# ON THE CLOSED GRAPH THEOREM

*by* J. O. POPOOLA† and I. TWEDDLE

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**1. Introduction.** Our main purpose is to describe those separated locally convex spaces which can serve as domain spaces for a closed graph theorem in which the range space is an arbitrary Banach space of (linear) dimension at most $c$, the cardinal number of the real line **R**. These are the $\delta$-barrelled spaces which are considered in §4. Many of the standard elementary Banach spaces, including in particular all separable ones, have dimension at most $c$. Also it is known that an infinite dimensional Banach space has dimension at least $c$ (see e.g. [**8**]). Thus if we classify Banach spaces by dimension we are dealing, in a natural sense, with the first class which contains infinite dimensional spaces.

Kalton [**7**] has considered this problem when the range space is an arbitrary *separable* Banach space. We show that the class of $\delta$-barrelled spaces forms a proper subclass of Kalton's domain spaces and strictly contains the class of separated barrelled spaces. Theorem 3 is the closed graph theorem which characterizes $\delta$-barrelled spaces and yields as corollaries various permanence properties of these spaces. We show also that a subspace of countable codimension of a $\delta$-barrelled space is again $\delta$-barrelled (Theorem 4).

In §5 we describe all possible separated locally convex range spaces for a closed graph theorem in which the domain space is an arbitrary $\delta$-barrelled Mackey space (Theorem 8). These are the infra-$\delta$-spaces, which are obtained by a variant of the hull idea due to Adasch [**1**]. They form a proper subclass of Adasch's infra-$s$-spaces and likewise they need not be weakly $t$-polar. We also consider a form of the open mapping theorem (Theorem 8).

The results of §3 are concerned with separability properties. Although they are developed for use in the sequel, we feel that they also have an independent interest.

**2. Preliminaries.** If $E$ is a real or complex vector space, $E^*$ will denote its algebraic dual. $K$ will stand for **R** or **C**. Let $\{x_\mu\}_{\mu\in M}$ be a basis in $E$. We will make frequent use of the well known fact that the mapping $x^* \to (\langle x_\mu, x^*\rangle)_{\mu\in M}$ is a topological isomorphism of $E^*$ onto the product $K^M$ when $E^*$ has the topology $\sigma(E^*, E)$ and $K^M$ has its product topology, for which its dual is the direct sum $K^{(M)}$. $e_\nu$ will stand for the element $(\delta_{\nu\mu})_{\mu\in M}$ of $K^M$ or $K^{(M)}$ and $p_\nu$ will be the projection of $K^M$ or $K^{(M)}$ onto its $\nu$-th component.

If $E$ is a locally convex space, $E'$ will represent its (continuous) dual. Generally we follow the topological vector space notation of [**12**], except that the term " Mackey space " is applied to any separated locally convex space $E$ with its Mackey topology $\tau(E, E')$.

The cardinality of a set $A$ will be denoted by $|A|$.

**3. Subsets of separable sets in E*.** We begin by considering the case $E = K^{(M)}$ for some non-empty index set $M$ so that $E^* = K^M$.

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THEOREM 1. *A non-empty subset $A$ of $K^M$ is contained in a separable subset of $K^M$ if and only if there is a subset $M_0$ of $M$ and a family $\{x'_\mu\}_{\mu \in M \setminus M_0}$ of elements of $K^{(M)}$ such that*

(i) $|M_0| \leqq c$,

(ii) *for each* $\mu \in M \setminus M_0$, $x'_\mu = \sum_{r=1}^{m(\mu)} \lambda(r, \mu)\, e_{\alpha(r,\mu)}$ *where* $\alpha(r, \mu) \in M_0$ *and* $\lambda(r, \mu) \in K$ $(r = 1, 2, \ldots, m(\mu))$,

(iii) *if* $x = (\xi_\mu)_{\mu \in M} \in A$, $\xi_\mu = \langle x, x_\mu \rangle$ *for each* $\mu \in M \setminus M_0$.

