Can. J. Math., Vol. XXXIII, No. 2, 1981, pp. 282–296

DIAMOND PRINCIPLES, IDEALS AND THE NORMAL MOORE SPACE PROBLEM

ALAN D. TAYLOR

1. Introduction. If $\mathscr{X}$ is a topological space then a sequence $(C_\alpha : \alpha < \lambda)$ of subsets of $\mathscr{X}$ is said to be *normalized* if for every $H \subseteq \lambda$ there exist disjoint open sets $\mathscr{U}$ and $\mathscr{V}$ such that

$$\bigcup \{C_\alpha : \alpha \in H\} \subseteq \mathscr{U} \quad \text{and} \quad \bigcup \{C_\alpha : \alpha \in \lambda - H\} \subseteq \mathscr{V}.$$

The sequence $(C_\alpha : \alpha < \lambda)$ is said to be *separated* if there exists a sequence $(\mathscr{U}_\alpha : \alpha < \lambda)$ of pairwise disjoint open sets such that $C_\alpha \subseteq \mathscr{U}_\alpha$ for each $\alpha < \lambda$. As is customary, we adopt the convention that all sequences $(C_\alpha : \alpha < \lambda)$ considered are assumed to be relatively discrete as defined in [**18**, p. 21]: if $x \in C_\alpha$ then there exists a neighborhood about x that intersects no C_β for $\beta \neq \alpha$.

Clearly every separated sequence is normalized. The question of whether every normal Moore space is metrizable has given rise to a general class of questions asking for topological and set theoretic assumptions sufficient to conclude that a given normalized sequence must be separated. In particular, Bing's theorem [**3**] guarantees that a given normal Moore space $\mathscr{X}$ is metrizable if and only if every normalized sequence in $\mathscr{X}$ is separated. For background on the normal Moore space problem see [**15**], [**16**] and [**18**].

Our concern here is with the special case in which $\lambda = \omega_1$ and each C_α is a singleton set $\{x_\alpha\}$, in which case we write $(x_\alpha : \alpha < \omega_1)$ instead of $(\{x_\alpha\} : \alpha < \omega_1)$. For such a sequence $S = (x_\alpha : \alpha < \omega_1)$, the *character of* S, denoted $\chi(S)$, is the least cardinal μ such that each x_α has a neighborhood base $(\mathscr{U}_\xi(x_\alpha) : \xi < \mu)$ of cardinality less than or equal to μ. Notice that if $\chi(S) \leqq \mu$ then S is normalized if, and only if, for every $X \subseteq \omega_1$ there exists a function $f_X : \omega_1 \to \mu$ such that

$$\mathscr{U}_{f(\alpha_1)}(x_{\alpha_1}) \cap \mathscr{U}_{f(\alpha_2)}(x_{\alpha_2}) = 0$$

whenever $\alpha_1 \in X$ and $\alpha_2 \in \omega_1 - X$. Similarly, S is separated if and only if there exists a function $f : \omega_1 \to \mu$ such that

$$\mathscr{U}_{f(\alpha_1)}(x_{\alpha_1}) \cap \mathscr{U}_{f(\alpha_2)}(x_{\alpha_2}) = 0$$

whenever $\alpha_1 \neq \alpha_2$.

We also wish to consider the obvious "σ-versions" of the above notions. That is, a sequence $(x_\alpha : \alpha < \omega_1)$ is *σ-normalized* (*σ-separated*) if there is a

---

Received March 10, 1979 and in revised form June 18, 1979. This research was supported by NSF grant MCS 77-04147.

pairwise disjoint partition $\{A_n : n \in \omega\}$ of ω_1 such that $(x_\alpha : \alpha \in A_n)$ is normalized (separated) for each $n \in \omega$. It is well known (and easy to prove) that if a normalized sequence is σ-separated then it is in fact separated.

In Section 2 we use the Devlin-Shelah weak version of $\blacklozenge$ [**6**] to show that if $2^{\aleph_0} < 2^{\aleph_1}$ then for every σ-normalized sequence $S = (x_\alpha : \alpha < \omega_1)$ of character $\leqq 2^{\aleph_0}$ there exists a stationary set $A \subseteq \omega_1$ such that $(x_\alpha : \alpha \in A)$ is separated. From this we quickly derive several known results related to the normal Moore space problem, including both Šapirovskiĭ's theorem [**17**] asserting the non existence (if $2^{\aleph_0} < 2^{\aleph_1}$) of countable discrete subsets of normal c.c.c. character $2^{\aleph_0}$ spaces, and Devlin and Shelah's recent result [**7**] concerning the non-normality (if $2^{\aleph_0} < 2^{\aleph_1}$) of any special Aronszajn tree.

In Section 3 we collect together several "weak-$\blacklozenge$-type" statements from [**6**] and several topological assertions related to the normal Moore space problem and show that they are equivalent. Many of these equivalences are known, but a few are new and the result in Section 2 allows us to give trivial proofs of implication of the form "weak-$\blacklozenge$-principle $\rightarrow 2^{\aleph_0} < 2^{\aleph_1}$" by appealing to known topological results.

In Section 4 we consider the ideal I_S of separated subsequences of the normalized sequence $S = (x_\alpha : \alpha < \omega_1)$, and we show that the sequence S can always be rearranged so that I_S contains no closed unbounded set. The relevance of this is that if $\blacklozenge_{\omega_1}{}^*$ holds, then the principle $\blacklozenge_{\omega_1}(I)$ (also introduced in Section 4) holds for a countably complete ideal I on ω_1 if and only if I contains no closed unbounded set. Moreover, it is shown here that if $\blacklozenge_{\omega_1}(I)$ holds for every such ideal I, then every normalized sequence $S = (x_\alpha : \alpha < \omega_1)$ of character at most ω_1 is separated. Taken together, these results yield a proof of Shelah's theorem that if $\blacklozenge_{\omega_1}{}^*$ holds, then every normal space of character at most ω_1 is ω_1–collectionwise Hausdorff.

Section 5 contains a proof that if $M[G]$ is obtained from a model M of $V = L$ by adding ω_2 random reals, then in $M[G]$ every normalized sequence of character ω_1 is separated. From this it follows that the normality of the special Aronszajn tree is not equivalent to $2^{\aleph_0} = 2^{\aleph_1}$.

