = h \quad (0<z<a). \tag{4}$$

This integral equation does not appear to be easily solvable in a closed form. However, on noting that $-\mathrm{cosech}^2 u$ is the derivative of $\coth u$ we are led to consider the following integral equation:

$$\int_0^a \phi(y)\,dy\int_0^\infty \left[\coth\left(\frac{u}{\lambda}\right)+\frac{u}{\lambda}\mathrm{cosech}^2\left(\frac{u}{\lambda}\right)\right]\sin uy\cos uz\,du = h \quad (0<z<a) \tag{5}$$

which, for $\lambda = 1$, reduces to (4).

Now (5) can be written as

$$\frac{d}{d\lambda}\left[\int_0^a \phi(y)\,dy\int_0^\infty \lambda\coth\left(\frac{u}{\lambda}\right)\sin uy\cos uz\,du\right] = h \quad (0<z<a).$$

Glasgow Math. J. **18** (1977) 175–177.

https://doi.org/10.1017/S0017089500003232 Published online by Cambridge University Press

Hence, on integrating with respect to λ we obtain

$$\int_0^a \phi(y)\,dy \int_0^\infty \coth\left(\frac{u}{\lambda}\right)\sin uy \cos uz\,du = h \quad (0 < z < a). \tag{6}$$

Now, by straightforward analysis we find that

$$\int_0^\infty \coth\left(\frac{u}{\lambda}\right)\sin uy \cos uz\,du = \tfrac{1}{2}\int_0^\infty \coth\frac{u}{\lambda}\,[\sin(y+z)u + \sin(y-z)u]\,du$$

$$= \frac{\pi\lambda}{4}\left[\coth\left\{\frac{(y+z)\pi\lambda}{2}\right\} + \coth\left\{\frac{(y-z)\pi\lambda}{2}\right\}\right].$$

By integrating with respect to z we obtain

$$\int_0^z dt \int_0^\infty \coth\left(\frac{u}{\lambda}\right)\sin uy \cos ut\,du = \tfrac{1}{2}\log\left|\frac{\sinh\frac{\pi\lambda}{2}(y+z)}{\sinh\frac{\pi\lambda}{2}(y-z)}\right|$$

$$= \tfrac{1}{2}\log\left|\frac{\tanh\gamma z + \tanh\gamma y}{\tanh\gamma z - \tanh\gamma y}\right|, \tag{7}$$

where $\gamma = \pi\lambda/2$. Thus, by integrating (6) with respect to z and using (7), we obtain

$$\int_0^a \phi(y)\log\left|\frac{\tanh\gamma z + \tanh\gamma y}{\tanh\gamma z - \tanh\gamma y}\right| dy = 2hz \quad (0 < z < a). \tag{8}$$

But $\tanh\gamma y$ is a positive monotonic increasing function on $(0, \infty)$, and so we can solve (8) by using a result due to Parihar. (See [1].) The solution is

$$\phi(y) = \frac{s'(y)}{m(y)}\left[\pi^{\frac{1}{2}}\int_0^a \frac{2hm(x)}{s(y)-s(x)}\,dx + \left(\tfrac{1}{4}B(s(a))^{\frac{1}{2}}/K\left\{\left(1-\frac{s(0)}{s(a)}\right)^{\frac{1}{2}}\right\}\right)\right] \quad (0 < y < a), \tag{9}$$

where $s(y) = \tanh^2\gamma y$, $m(y) = \tanh^2\gamma y\,[\tanh^2\gamma a - \tanh^2\gamma y]^{\frac{1}{2}}$,

$$K\left\{\left(1-\frac{s(0)}{s(a)}\right)^{\frac{1}{2}}\right\} = K(1),$$

where $K(k)$ denotes the complete elliptic of the first kind; (thus $K(1)$ is a divergent integral), and

$$B = \left\{\frac{\tanh\gamma a}{\pi K(0)}\right\}\left[\int_0^a \frac{2hxs'(x)}{m(x)}\,dx - 2\int_0^a \frac{s'(y)}{m(y)}\,dy \int_0^a \frac{2hm(x)}{s(y)-s(x)}\,dx\right] \tag{10}$$

where the first integral in (9) and the last integral in (10) are to be understood in the sense of their principal values. On carrying out the integrations involved we find after considerable effort that

$$\phi(y) = \frac{2h[\operatorname{sech}\gamma a - \operatorname{sech}^2\gamma y]}{\pi\tanh\gamma y[\tanh^2\gamma a - \tanh^2\gamma y]^{\frac{1}{2}}}. \tag{11}$$

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Hence (3) gives

$$\psi(u)=\frac{2h}{\pi}\int_0^a \frac{[\operatorname{sech}\gamma a-\operatorname{sech}^2\gamma y]\sin uy}{\tanh\gamma y[\tanh^2\gamma a-\tanh^2\gamma y]^{\frac{1}{2}}}\,dy. \tag{12}$$

ACKNOWLEDGEMENT. The author wishes to express his thanks to his colleague Dr L. W. Longdon for his valuable criticisms of the first draft of this paper.

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PRINTERS TO THE UNIVERSITY OF GLASGOW

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https://doi.org/10.1017/S0017089500003232 Published online by Cambridge University Press