*Proof.* (a) *Sufficiency.* Let $\{(\eta_\mu^{(n)})_{\mu \in M_0} : n \in \mathrm{N}\}$ be an at most countable dense subset of the separable product $\Pi\{p_\mu(A) : \mu \in M_0\}$ [4, Chapter VIII, 7.2]. For each $n \in \mathrm{N}$ define $x_n = (\xi_\mu^{(n)})_{\mu \in M} \in K^M$ by

$$\xi_\mu^{(n)} = \begin{cases} \eta_\mu^{(n)} & \text{if} \quad \mu \in M_0, \\ \sum_{r=1}^{m(\mu)} \lambda(r, \mu) \eta_{\alpha(r,\mu)}^{(n)} & \text{if} \quad \mu \in M \setminus M_0. \end{cases}$$

Then $\langle x_n, x'_\mu \rangle = \xi_\mu^{(n)}$ for all $n \in \mathrm{N}$, $\mu \in M \setminus M_0$.

Now let $\{\mu(1), \ldots, \mu(s)\}$ be any non-empty finite subset of $M$ and let $\varepsilon$ be a positive real number. If at least one $\mu(t) \in M \setminus M_0$, let

$$\sigma = \max\{|\lambda(r, \mu(t))| : r = 1, 2, \ldots, m(\mu(t));\ \mu(t) \in M \setminus M_0\}$$

and $N = \max\{m(\mu(t)) : \mu(t) \in M \setminus M_0\}$. Otherwise let $\sigma = N = 1$. If $x = (\xi_\mu)_{\mu \in M} \in A$, then $y = (\xi_\mu)_{\mu \in M_0} \in \Pi\{p_\mu(A) : \mu \in M_0\}$ and so there is a positive integer $n$ such that

$$\left|\xi_{\alpha(r,\mu(t))} - \eta_{\alpha(r,\mu(t))}^{(n)}\right| \leqq \frac{\varepsilon}{N(\sigma+1)} \quad (r = 1, 2, \ldots, m(\mu(t))\,;\ t = 1, 2, \ldots, s)$$

(of course if $\mu(t) \in M_0$ we put $m(\mu(t)) = 1$, $\lambda(1, \mu(t)) = 1$ and $\alpha(1, \mu(t)) = \mu(t)$). It now follows easily that $|\xi_{\mu(t)} - \xi_{\mu(t)}^{(n)}| \leqq \varepsilon$ $(t = 1, 2, \ldots, s)$ so that $A$ is contained in the closure of $\{x_n : n \in \mathrm{N}\}$.

(b) *Necessity.* Let $\{x_n : n \in \mathrm{N}\}$ be an at most countable subset of $K^M$ whose closure contains $A$ and let $F$ be the closed vector subspace of $K^M$ generated by $\{x_n : n \in \mathrm{N}\}$. Let $\{f_\phi : \phi \in \Phi\}$ be a basis in $E = K^{(M)}/F^0$. $F$ is the algebraic dual of $E$ and it is $\sigma(F, E)$-separable. It is also topologically isomorphic to $K^\Phi$ so that we must have $|\Phi| \leqq c$ [4, Chapter VIII, 7.2].

Let $q : K^{(M)} \to E$ be the quotient map. For each $\phi \in \Phi$ choose $e_{\mu(r,\phi)} \in K^{(M)}$ and scalars $\gamma(r, \phi)$ $(r = 1, 2, \ldots, n(\phi))$ such that

$$f_\phi = q\left(\sum_{r=1}^{n(\phi)} \gamma(r, \phi) e_{\mu(r,\phi)}\right).$$

If $M_0 = \{\mu(r, \phi) : r = 1, 2, \ldots, n(\phi);\ \phi \in \Phi\}$, $|M_0| \leqq c$.

Let $\nu \in M \setminus M_0$ and choose $\phi(s, \nu) \in \Phi$, $\beta(s, \nu) \in K$ $(s = 1, 2, \ldots, m(\nu))$ such that

$$q(e_\nu) = \sum_{s=1}^{m(\nu)} \beta(s, \nu) f_{\phi(s,\nu)}.$$

Then

$$q(e_\nu) = q\left(\sum_{s=1}^{m(\nu)} \sum_{r=1}^{n(\phi(s,\nu))} \beta(s,\nu)\gamma(r,\phi(s,\nu))\, e_{\mu(r,\phi(s,\nu))}\right),$$

and, if

$$x'_\nu = \sum_{s=1}^{m(\nu)} \sum_{r=1}^{n(\phi(s,\nu))} \beta(s,\nu)\gamma(r,\phi(s,\nu))\, e_{\mu(r,\phi(s,\nu))},$$

for all $x = (\xi_\mu)_{\mu \in M} \in F$,

$$\xi_\nu = \langle x, e_\nu \rangle = \langle x, x'_\nu \rangle.$$

This gives us the required set $M_0$ and the family $\{x'_\mu\}_{\mu \in M \setminus M_0}$.