We are grateful to Frank Tall for several discussions concerning the problems considered here. His specific contributions will be pointed out at the appropriate places. We would also like to thank the referee for several suggestions that have been incorporated into this paper.

2. Separating stationary subsequences.

THEOREM 2.1. *Suppose* $2^{\aleph_0} < 2^{\aleph_1}$ *and let* $S = (x_\alpha : \alpha < \omega_1)$ *be a* σ*-normalized sequence such that* $\chi(S) \leqq 2^{\aleph_0}$. *Then* $(x_\alpha : \alpha \in A)$ *is separated for some stationary set* $A \subseteq \omega_1$.

Proof. If $X, Y \subseteq \omega_1$ and $X \neq Y$ then the "discrepancy of X and Y", denoted $\delta(X, Y)$, is the least element of $X \Delta Y$ (i.e., the first α at which the characteristic functions of X and Y disagree). Let θ denote the following assertion:

θ: Suppose that for each $X \subseteq \omega_1$ we have a function $f_X : \omega_1 \to 2^\omega$ and for each $X \subseteq \omega_1$ let

$$G_X = \{\delta < \omega_1 : \exists\, Y \subseteq \omega_1 [\delta = \delta(X, Y) \quad \text{and} \quad f_X|\delta = f_Y|\delta]\}.$$

Then G_X is a stationary subset of ω_1 for some $X \subseteq \omega_1$.

By way of motivation for the assertion θ, consider the following "canonical" way to get a family $\{f_X : X \subseteq \omega_1\}$ of functions mapping ω_1 to 2^ω: for each $\alpha < \omega_1$ let

$$\{X_\beta{}^\alpha : \beta < |\mathscr{P}(\alpha)|\}$$

be an enumeration of $\mathscr{P}(\alpha)$ and for each $X \subseteq \omega_1$ define $f_X : \omega_1 \to 2^\omega$ by $f_X(\alpha) = \beta$ if $X \cap \alpha = X_\beta{}^\alpha$. Then $G_X = \omega_1$ for each $X \subseteq \omega_1$ and so we can regard θ as saying that every collection $\{f_X : X \subseteq \omega_1\}$ of functions mapping ω_1 to 2^ω locally resembles this canonical collection.

The assertion θ occurs in Section 6.1 of [**6**], and they show there that θ holds if $2^{\aleph_0} < 2^{\aleph_1}$. (We will later show that the converse holds also.) Hence, to complete the proof of Theorem 2.1 we need only verify the following:

LEMMA 2.2. *Assume θ holds and let $S = (x_\alpha : \alpha < \omega_1)$ be a σ-normalized sequence such that $\chi(S) \leqq 2^{\aleph_0}$. Then $(x_\alpha : \alpha \in A)$ is separated for some stationary set $A \subseteq \omega_1$.*

Proof. Let $\{A_n : n \in \omega\}$ be a pairwise disjoint partition of ω_1 such that for each $n \in \omega$ $S_n = (x_\alpha : \alpha \in A_n)$ is normalized. For each $\alpha < \omega_1$ let $(\mathscr{U}_\xi(x_\alpha) : \xi < 2^\omega)$ be a neighborhood base for x_α. If $f : \omega_1 \to 2^\omega$ then let $f^\#$ be the function such that

$$f^\#(\alpha) = \mathscr{U}_{f(\alpha)}(x_\alpha).$$

Since each S_n is normalized there exists, for each $X \subseteq \omega_1$, a function $f_X : \omega_1 \to 2^\omega$ such that if $\alpha \in A_n \cap X$ and $\beta \in A_n - X$ (for some $n \in \omega$) then

$$f_X{}^\#(\alpha) \cap f_X{}^\#(\beta) = 0.$$

By θ, there exists $X \subseteq \omega_1$ such that G_X is stationary, and hence there exists some $n \in \omega$ such that $G_X \cap A_n$ is stationary.

Claim. If $\delta \in G_X \cap A_n$ then x_δ is not an element of the closure of $\bigcup \{f_X{}^\#(\alpha) : \alpha \in A_n \cap \delta\}$.

Proof. Suppose not. Since $\delta \in G_X$ there exists a set $Y \subseteq \omega_1$ such that $\delta = \delta(X, Y)$ and $f_X|\delta = f_Y|\delta$. Let $g = f_X|\delta = f_Y|\delta$ and let $Z = X \cap \delta = Y \cap \delta$. Since x_δ is in the closure of $\cup \{f_X^\#(\alpha) : \alpha \in A_n \cap \delta\}$, we have that either x_δ is in the closure of $\cup \{f_X^\#(\alpha) : \alpha \in A_n \cap Z\}$ or x_δ is in the closure of $\cup \{f_X^\#(\alpha) : \alpha \in A_n \cap \delta - Z\}$. If the former holds then we must have $\delta \in X$ and $\delta \in Y$ and if the latter holds then we must have $\delta \notin X$ and $\delta \notin Y$. Either way contradicts the fact that $\delta \in X$ if and only if $\delta \notin Y$ (since $\delta = \delta(X, Y)$), and so the claim is proved.

The lemma, and hence the theorem, now follows easily. That is, we choose X and n such that $G_X \cap A_n$ is stationary. Now, using the claim it is easy to see that there is a function $h : \omega_1 \to 2^\omega$ such that $\delta \in G_X \cap A_n$ and $\alpha \in A_n \cap \delta$ then

$$f_X^\#(\alpha) \cap h^\#(\delta) = 0.$$

For each $\delta \in G_X \cap A_n$ choose $k(\delta) < 2^\omega$ such that

$$k^\#(\delta) \subseteq f_X^\#(\delta) \cap h^\#(\delta).$$

Then clearly $k : G_X \cap A_n \to 2^\omega$ shows that $(x_\alpha : \alpha \in G_X \cap A_n)$ is separated. Since $G_X \cap A_n$ is stationary, this completes the proof.