REMARKS. Suppose that the non-empty set $A$ is contained in a separable subset of $K^M$.

(i) If $A$ is bounded, the sets $\Pi\{p_\mu(A): \mu \in M_0\}$ and $\{x_n : n \in \mathbf{N}\}$ constructed in part (a) of the proof will also be bounded. Thus $A$ is contained in a bounded separable subset of $K^M$.

(ii) Let $G$ be the closed linear span of $A$ in $K^M$. Then in the notation of part (b) of the proof, $G \subseteq F$ and $K^{(M)}/G^0 \simeq (K^{(M)}/F^0)/(G^0/F^0)$. Since $K^{(M)}/F^0$ has dimension at most $c$, the dimension of $K^{(M)}/G^0$ cannot exceed $c$. $G$, being the algebraic dual of $K^{(M)}/G^0$, is then topologically isomorphic to a product of at most $c$ copies of $K$ and so is separable.

DEFINITION. Let $E$ be a vector space over $K$. We shall say that a subset of $E^*$ is *essentially separable* if it is contained in a $\sigma(E^*, E)$-separable set.

To avoid possible confusion later on when applying this definition to a subset of the dual space $E'$ of a separated locally convex space $E$, we stress that we shall regard it as a subset of $E^*$ and require that it be contained in a $\sigma(E^*, E)$-separable set.

Since separability and boundedness are preserved by continuous linear mappings and since any algebraic dual is topologically isomorphic to a product of copies of the scalar field, we obtain immediately from the above remarks:

COROLLARY. *Let $E$ be a vector space over $K$. If $A$ is a non-empty essentially separable subset of $E^*$, the $\sigma(E^*, E)$-closed linear span $G$ of $A$ is $\sigma(E^*, E)$-separable. If $A$ is also $\sigma(E^*, E)$-bounded, there is a $\sigma(E^*, E)$-bounded separable subset of $G$ which contains $A$.*

**4. $\delta$-barrelled spaces.** We shall say that a separated locally convex space $E$ is *$\delta$-barrelled* if each essentially separable $\sigma(E', E)$-bounded set is equicontinuous.

Certainly every separated barrelled space is $\delta$-barrelled and every $\delta$-barrelled space is $\sigma$-barrelled [3], (also called $\omega$-barrelled [9]). The following two examples show that $\delta$-barrelled spaces are strictly intermediate to these classes.

EXAMPLE 1. Let $E = \mathbf{R}^{(M)}$ where $|M| = 2^{2^c}$ and let

$$E' = \{(\xi_\mu)_{\mu \in M} \in \mathbf{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leqq 2^c\}.$$

$(E, E')$ is a dual pair and $E^* = \mathbf{R}^M$. Let $A$ be any non-empty essentially separable $\sigma(E', E)$-bounded set. By Theorem 1 there is a subset $M_0$ of $M$ such that $|M_0| \leqq c$ and any component

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of any element of $A$ is uniquely determined by its components corresponding to elements of $M_0$. It follows that $|A| \leqq c^c = 2^c$ (alternatively use [4, Chapter VIII, 7 Ex. 4]) so that

$$|\{\nu : \exists\, x = (\xi_\mu)_{\mu \in M} \in A \quad \text{such that} \quad \xi_\nu \neq 0\}| \leqq \sum_{x \in A} |\{\mu : p_\mu(x) \neq 0\}| \leqq 2^c 2^c = 2^c.$$

Thus $A \subseteq \Pi\{\text{cl}\,(p_\mu(A)) : \mu \in M\} \subseteq E'$ and by Tychonoff's theorem the $\sigma(E', E)$-closure of $A$ is $\sigma(E', E)$-compact.

It follows that if we give $E$ its Mackey topology $\tau(E, E')$ it is $\delta$-barrelled. However $[0, 1]^M \cap E'$ is a $\sigma(E', E)$-bounded set which is not $\sigma(E', E)$-relatively compact since it is dense in $[0, 1]^M \nsubseteq E'$. Thus $E$ is not barrelled under $\tau(E, E')$.