A sequence $S = (x_\alpha : \alpha < \omega_1)$ in a space $\mathscr{X}$ is called *discrete* (in $\mathscr{X}$) if every point of $\mathscr{X}$ has a neighborhood intersecting at most one point in S. It is easy to see that S is discrete if $\{x_\alpha : \alpha < \omega_1\}$ is a closed set which inherits the discrete topology when considered as a subspace of $\mathscr{X}$. It is well known (and easy to prove) that if a space $\mathscr{X}$ is normal, then every discrete sequence in $\mathscr{X}$ is normalized. The space $\mathscr{X}$ is said to be $\aleph_1$-*compact* if every uncountable subset of $\mathscr{X}$ has a limit point in $\mathscr{X}$. It is easy to see that $\mathscr{X}$ is $\aleph_1$-compact if and only if $\mathscr{X}$ contains no uncountable discrete sequence.

COROLLARY 2.3 [**19**]. *Suppose $2^{\aleph_0} < 2^{\aleph_1}$ and $\mathscr{X}$ is a normal space of character at most $2^{\aleph_0}$. Then $\mathscr{X}$ is weakly ω_1-collectionwise Hausdorff.*

To say $\mathscr{X}$ is weakly ω_1-collectionwise Hausdorff means that for every closed discrete subspace of $\mathscr{X}$ of cardinality ω_1, there exist mutually disjoint open sets about ω_1 of the points of the subspace. As pointed out in [**19**], Corollary 2.3 above is enough to yield the following:

COROLLARY 2.4 [**17**]. *Suppose $2^{\aleph_0} < 2^{\aleph_1}$ and $\mathscr{X}$ is a normal countable chain condition space of character at most $2^{\aleph_0}$. Then $\mathscr{X}$ is $\aleph_1$-compact.*

When we first proved Theorem 2.1 (with "normalized" in place of "σ-normalized"), we were unaware of Devlin and Shelah's result [**7**] that $2^{\aleph_0} < 2^{\aleph_1}$ implies that the special Aronszajn tree is not normal (equivalently: Jones' "road space" is not normal). Frank Tall brought this result to our attention, and pointed out that it can easily be derived from Theorem 2.1. For background, the reader should consult [**7**] or [**16**].

COROLLARY 2.5 [**7**]. *If* $2^{\aleph_0} < 2^{\aleph_1}$ *then the tree topology on a special Aronszajn tree is not normal.*

Proof. Let T be a special Aronszajn tree and identify T with ω_1 so that if α is less than β in the tree ordering $<_T$ then $\alpha < \beta$ and so that if α is a limit ordinal then α occurs at level α of the tree. Since T is special there exists a pairwise disjoint partition $\{A_n : n \in \omega\}$ of ω_1 such that each set A_n is an antichain in T. It is easy to see that each A_n is a discrete subset of T.

Suppose by way of contradiction that T is normal. Then each discrete set A_n is normalized and so ω_1 is σ-normalized. Hence, by Theorem 2.1 some stationary set $A \subseteq \omega_1$ is separated. We can clearly assume that every $\alpha \in A$ is a limit ordinal. Thus, for each $\alpha \in A$ there exists $\beta_\alpha <_T \alpha$ such that if $\alpha_1 \neq \alpha_2$ then

$$(\beta_{\alpha_1}, \alpha_1] \cap (\beta_{\alpha_2}, \alpha_2] = 0.$$

For each $\alpha \in A$ choose ξ_α such that $\beta_\alpha <_T \xi_\alpha <_T \alpha$. Then $\xi_\alpha < \alpha$ so this yields a regressive function which must therefore be constant on a stationary subset of A. This contradicts the fact that A is separated and completes the proof.

3. Equivalents of $2^{\aleph_0} < 2^{\aleph_1}$.

THEOREM 3.1. *The following are equivalent.*

1. $2^{\aleph_0} < 2^{\aleph_1}$.
2. Φ: $\forall F: {}^{\omega_1}2 \to 2 \, \exists \, g : \omega_1 \to 2$ *such that* $\forall f : \omega_1 \to 2 \, \{\alpha < \omega_1 : F(f|\alpha) = g(\alpha)\}$ *is stationary.*
3. Φ_{2^ω}: $\forall F: {}^{\omega_1}2^\omega \to 2 \, \exists \, g : \omega_1 \to 2$ *such that* $\forall f : \omega_1 \to 2^\omega \, \{\alpha < \omega_1 : F(f|\alpha) = g(\alpha)\}$ *is stationary.*
4. Θ: (*as stated in the proof of Theorem* 2.1).
5. *If* $S = (x_\alpha : \alpha < \omega_1)$ *is a* σ*-normalized sequence such that* $\chi(S) \leqq 2^\omega$ *then* $(x_\alpha : \alpha \in A)$ *is separated for some stationary set* $A \subseteq \omega_1$.
6. *If* $S = (x_\alpha : \alpha < \omega_1)$ *is a normalized sequence such that* $\chi(S) \leqq 2^\omega$ *then* $(x_\alpha : \alpha \in A)$ *is separated for some uncountable set* $A \subseteq \omega_1$.
7. *No normal c.c.c. space of character* $\leqq 2^{\aleph_0}$ *has an uncountable discrete subspace.*
8. *Every uncountable set in a separable normal space has a limit point in the space.*

Remark. The assertions Φ, Φ_{2^ω} and θ are due to Devlin and Shelah [**6**] as is the proof that each follows from $2^{\aleph_0} < 2^{\aleph_1}$. The fact that Φ also implies $2^{\aleph_0} < 2^{\aleph_1}$ was noticed independently by several people, including Baumgartner who pointed it out to us. The equivalence of 1, 7 and 8 are

https://doi.org/10.4153/CJM-1981-023-4 Published online by Cambridge University Press

well known. In fact (as pointed out by the referee), it was F. B. Jones' proof [**14**] of $1 \to 8$ that initiated just the kind of investigations that we are pursuing here.

Proof. ($1 \Rightarrow 2$) This is in [**6**].