EXAMPLE 2. Let $E = \mathbf{R}^{(M)}$ where $|M| = c$ and let $E' = \{(\xi_\mu)_{\mu \in M} : |\{\mu : \xi_\mu \neq 0\}| \leqq \aleph_0\}$. Again $(E, E')$ is a dual pair and $E^* = \mathbf{R}^M$. Given an at most countable subset $\Lambda$ of $M$ and a family $(\alpha_\lambda)_{\lambda \in \Lambda}$ of non-negative real numbers, the set

$$\{(\xi_\mu)_{\mu \in M} : |\xi_\lambda| \leqq \alpha_\lambda (\lambda \in \Lambda), \quad \xi_\mu = 0 \ \text{otherwise}\}$$

is $\sigma(E', E)$-compact and absolutely convex. The topology on $E$ of uniform convergence on all such sets is a topology of the dual pair $(E, E')$ under which $E$ is $\sigma$-barrelled and even countably barrelled [5].

However $E$ cannot have a $\delta$-barrelled topology for this dual pair since $E^*$ is $\sigma(E^*, E)$-separable and $[0, 1]^M \cap E'$ is $\sigma(E', E)$-bounded but not $\sigma(E', E)$-relatively compact, being dense in $[0, 1]^M$.

THEOREM 2. *Let $E$ be a $\delta$-barrelled space. The topology $\delta(E, E')$ of uniform convergence on the family $\mathscr{A}$ of essentially separable $\sigma(E', E)$-bounded sets is a topology of the dual pair $(E, E')$ and $E$ is $\delta$-barrelled under any topology of this dual pair which is finer than $\delta(E, E')$. $E$ is also countably barrelled under $\delta(E, E')$.*

*Proof.* The polars of the elements of $\mathscr{A}$ form a base of neighbourhoods of the origin for $\delta(E, E')$ since $\mathscr{A}$ clearly contains the union of any finite collection of its elements and any scalar multiple of an element. Since $\mathscr{A}$ contains all finite subsets of $E'$, $\delta(E, E')$ is separated. Since each element of $\mathscr{A}$ is equicontinuous in the initial topology of $E$, by the Mackey–Arens theorem, the $\sigma(E', E)$-closed absolutely convex envelope of each element of $\mathscr{A}$ is $\sigma(E', E)$-compact and $\delta(E, E')$ is a topology of the dual pair $(E, E')$.

It is clear that $E$ is $\delta$-barrelled under any topology of the dual pair $(E, E')$ which is finer than $\delta(E, E')$.

Finally let $(A_n)$ be a sequence of $\delta(E, E')$-equicontinuous sets whose union is $\sigma(E', E)$-bounded. Since a countable union of separable sets is again separable, $\bigcup_{n=1}^{\infty} A_n$ is contained in a $\sigma(E^*, E)$-separable set. Thus $\bigcup_{n=1}^{\infty} A_n$ is $\delta(E, E')$-equicontinuous so that $E$ is countably barrelled under $\delta(E, E')$.

The following result provides our principal reason for introducing $\delta$-barrelled spaces. The corresponding characterization of barrelled spaces as the domain spaces for a closed graph theorem in which an arbitrary Banach space may serve as range space is of course well known.

THEOREM 3. *A separated locally convex space $E$ is $\delta$-barrelled if and only if whenever $F$ is a Banach space with dimension at most $c$ and $t: E \to F$ is a linear mapping whose graph is closed, $t$ is necessarily continuous.*

*Proof.* Consider a linear mapping $t: E \to F$ where $E$ is a $\delta$-barrelled space, $F$ is a Banach space of dimension at most $c$ and the graph of $t$ is closed in $E \times F$. Let $t^*: F^* \to E^*$ be the transpose of $t$. $t^*$ is continuous under $\sigma(F^*, F)$ and $\sigma(E^*, E)$ and $t^{*-1}(E') \cap F'$ is a $\sigma(F', F)$-dense vector subspace of $F'$. Let $B$ be the closed unit ball of $F'$ and consider $t^{*-1}(E') \cap B$. Since $F^*$ is $\sigma(F^*, F)$-separable, being topologically isomorphic under $\sigma(F^*, F)$ to a product of at most $c$ copies of $K$, $A = t^*(t^{*-1}(E') \cap B)$ is an essentially separable $\sigma(E', E)$-bounded set. It is therefore equicontinuous and so $\sigma(E', E)$-relatively compact. The standard closed graph theorem argument now shows that $t^{*-1}(E') \cap B$ is $\sigma(F', F)$-closed so that $t^{*-1}(E') \cap F'$ is $\sigma(F', F)$-closed, $F$ being fully complete. Since $t^{*-1}(E') \cap F'$ is $\sigma(F', F)$-dense, it coincides with $F'$ and $t^*(B)$ is equicontinuous. (See [**12**, Chapter VI, §2] for details.) The mapping $t$ is therefore continuous as required.

We now suppose that $E$ is a separated locally convex space which satisfies the condition of the enunciation. Let $A$ be any non-empty essentially separable $\sigma(E', E)$-bounded set, let $B$ be the $\sigma(E^*, E)$-closed absolutely convex envelope of $A$ and let $F'$ be the linear span of $B$ in $E^*$. Since $B$ is $\sigma(E^*, E)$-compact, $E/F'^0$ is a normed space under $\tau(E/F'^0, F')$. $(E/F'^0)^*$ is the $\sigma(E^*, E)$-closed linear span of $A$ in $E^*$ which, by the corollary to Theorem 1, is $\sigma(E^*, E)$-separable. It now follows from the usual topological isomorphism that $E/F'^0$ has dimension at most $c$. Let $F$ be the completion of $E/F'^0$ in the norm topology. Since each element of $F$ is the limit of a sequence of elements of $E/F'^0$, it is easy to see that

$$|F| \leqq \left|\prod_{\mathbf{N}}(E/F'^0)\right| \leqq c^{\aleph_0} = c,$$

from which it follows that the Banach space $F$ has dimension at most $c$. Again we can refer to standard arguments to observe that the quotient map $q: E \to E/F'^0$, regarded as a mapping of $E$ into $F$, has a closed graph in $E \times F$. By hypothesis $q$ is then continuous and so its transpose must map $B$ onto an equicontinuous subset of $E'$. Since the transpose of $q$ is just the natural injection of $F'$ into $E'$, it follows that $B$ and therefore also $A$ are equicontinuous. Thus $E$ is $\delta$-barrelled.

COROLLARY 1. (a) *A separated inductive limit of $\delta$-barrelled spaces is $\delta$-barrelled.*

(b) *Any product of $\delta$-barrelled spaces is $\delta$-barrelled.*

(c) *The completion of a $\delta$-barrelled space is $\delta$-barrelled.*

*Proof.* (a) This may be proved directly using [**12**, Chapter V, Proposition 5] or it may be deduced immediately from the theorem using [**6**, Theorem 2.1] (this is stated without proof but the method is standard).

(b) This follows from the theorem and [**6**, Theorem 2.2].

(c) Let $E$ be a $\delta$-barrelled space and $G$ its completion. Suppose that $t: G \to F$ is a linear mapping with a closed graph of $G$ into a Banach space $F$ whose dimension is at most $c$. $t|_E$ has a closed graph and so is continuous by the theorem. Let $s$ be the unique extension by continuity of $t|_E$ to a continuous linear mapping of $G$ into $F$. Since the graph of $s-t$ is closed, $\{x\in G: s(x)=t(x)\}$ is a closed vector subspace of $G$ which contains $E$ and therefore must be the whole of $G$. $t$ is therefore continuous and the result follows by the theorem.

COROLLARY 2. *If a $\delta$-barrelled space $E$ has a $\beta(E, E')$-dense subset of cardinality at most $c$, then $E$ is barrelled.*

*Proof.* If we carry out the construction of the Banach space $F$ as in the second part of the proof of the theorem but start with an arbitrary non-empty $\sigma(E', E)$-bounded set $A$, $E/F'^0$ has a norm dense subset $G$ of cardinality at most $c$. It follows, by considering sequences in $G$, that $F$ has dimension at most $c$. Since $E$ is $\delta$-barrelled, the theorem shows that the quotient map is continuous and that $A$ is equicontinuous as before.

REMARKS. (i) We can also show that any linear mapping with a closed graph of a Banach space of dimension at most $c$ onto a $\delta$-barrelled space is open. This is deduced from the closed graph theorem in the usual way on observing that any quotient of such a Banach space by a closed vector subspace (in this case the null-space of the mapping) is also a Banach space of dimension at most $c$.