($2 \Rightarrow 3$). If $f:\alpha \to {}^{\omega}2$ for some $\alpha < \omega_1$ then let $\bar{f}:\alpha \times \omega \to 2$ be given by $\bar{f}(\beta, n) = f(\beta)(n)$. Let $\rho:\omega_1 \times \omega_1 \to \omega_1$ be Gödel's pairing function and let C be a closed unbounded subset of ω_1 such that if $\alpha \in C$ and $\beta_1, \beta_2 < \alpha$ then $\rho(\beta_1, \beta_2) < \alpha$. Let $\bar{C} = C \cup \{\omega_1\}$. Suppose now that

$$F: \bigcup \{{}^{\alpha}({}^{\omega}2):\alpha < \omega_1\} \to 2$$

is given. If $\alpha \in \bar{C}$ and $f:\alpha \to {}^{\omega}2$ then let $f':\alpha \to 2$ be defined by

$$f'(\beta) = \begin{cases} \bar{f}(\xi, \eta) & \text{if } \rho(\xi, \eta) = \beta \text{ for some } \xi < \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Now define $F': \bigcup \{{}^{\alpha}2:\alpha < \omega_1\} \to 2$ by $F'(g) = 1$ if $\exists\, \alpha \in C$ and $f:\alpha \to {}^{\omega}2$ such that $g = f'$ and $F(f) = 1$. Let $g:\omega_1 \to 2$ be for F' as guaranteed to exist by Φ. We claim that g also works for F. To see this, suppose $f:\omega_1 \to 2^{\omega}$ and consider f'. Then there exists a stationary set S such that $F'(f'|\alpha) = g(\alpha)$ for every $\alpha \in S$. Let $S' = S \cap C$. Then for $\alpha \in S'$ we have

$$g(\alpha) = F'(f'|\alpha) = F(f|\alpha).$$

($3 \Rightarrow 4$). This is in Section 6.1 of [**6**].

($4 \Rightarrow 5$). This is Lemma 2.2.

($5 \Rightarrow 6$). This is trivial.

($6 \Rightarrow 7$). This follows from the proof of Corollary 2.3.

($7 \Rightarrow 8$). This is easy, given the observation that a separable normal space has c.c.c. and character $\leqq 2^{\omega}$.

($8 \Rightarrow 1$). This is a well known theorem [**12**]. (See also [**4**]).

Of course, several other equivalents could be added to the list. For example, if we let θ' be the result of changing the word "stationary" in θ to "uncountable," then the proof of Lemma 2.2 goes through to show that $\theta' \Rightarrow 6$. Hence $\theta \Rightarrow \theta' \Rightarrow 6 \Rightarrow \theta$ and so θ and θ' are equivalent.

4. The ideal of separated subsequences. Let $\mathscr{S}_{\kappa}$ denote the collection of all normalized sequences of length ω_1 and of character at most κ. If $S = (x_{\alpha}:\alpha < \omega_1)$ is in $\mathscr{S}_{\kappa}$ then we define the *ideal* I_S *of separated subsequences of* S by

$$I_S = \{X \subseteq \omega_1: (x_{\alpha}:\alpha \in X) \text{ is separated}\}.$$

It is relatively straightforward to check that I_S is a countably complete ideal on ω_1 containing all singletons, and clearly I_S is proper if and only if S fails to be separated.

https://doi.org/10.4153/CJM-1981-023-4 Published online by Cambridge University Press

The ideal I_S provides a rough measure of the extent to which S is or is not separated. For example, the results in Sections 2 and 3 show that the following are both equivalent to $2^{\aleph_0} < 2^{\aleph_1}$.

(1) For every $S \in \mathscr{S}_{2^\omega}$ the ideal I_S contains an uncountable set.

(2) For every $S \in \mathscr{S}_{2^\omega}$ the ideal I_S contains a stationary set.

This motivates the following question. When can we guarantee that for every $S \in \mathscr{S}_{2\omega}$ the ideal I_S contains a closed unbounded set? The answer is given by the following:

THEOREM 4.1. *For any cardinal* κ *and any* $S = (x_\alpha : \alpha < \omega_1) \in \mathscr{S}_{\kappa'}$ *the following are equivalent.*

(1) *S is separated (i.e., I_S fails to be proper).*

(2) *Every rearrangement of S is separated (i.e., $I_{S'}$ fails to be proper whenever $S' = (x_{f(\alpha)} : \alpha < \omega_1)$ and $f : \omega_1 \to \omega_1$ is a bijection).*

(3) *For every rearrangement S' of S, the ideal $I_{S'}$ contains a closed unbounded set.*

COROLLARY 4.2. *The following are equivalent.*

(1) *For every $S \in \mathscr{S}_{2^\omega}$ the ideal I_S contains a closed unbounded set.*

(2) *For every $S \in \mathscr{S}_{2^\omega}$ the ideal I_S is improper (and hence S is separated).*

Proof of Theorem 4.1. Clearly (1) implies (2), since the notion of S being separated is really a property of the underlying set $\{x_\alpha : \alpha < \omega_1\}$ and not the particular ordering of the elements that determined the sequence S. Of course (2) implies (3) is obvious, and so the heart of the proof involves showing that (3) implies (1).

Suppose that (1) does not hold, and hence I_S is proper. Notice that every bijection $f : \omega_1 \to \omega_1$ gives rise to both a rearrangement $S' = (x_{f(\alpha)} : \alpha < \omega_1)$ of S and to an isomorph $f_*(I_S)$ of I_S. (The ideal $f_*(I_S)$ is defined to be $\{X \subseteq \omega_1 : f^{-1}(X) \in I_S\}$.) Moreover, it is easy to see that $I_{S'} = f_*(I_S)$. Thus, in order to prove that (3) fails it is sufficient (and necessary) to produce a bijection $f : \omega_1 \to \omega_1$ so that the isomorph $f_*(I_S)$ of I_S contains no closed unbounded set. But Theorem II.3.6(b) of [**2**] guarantees that for every κ-complete proper ideal I on the successor cardinal κ, some isomorph of I contains no closed unbounded set. An appeal to this thus completes the proof.

At the suggestion of the referee, we will include a brief sketch of the ideal theoretic result employed in the proof of Theorem 4.1. This sketch (as well as the upcoming discussion of diamond principles) requires the introduction of a bit of notation and terminology.

If I is an ideal on κ, then I^+ denotes $\mathscr{P}(\kappa) - I$ (the sets of positive I-measure), and I^* denotes $\{X \subseteq \kappa : \kappa - X \in I\}$ (the sets of I-measure one). If $A \in I^+$ then the restriction of I to A is the ideal $I|A = \{X \subseteq \kappa : X \cap A \in I\}$. NS_κ denotes the ideal of non-stationary subsets of κ. Finally, if $f:\kappa \to \kappa$ and $f^{-1}(\{\alpha\}) \in I$ for every $\alpha < \kappa$ then $f_*(I)$ denotes the ideal of subsets X of κ such that $f^{-1}(X) \in I$.

Now for the sketch of the proof of the result used in establishing Theorem 4.1: We have an ideal I on ω_1 and we would like to find an isomorph J of I so that J contains no closed unbounded set. It is not hard to see that this condition on J is equivalent to demanding that no regressive function $g:X \to \omega_1$, where $X \in J^*$ is $<\omega_1$ to 1. Thus, we define an ordering $<_I$ on the set of all $<\omega_1$ to 1 functions mapping ω_1 to ω_1 by $f <_I g$ if and only if

$$\{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} \in I^*.$$

Since I is countably complete, this ordering is well founded and hence we can choose a $<\omega_1$ to 1 function f that is minimal with respect to $<_I$.

Since f is $<\omega_1$ to 1, there is a partition $\{A_n : n \in \omega\}$ of ω_1 such that for each $n \in \omega$, $f|A_n$ is one to one. The minimality of f with respect to $<_I$ now guarantees that there is at least one $n \in \omega$ such that $A_n \in I^+$ and f is $<_{I|A_n}$-minimal for the ideal $I|A_n$. Let $J' = f_*(I|A_n)$. Then

$$I \subseteq I|A_n \cong f_*(I|A_n) = J',$$

where we have used the fact that $f|A_n$ is one to one. It is now easy to check that J' contains no closed unbounded set. Thus, we have shown that some ideal extending I (namely $I|A$) has an isomorph containing no closed unbounded set, and from this it easily follows that I does also. (It should be pointed out that we are glossing over some details involved in passing from one to one functions defined on sets of measure one to actual bijections on ω_1. These details are provided in [**2**].)

There is an alternative motivation for Theorem 4.1 that seems to have some benefit. That is, as remarked above, being separated or normalized is really a property of the set $\{x_\alpha : \alpha < \omega_1\}$ as opposed to the sequence $(x_\alpha : \alpha < \omega_1)$. Thus, we should not really be thinking of the ideal I_S as giving us a measure of the extent to which S fails to be separated, but rather it is the isomorphism type of I_S (i.e., the class of ideals isomorphic to I_S) that more clearly reflects this desired measure. The practical importance of this is that a comparison of the ideals isomorphic to I_S may suggest a particular rearrangement of the sequence S (or a particular ordering of $\{x_\alpha : \alpha < \omega_1\}$ if one wishes to think of starting with the set) that is more suited to one's immediate needs. The following sequence of results is offered as an illustration of this.

Definition 4.3. If I is a countably complete proper ideal on ω_1, then $\blacklozenge_{\omega_1}(I)$ is the assertion that there exists a sequence $(f_\alpha : \alpha < \omega_1)$ such that

https://doi.org/10.4153/CJM-1981-023-4 Published online by Cambridge University Press

$f_\alpha : \alpha \to \alpha$ and for every $f : \omega_1 \to \omega_1$ we have

$$\{\alpha < \omega_1 : f \mid \alpha = f_\alpha\} \in I^+.$$

Notice that $\blacklozenge_{\omega_1}$ is equivalent to $\blacklozenge_{\omega_1}(\mathrm{NS}_{\omega_1})$, and if A is a stationary subset of ω_1, then $\blacklozenge_{\omega_1}(A)$ is equivalent to $\blacklozenge_{\omega_1}$ $(\mathrm{NS}_{\omega_1}|A)$. Our interest in $\blacklozenge_{\omega_1}(I)$ for other ideals stems from the following analogue of Fleissner's result in [**9**].

THEOREM 4.4. *Suppose that $\blacklozenge_{\omega_1}(I)$ holds for every countably complete proper ideal I on ω_1 which does not contain any closed unbounded sets. Then every normalized sequence $S = (x_\alpha : \alpha < \omega_1)$ of character at most ω_1 is separated.*

Proof. Assume that $S = (x_\alpha : \alpha < \omega_1)$ is of character at most ω_1 but not separated. Then, by Theorem 4.1 there exists a bijection $f : \omega_1 \to \omega_1$ such that if $S' = (x_{f(\alpha)} : \alpha < \omega_1)$, then I_S contains no closed unbounded set. For notational simplicity, we assume that f is the identity map, so $S = S'$ and $I_S = I_{S'}$. Notice that the hypothesis of the theorem now guarantees that $\blacklozenge_{\omega_1}(I_S)$ holds.

We now mimic Fleissner's argument in [**9**] to show that S is not normalized. For each $\alpha < \omega_1$ let $\{\mathscr{U}_\xi{}^\alpha : \xi < \omega_1\}$ be a neighborhood base of x_α, and if $f : \alpha \to \omega_1$ for some $\alpha \leqq \omega_1$ and if $\beta < \alpha$ then let $f^\#(\beta) = \mathscr{U}_{f^{\beta}(\beta)}$.

Claim. If $f : \omega_1 \to \omega_1$ and

$$T_f = \{\alpha < \omega_1 : x_\alpha \notin \text{closure } (\bigcup \{ f^\#(\beta) : \beta < \alpha\})\},$$

then $T_f \in I_S$.

Proof. The subsequence $(x_\alpha : \alpha \in T_f)$ can be easily separated by choosing $g(\alpha)$ for each $\alpha \in T_f$ so that

$$\mathscr{U}^\alpha_{g(\alpha)} \cap f^\#(\beta) = 0 \text{ for every } \beta < \alpha,$$

and such that

$$\mathscr{U}^\alpha_{g(\alpha)} \subseteq f^\#(\alpha).$$

(i.e., $(\mathscr{U}^\alpha_{g(\alpha)} : \alpha \in T_f)$ separates $(x_\alpha : \alpha \in T_f)$.)