(ii) In Theorem 2.6 of [7], Kalton has characterized those spaces which can be domain spaces for a closed graph theorem in which the range space is an arbitrary separable Banach space. Of course every separable Banach space has dimension at most $c$, but the converse is not true (e.g. $\ell_\infty$). Our $\delta$-barrelled spaces therefore form a subclass of Kalton's class $\mathscr{C}(\zeta_B)$. As an example of a non-barrelled element of $\mathscr{C}(\zeta_B)$ he gives $\ell_\infty$ with the topology $\tau(\ell_\infty, \ell_1)$. This space is not $\delta$-barrelled since the closed unit ball of $\ell_1$ is a $\sigma(\ell_1, \ell_\infty)$-separable bounded set which is not $\sigma(\ell_1, \ell_\infty)$-relatively compact. The inclusion is therefore strict.

(iii) Since a $\delta$-barrelled space is also $\sigma$-barrelled, a separable $\delta$-barrelled space is necessarily barrelled [3, Corollary 4.a].

It is known that a subspace of countable codimension of a barrelled, $\sigma$-barrelled or countably barrelled space is again of the same type [**13**, **15**, **9**, **16**]. We now establish the corresponding result for $\delta$-barrelled spaces.

THEOREM 4. *Let $E$ be a $\delta$-barrelled space and let $F$ be a subspace of countable codimension. Then $F$ is $\delta$-barrelled in the induced topology.*

*Proof.* Let $A$ be a non-empty essentially separable $\sigma(F', F)$-bounded set and let $\hat{F}$ be the completion of $F$ in the topology induced from $E$. We show first of all that $A$ is $\sigma(F', \hat{F})$-bounded. If this were not the case, there would be an element $x\in\hat{F}$ and a sequence $(x'_n)$ in $A$ such that $\{\langle x, x'_n\rangle : n\in\mathbb{N}\}$ is an unbounded set of scalars. Now by [**9**, Theorem], $F$ is certainly $\sigma$-barrelled in the induced topology so that $\{x'_n : n\in\mathbb{N}\}$ is equicontinuous on $F$. A contradiction is obtained on observing that the same subsets of $F'$ are equicontinuous for the initial topology of $F$ and its extension to $\hat{F}$ [**12**, Chapter VI, Theorem 3].

Let $G$ be the closure of $F$ in $E$ and let $B\subseteq G'$ be the set of extensions by continuity of the elements of $A$. Since $G$ can be embedded in $\hat{F}$, it follows that $B$ is $\sigma(G', G)$-bounded. The codimension of $G$ in $E$ is also countable (perhaps finite) and so we can choose an (at most) countable family $\{x_n\}$ of linearly independent elements of $E\backslash G$ which spans a supplement of

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$G$ in $E$. We extend the elements of $B$ to the whole of $E$ by putting $\langle x_n, x' \rangle = 0$ for each $n$ and each $x' \in B$. Let $C$ be the set of all these extensions. By the Lemma of **[13**, §2], $C \subseteq E'$ and clearly $C$ is $\sigma(E', E)$-bounded.

We now show that $C$ is essentially separable. By hypothesis $C$ must then be equicontinuous and the result follows on observing that $C^0 \cap F = A^0$. We consider the usual topological isomorphism of $F^*$ onto $K^M$ where $|M|$ is the dimension of $F$. Since $F$ has countable codimension in $E$, this extends to a topological isomorphism of $E^*$ onto $K^{M \cup N}$ where $N$ is at most countable. Now determine $M_0 \subseteq M$, $x'_\mu \in K^{(M)}$ $(\mu \in M \backslash M_0)$ as in Theorem 1 for the image of $A$ in $K^M$. With $M_0$ replaced by $M_0 \cup N$ and the same linear forms $x'_\mu$ (more correctly, the elements of $K^{(M \cup N)}$ defined by the same formulae with the $e_{\alpha(r,\mu)}$ now in $K^{(M \cup N)}$), Theorem 1 shows that the image of $C$ in $K^{M \cup N}$ is contained in a separable subset of $K^{M \cup N}$. This completes the proof.