Now we inductively define disjoint sets $H, K \subseteq \omega_1$ as follows. At stage α we consider the αth function f_α in the $\blacklozenge_{\omega_1}(I_S)$ sequence. If $x_\alpha \notin$ closure $(\bigcup \{ f_\alpha{}^\#(\beta) : \beta < \alpha\})$, then put α in (say) H. If $x_\alpha \in$ closure $(\bigcup \{ f^\#(\beta) : \beta < \alpha\})$ and if $H \cap \alpha$ and $K \cap \alpha$ have already been defined then either

$$x_\alpha \in \text{closure } (\{ f_\alpha{}^\#(\beta) : \beta \in H \cap \alpha\}) \quad \text{or}$$

$$x_\alpha \in \text{closure } (\{ f^\#(\beta) : \beta \in K \cap \alpha\}).$$

https://doi.org/10.4153/CJM-1981-023-4 Published online by Cambridge University Press

If it is the former, then we put $\alpha \in K$, and if it is the latter then we put $\alpha \in H$.

We claim that H and K show that S is not normalized. To see this, suppose $f:\omega_1 \to \omega_1$ and $f^\#(\alpha) \cap f^\#(\beta) = 0$ whenever $\alpha \in H$ and $\beta \in K$. By the claim, $T_f \in I$ so there exists an $\alpha < \omega_1$ such that $\alpha \notin T_f$ and $f|\alpha = f_\alpha$. Since $\alpha \notin T_f$ and $f|\alpha = f_\alpha$ it must have been the case at stage α of our construction that

$$x_\alpha \in \text{closure } (\bigcup \{f_\alpha^\#(\beta):\beta < \alpha\}).$$

But then clearly we arranged things at that stage so that no function extending f_α (in particular, f) could separate H from K.

The question arises, of course, as to whether our choice of a particular rearrangement of S in the first paragraph of the proof is really necessary. That is, why not just restate Theorem 4.4 with the stronger hypothesis that $\blacklozenge_{\omega_1}(I)$ holds for every countably complete ideal I on ω_1? The answer is given by the following:

THEOREM 4.5. (i) *If* $\blacklozenge_{\omega_1}(I)$ *holds, then* I *contains no closed unbounded set.*

(ii) *If* $\blacklozenge_{\omega_1}{}^*$ *holds, then* $\blacklozenge_{\omega_1}(I)$ *holds for every countably complete ideal* I *on* ω_1 *which contains no closed unbounded set.*

Proof. (i) Suppose $C \in I$ where C is closed and unbounded. Define $f:\omega_1 \to \omega_1$ by

$$f(\alpha) = \inf \{\beta \in C:\beta \geqq \alpha\} + 1.$$

We claim that if $f|\alpha:\alpha \to \alpha$ then $\alpha \in C$. That is, if $\alpha = \beta + 1$ then $f(\beta) \geqq \alpha$, and if α is a limit ordinal and $\alpha \notin C$ then $[\beta, \alpha] \cap C = 0$ for some $\beta < \alpha$ and so $f(\beta) \geqq \alpha$. This proves (i).

(ii) The proof here is just a reworking of Kunen's proof in [**13**] that if $\blacklozenge'(E)$ holds then $\blacklozenge(E)$ holds. Since [**13**] has not appeared as far as we know, we will give the proof.

Recall that $\blacklozenge_{\omega_1}{}^*$ asserts the existence of a sequence $(\mathscr{F}_\alpha:\alpha < \omega_1)$ such that $\mathscr{F}_\alpha \subseteq {}^\alpha\alpha$, $|\mathscr{F}_\alpha| \leqq \omega$, and for every $f:\omega_1 \to \omega_1$ we have that $\{\alpha < \omega_1:f|\alpha \in \mathscr{F}_\alpha\}$ contains a closed unbounded subset of ω_1. For each $\alpha < \omega_1$ let $\{f_n^\alpha:n \in \omega\}$ be an enumeration of $\mathscr{F}_\alpha$, and let $(\xi_\alpha:\alpha < \omega_1)$ be the strictly increasing enumeration of all limit ordinals less than ω_1. Define $\Phi:\omega \times \omega \to \omega_1$ by

$$\Phi(\alpha, n) = \xi_\alpha + n,$$

and let

$$C = \{\alpha < \omega_1:\xi_\alpha = \alpha\}.$$

https://doi.org/10.4153/CJM-1981-023-4 Published online by Cambridge University Press

Then C is closed unbounded and hence in I^+ by our assumption. Notice that if $\alpha \in C$ then

$$\Phi|(\alpha \times \omega) : a \times \omega \to \alpha.$$

If $\alpha \in C \cup \{\omega_1\}$ and $h : \alpha \to \omega_1$ then let $\bar{\Phi}h : \alpha \times \omega \to \omega_1$ be given by

$$\Phi h(\beta, n) = h(\Phi(\beta, n)).$$

Notice that if $\alpha \in C$ and $h : \alpha \to \alpha$ then $\bar{\Phi}h : \alpha \times \omega \to \alpha$. The heart of the proof lies in the following:

Claim. $\exists\ n \in \omega\ \forall f : \omega_1 \to \omega_1\ \exists\ g : \omega_1 \times \omega \to \omega_1$ such that $f(\alpha) = g(\alpha, n)$ for every $\alpha < \omega_1$ and

$$\{\alpha \in C : g|(\alpha \times \omega) = \bar{\Phi}f_n{}^\alpha\} \in I^+.$$

Proof. Suppose not, and for each $n \in \omega$ let $f_n : \omega_1 \to \omega_1$ be a counterexample. Define $g : \omega_1 \times \omega \to \omega_1$ by $g(\alpha, n) = f_n(\alpha)$. Define $h : \omega_1 \to \omega_1$ by $h(\beta) = g(\Phi^{-1}(\beta))$. Now since $(\mathscr{F}_\alpha : \alpha < \omega_1)$ is a $\blacklozenge_{\omega_1}{}^*$ sequence, we know that there is a closed unbounded set C' such that if $\alpha \in C'$ then $h|\alpha \in \mathscr{F}_\alpha$. Let $C'' = C \cap C'$ and for each $\eta \in \omega$ let

$$X_n = \{\alpha \in C'' : h|\alpha = f_n{}^\alpha\}.$$

Then $C'' \in I^+$ (since we are assuming that I contains no closed unbounded set) and hence $X_n \in I^+$ for at least one $\eta \in \omega$ (since I is countably complete). But now it is easy to see that g and X_n contradict the fact that f_n was a counterexample, since if $\alpha \in X_n$ then

$$g|\alpha \times \omega = \bar{\Phi}f_n{}^\alpha.$$

That is, if $\alpha \in X_n$ and $\beta < \alpha$ and $k \in \omega$, then

$$g(\beta, k) = g(\Phi^{-1}(\Phi(\beta, k)) = h(\Phi(\beta, k)) = f_n{}^\alpha(\Phi(\beta, k)) = \Phi f_n{}^\alpha(\beta, k).$$

This proves the claim.