**5. The range space.** In our closed graph theorem (Theorem 3) the range space is a Banach space of dimension at most $c$. It is clear that the result continues to hold and the proof is essentially the same if the range space $F$ is taken to be a $B_r$-complete space **[11]** or a weakly $t$-polar space **[10]** in which each equicontinuous set, respectively each $\sigma(F', F)$-bounded set, is essentially separable. By suitably adapting Adasch's hull approach **[1]**, we are able to describe those separated locally convex spaces which can serve as range spaces for a closed graph theorem in which the domain space is an arbitrary $\delta$-barrelled space endowed with its Mackey topology.

DEFINITIONS. Let $F$ be a separated locally convex space. For each vector subspace $H$ of $F'$ let $H^\delta$ be the intersection of all vector subspaces $G$ of $F^*$ such that

(i) $H \subseteq G$,

(ii) the $\sigma(F^*, F)$-closure of each essentially separable $\sigma(F^*, F)$-bounded subset of $G$ is contained in $G$. (Note that this closure will be $\sigma(F^*, F)$-compact.)

We shall say that (a) $F$ is a *$\delta$-space* if the $\sigma(F', F)$-closure of each vector subspace $H$ of $F'$ coincides with $F' \cap H^\delta$, and (b) $F$ is an *infra-$\delta$-space* if for each $\sigma(F', F)$-dense vector subspace $H$ of $F'$, $F' \cap H^\delta = F'$.

We need the following analogues of **[1**, §5 (4), (5)].

THEOREM. 5. *Let $F$ be an (infra-)$\delta$-space and let $G'$ be a $\sigma(F', F)$-dense vector subspace of $F'$. Then $F$ is an (infra-)$\delta$-space for any topology of the dual pair $(F, G')$.*

*Proof.* This is immediate since the $\sigma(G', F)$-closure of any vector subspace of $G'$ is the intersection with $G'$ of its $\sigma(F', F)$-closure and any $\sigma(G', F)$-dense vector subspace of $G'$ is also $\sigma(F', F)$-dense in $F'$.

THEOREM 6. *Let $F$ be a $\delta$-space and $M$ a closed vector subspace of $F$. Then $F/M$ is a $\delta$-space under its quotient topology.*

*Proof.* $(F/M)' = M^0$, the polar of $M$ in $F'$, and $(F/M)^* = M^+$, the polar of $M$ in $F^*$. The problem is to show that for any vector subspace $H$ of $M^0$, the hull $H^{\delta_1}$ constructed for the dual pair $(F/M, M^0)$ and the hull $H^{\delta_2}$ constructed for the dual pair $(F, F')$ coincide.

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Let $G$ be any vector subspace of $M^+$ which contains $H$ and suppose that the $\sigma(M^+, F/M)$-closure of each $\sigma(M^+, F/M)$-bounded subset of $G$ which is contained in a $\sigma(M^+, F/M)$-separable set is contained in $G$. Let $A$ be a $\sigma(F^*, F)$-bounded subset of $G$ which is contained in a $\sigma(F^*, F)$-separable set. By the corollary to Theorem 1, the $\sigma(F^*, F)$-closed vector subspace of $F^*$ generated by $A$ is $\sigma(F^*, F)$-separable. It is also contained in $M^+$. Since $\sigma(M^+, F/M)$ and $\sigma(F^*, F)$ coincide on $M^+$, it follows that the $\sigma(F^*, F)$-closure of $A$ is contained in $G$. Thus $H^{\delta_2} \subseteq H^{\delta_1}$.

Conversely let $G$ be any vector subspace of $F^*$ which contains $H$ and suppose that the $\sigma(F^*, F)$-closure of each $\sigma(F^*, F)$-bounded subset of $G$ which is contained in a $\sigma(F^*, F)$-separable set is contained in $G$. Since $M^+$ is $\sigma(F^*, F)$-complete, $G \cap M^+$ has the same property. It now follows easily that $H^{\delta_1} \subseteq H^{\delta_2}$ and the result is immediate.

We now give our general closed graph and open mapping theorems. The proofs have much in common with the proofs of the corresponding results **A**, $\mathbf{A}_1$, $\mathbf{A}_2$ and **B**, $\mathbf{B}_1$ of [**1**]. We therefore confine ourselves to the essential details and differences.

THEOREM 7. (a) *A linear mapping with a closed graph of a $\delta$-barrelled Mackey space into an infra-$\delta$-space is continuous.*

(b) *Let $F$ be a separated locally convex space with the property that any linear mapping with a closed graph of a $\delta$-barrelled Mackey space into $F$ is continuous. Then $F$ is an infra-$\delta$-space.*

(c) *Let $E$ be a separated locally convex Mackey space with the property that any linear mapping with a closed graph of $E$ into an infra-$\delta$-space is continuous. Then $E$ is $\delta$-barrelled.*

*Proof.* (a) If $t: E \to F$ is such a mapping and $t^*: F^* \to E^*$ is its transpose, we observe that $G = t^{*-1}(E')$ is one of the subspaces of $F^*$ in the intersection which determines $(t^{*-1}(E') \cap F')^\delta$. As in [**1**, proof of **A**] it follows that

$$t^{*-1}(E') \cap F' = (t^{*-1}(E') \cap F')^\delta \cap F' = F'$$

so that $t$ is weakly continuous and therefore continuous since $E$ is a Mackey space.

(b) Let $H$ be any $\sigma(F', F)$-dense vector subspace. $F$ is $\delta$-barrelled under $\tau(F, H^\delta)$. The result follows from (a) on considering the identity mapping of $F(\tau(F, H^\delta))$ onto $F$ with its given topology. (See [**1**, proof of $\mathbf{A}_1$].)

(c) This is immediate from Theorem 3 and (b).

THEOREM. 8 (a) *A linear mapping with a closed graph of a $\delta$-space onto a $\delta$-barrelled Mackey space is open.*

(b) *Let $F$ be a separated locally convex space with the property that any linear mapping with a closed graph of $F$ onto a $\delta$-barrelled Mackey space is open. Then $F$ is a $\delta$-space.*

*Proof.* (a) This is deduced from Theorem 7 (a) and Theorem 6 in the usual way. (See also [**1**, proof of **B**].)

(b) Let $H$ be a vector subspace of $F'$. We consider the quotient mapping $q$ of $F$ onto $F/H^0$ when $F/H^0$ has the $\delta$-barrelled topology $\tau(F/H^0, H^\delta)$. As in [**1**, proof of $\mathbf{B}_1$], the hypothesis implies that $q$ is open so that $\tau(F/H^0, H^\delta)$ must be finer than the quotient topology on $F/H^0$, for which the dual is the $\sigma(F', F)$-closure $H^{00}$ of $H$. Then $H^{00} \subseteq H^\delta$ and the result is now immediate.

REMARKS. (i) It is not possible to allow $E$ to be an arbitrary $\delta$-barrelled space in Theorems 7 and 8. Consider for example the Hilbert space $l_2(\Lambda)$ where $|\Lambda| > c$. $l_2(\Lambda)$ is $\delta$-barrelled under $\delta(l_2(\Lambda), l_2(\Lambda))$ (Theorem 2) which is strictly coarser than the norm topology. For any topology of the dual pair $(l_2(\Lambda), l_2(\Lambda))$, a point of closure of a vector subspace $H$ of $l_2(\Lambda)$ is the limit of a sequence in $H$, from which it follows that $l_2(\Lambda)$ is a $\delta$-space for any such topology. We now observe that the identity mapping of $l_2(\Lambda)$ under $\delta(l_2(\Lambda), l_2(\Lambda))$ onto $l_2(\Lambda)$ with its norm topology has a closed graph but it is not continuous. Also its inverse is not open.

(ii) It follows from Theorem 7 (a), [**1**, $A_1$], Example 1 and [**1**, $A_2$] that the infra-$\delta$-spaces form a proper subclass of the class of infra-$s$-spaces.

(iii) In [**14**], Sulley gives an example of an infra-$s$-space which is not weakly $t$-polar. He starts with a Banach space $E$ and a proper $\sigma(E', E)$-dense vector subspace $M$ described in [**2**, p. 121 Ex. 14] and puts them in duality. Since $E$ is clearly separable in its norm topology, we have by Theorem 3, Theorem 7 (b) and Theorem 5 that $E$ is an infra-$\delta$-space for any topology of the dual pair $(E, M)$. Thus infra-$\delta$-spaces need not be weakly $t$-polar.

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UNIVERSITY OF LAGOS
LAGOS
NIGERIA

UNIVERSITY OF STIRLING
STIRLING, FK9 4LA

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