Let n be the natural number guaranteed to exist by the claim. The desired $\blacklozenge_{\omega_1}(I)$ sequence $(f_\alpha : \alpha < \omega_1)$ is obtained by setting

$$f_\alpha(\beta) = \bar{\Phi}f_\alpha{}^n(\beta, n) \quad \text{if } \alpha \in C$$

(and set $f_\alpha(\beta) = 0$ if $\alpha \notin C$). To see that this works, suppose $f : \omega_1 \to \omega_1$. Choose $g : \omega_1 \times \omega \to \omega$ as guaranteed to exist by the claim and let

$$X = \{\alpha \in C : g|\alpha \times \omega = \bar{\Phi}f_n{}^\alpha\}.$$

Then $X \in I^+$ and if $\alpha \in X$ and $\beta < \alpha$ then

$$f(\beta) = g(\beta, n) = \bar{\Phi}f_n{}^\alpha(\beta, n) = f_\alpha(\beta)$$

as desired.

COROLLARY 4.6. (Shelah) *If* $\diamondsuit_{\omega_1}{}^*$ *holds then every normalized sequence of character at most* ω_1 *is separated.*

Proof. This follows immediately from Theorem 4.4 and 4.5 (ii).

While we have not seen Shelah's proof of the above result, Fleissner points out in [**11**] that it does not generalize to (say) sequences of length ω_2 unless one assumes that every normalized sequence $(Y_\alpha : \alpha < \omega_1)$ of sets is separated. The same is true of the above proof. That is, we used the fact that every κ-complete proper ideal on the successor cardinal κ is isomorphic to one that contains no closed unbounded subset of κ. This applied in the present situation, since I_S is ω_1-complete regardless of the length of S. Unfortunately, "κ-complete" and "ω_1-complete" fail to coincide if $\kappa > \omega_1$.

5. Some non equivalents of $2^{\aleph_0} < 2^{\aleph_1}$**.** Fleissner has shown [**10**] that MA_{ω_1} implies that every special Aronszajn tree is normal. If the assumption of MA_{ω_1} could be weakened to $2^{\aleph_2} = 2^{\aleph_1}$, then by Devlin and Shelah's theorem we would have another equivalence for our list in Section 3. That this is not the case is a consequence of the following:

THEOREM 5.1. *If* ZF *is consistent then so is* ZFC + $2^{\aleph_0} = 2^{\aleph_1}$ + *every normalized sequence of character* $\leqq \omega_1$ *is separated.*

Proof. Suppose M is a model of ZFC + $V = L$ and let $M[G]$ be the result of adding ω_2 random reals. To show that $M[G]$ is the desired model, we argue by contradiction. Suppose then that in $M[G]$, we have a normalized sequence $S = (x_\alpha : \alpha < \omega_1)$ of character $\leqq \omega_1$ in some space $\mathscr{X}$ such that S is not separated. As pointed out to us by Frank Tall, Theorem 2.1 of [**18**] guarantees that we lose no generality in assuming that $|\mathscr{X}| = \aleph_1$ and that $\mathscr{X}$ has a basis $\mathscr{B}$ of size $\aleph_1$.

Now, by some well known results of Solovay, there exists a set $A \subseteq \omega_1$ such that if $M' = M[A]$ then (i)–(iii) hold.

(i) $M' \models \mathrm{ZFC} + V = L[A]$.
(ii) $\mathscr{X}$, S and $\mathscr{B}$ are elements of M'.
(iii) $M[G]$ is a random real extension $M'[G]$ of M'.

(Proofs of results such as these can be found in Section 5 of Chapter 6 of [**1**].) It is also well known (see [**5**, p. 211]) that if $V = L[A]$ for some $A \subseteq \omega_1$ then $\diamondsuit_{\omega_1}{}^*$ holds. In particular then $\diamondsuit_{\omega_1}{}^*$ holds in M'.

We now work in M'. For each $\alpha < \omega_1$ let $\{\mathscr{U}_\xi{}^\alpha : \xi < \omega_1\}$ be a neighborhood base at x_α. Clearly S is not separated (in M') or else S would be separated in $M'[G']$. Hence, since $\diamondsuit_{\omega_1}{}^*$ holds in M', we can use Corollary 4.6 to conclude that S is not normalized. Let H and K be disjoint subsets of ω_1 that serve as a witness to the fact that S is not normalized.

Now, still in M', choose a term $\dot{f}$ and a forcing condition B such that

$$B \mid\mid\!- \dot{f}: \omega_1 \to \omega_1$$

and if $\alpha \in H$ and $\beta \in K$ then

$$\mathscr{U}^{\alpha}_{\dot{f}(\alpha)} \cap \mathscr{U}^{\beta}_{\dot{f}(\beta)} = 0.$$

Now, for each $\alpha < \omega_1$ let $\{(B_i^{\alpha}, \xi_i^{\alpha}) : i \in I_{\alpha}\}$ be a maximal collection satisfying (i)-(iii) below.

(i) $B_i^{\alpha} \subseteq B$.
(ii) $B_i^{\alpha} \mid\mid\!- \dot{f}(\alpha) = \xi_i^{\alpha}$.
(iii) if $\{i, j\} \in [I_{\alpha}]^2$ then $\xi_i^{\alpha} \neq \xi_j^{\alpha}$.

Since we are dealing with a countable chain condition notion of forcing, we have that $|I_{\alpha}| \leqq \aleph_0$ since B_i^{α} and B_j^{α} must be incompatible if $i \neq j$. Using this, it is easy to see that we get a set $B_{\alpha} \subseteq B$ and a finite set $s_{\alpha} \subseteq \omega_1$ such that

$$\mu(B_{\alpha}) > \tfrac{1}{2}\mu(B) \quad \text{and} \quad B \mid\mid\!- \dot{f}(\alpha) \in s_{\alpha}.$$

Now we define $h: \omega_1 \to \omega_1$ as follows. For $\alpha \in \omega_1$ choose $h(\alpha)$ such that

$$\mathscr{U}^{\alpha}_{h(\alpha)} \subseteq \bigcap \{\mathscr{U}_{\xi}^{\alpha} : \xi \in s_{\alpha}\}.$$

This is possible since s_{α} is finite and $x_{\alpha} \in \mathscr{U}_{\xi}^{\alpha}$ for every $\xi \in s_{\alpha}$.

Since H and K showed S is not normalized, there must exist $\alpha \in H$ and $\beta \in K$ such that

$$\mathscr{U}^{\alpha}_{h(\alpha)} \cap \mathscr{U}^{\beta}_{h(\beta)} \neq 0.$$

Since $\mu(B_{\alpha}) > \frac{1}{2}\mu(B)$ and $\mu(B_{\beta}) > \frac{1}{2}\mu(B)$ we have that $\mu(B_{\alpha} \cap B_{\beta}) > 0$. But clearly

$$B_{\alpha} \cap B_{\beta} \mid\mid\!- \mathscr{U}^{\alpha}_{h(\alpha)} \subseteq \mathscr{U}^{\alpha}_{\dot{f}(\alpha)} \quad \text{and} \quad \mathscr{U}^{\beta}_{h(\beta)} \subseteq \mathscr{U}^{\beta}_{\dot{f}(\beta)}.$$

Hence,

$$B_{\alpha} \cap B_{\beta} \mid\mid\!- \mathscr{U}^{\alpha}_{\dot{f}(\alpha)} \cap \mathscr{U}^{\beta}_{\dot{f}(\beta)} \neq 0.$$

This contradicts the fact that

$$B \mid\mid\!- \mathscr{U}^{\alpha}_{\dot{f}(\alpha)} \cap \mathscr{U}^{\beta}_{\dot{f}(\beta)} = 0,$$

and completes the proof.

COROLLARY 5.2. *If* ZF *is consistent then so is* ZFC $+ 2^{\aleph_0} = 2^{\aleph_1} +$ *no special Aronszajn tree is normal.*

Proof. This follows immediately from the theorem as in the proof of Corollary 2.4.

Frank Tall has pointed out that Theorem 5.1 can be improved to allow normalized sequences of arbitrary cardinality. His argument runs as follows: We proceed by induction on the length κ of the sequence, with the initial case $\kappa = \omega_1$ being handled as in the proof of Theorem 5.1. For the inductive step at regular cardinals one uses a result of Fleissner asserting that the addition of a subset of ω_2 to L preserves diamond for stationary systems for regular $\kappa \geqq \omega_2$. Finally, at singular cardinals one observes that Fleissner's argument in [9] works if $2^{\aleph_0} \leqq \aleph_2$ and for all $\kappa \geqq \omega_1$, $2^\kappa = \kappa^+$.

It should be noted that if one assumes the consistency of a strongly compact cardinal then the consistency of "$2^{\aleph_0} = 2^{\aleph_1}$ + every normalized sequence of character ω_1 is separated" follows from the much stronger recent result of [**15**] asserting (somewhat more than) the validity of the normal Moore space conjecture in Kunen's model of what Nyikos calls the "product measure extension axiom." In fact, both Kunen's model and the one of Theorem 5.1 are obtained by adding random reals (with the proof of the stronger result requiring the addition of a few more).

References

1. J. Baumgartner, *Results and independence proofs in combinatorial set theory*, thesis, University of California at Berkeley (1970).

2. J. Baumgartner, A. Taylor and S. Wagon, *Structural properties of ideals*, to appear in Dissertationes Mathematicae.

3. R. H. Bing, *Metrization of topological spaces*, Can. J. Math. *3* (1951), 175–186.

4. A. Charlesworth, R. Hodel and F. Tall, *On a theorem of Jones and Heath concerning discrete subspaces*, Colloq. Math. *34* (1975), 33–37.

5. K. Devlin, *Aspects of constructibility*, Lecture notes in Mathematics *354* (Springer-Verlag, Berlin-Heidelberg-New York, 1973).

6. K Devlin and S. Shelah, *A weak version of ◆ which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Israel Journal of Math. *29* (1978), 239–247.

7. ——— *A note on the normal Moore space conjecture*, Can. J. Math. *31* (1979), 241–251.

8. ——— *Souslin properties and tree topologies*, Proc. of the London Math. Soc. (to appear).

9. W. Fleissner, *Normal Moore spaces in the constructible universe*, Proc. Amer. Math. Soc. *46* (1974), 294–298.

10. ——— *When is Jones' space normal?* Proc. Amer. Math. Soc. *50* (1975), 375–378.

11. ——— *Current research on Q-sets*, Proc. Bolyai János Society Colloquium on Topology (Budapest, 1978), to appear.

12. R. Heath, *Screenability, pointwise paracompactness and metrization of Moore spaces*, Can. J. Math. *16* (1964), 763–770.

13. R. Jensen and K. Kunen, *Some combinatorial properties of L and V*, mimeograph.

14. F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. *43* (1937), 671–677.

15. P. Nyikos, *A provisional solution to the normal Moore space problem*, Proc. Amer. Math. Soc. (to appear).

16. M. E. Rudin, *Lectures on set theoretic topology*, Regional Conf. Ser. *23* (Amer. Math. Soc., Providence, 1975).

17. B. Šapirovskiĭ, *On separability and metrizability of spaces with Souslin's condition*, Soviet Math. Dokl. *13* (1972), 1633–1638.

18. F. Tall, *Set-theoretic consistency results and topological theorems concerning the normal Moore space conjecture and related problems*, Dissertationes Mathematicae *148* (1977).

19. ——— *Weakly collectionwise Hausdorff spaces*, Topology Proceedings *1* (1976), 295–304.

20. ——— *The normal Moore space problem, topological structures II*, Math. Res. Centre (Amsterdam), to appear.

Union College,
Schenectady, New York