A NOTE ON GROUP EXTENSIONS

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In [2] Hauptfleisch proved that if A, B, H, K are Abelian groups, $\phi : A \to H$ and $\psi : B \to K$ are epimorphisms, then every central group extension G of H by K is homomorphic image of a central loop extension L of A by B. The aim of the present note is to prove (using almost the same argument as in [2])

THEOREM. *Let B, K be any groups, A a left B-module, H a left K-module, $\psi : B \to K$ an epimorphism and $\phi : A \to H$ an onto B-homomorphism. Then every group extension G of H by K which induces on H the given K-module structure is the homomorphic image of a loop extension L of A by B which induces on A the given B-module structure.*

(The B-module structure on H is that induced through ψ.)

PROOF. Let $1 \to H \xrightarrow{i} G \xrightarrow{\alpha} K \to 1$, i the inclusion map, be an extension of H by K which induces the given K-module structure on H. Let $\{u(k)\}_{k \in K}$ be a set of representatives of the elements of K in G with $u(1) = 1$ (1 denotes the identity of the group concerned) and $g : K \times K \to H$ be the corresponding 2-cocycle. Then

$$g(k, 1) = g(1, k) = 1 \qquad \text{for every } k \in K.$$

Also

$$k \cdot h = u(k)hu(k)^{-1}, \qquad k \in K,\ h \in H.$$

$$1 \to \ker\phi \to A \xrightarrow{\phi} H \to 1$$

is a central group extension of the Abelian group $\ker\phi$ by the Abelian group H (in fact it is a B-module extension). Let $\{r(h)\}_{h \in H}$ with $r(1) = 1$ be a set of representatives of the elements of H in A. Define a map $f : B \times B \to A$ by

$$f(b, b') = r(g(\psi(b), \psi(b'))), \qquad b, b' \in B.$$

It is then clear that $f(b, 1) = f(1, b) = 1$ for every $b \in B$. Therefore $L = \{(a, b) \,/\, a \in A, b \in B\}$ with multiplication defined by

$$(a,b)(a',b') = (a(b\cdot a')f(b,b'),bb'), \quad a,a'\in A, \quad b,b'\in B,$$

is a loop containing a normal subgroup isomorphic to A and the loop extension (prolongation in the terminology of [1])

$$1 \to A \xrightarrow{j} L \xrightarrow{\beta} B \to 1 \quad \text{with}$$

$j(a) = (a,1)$, $\beta(a',b) = b$, $a,a'\in A$, $b\in B$, induces the given B-module structure on A ([1], §11). Define a map $\theta : L \to G$ by

$$\theta(a,b) = \phi(a)u(\psi(b)), \quad a\in A, \quad b\in B.$$

Since ϕ and ψ are epimorphisms and every element of G can be uniquely written as $hu(k)$, $h\in H$, $k\in K$, θ is onto. Again

$$\begin{aligned}\theta((a,b)(a',b')) &= \theta(a(b\cdot a')f(b,b'),bb') = \phi(a(b\cdot a')f(b,b'))u(\psi(bb'))\\ &= \phi(a)(b\cdot\phi(a'))\phi(f(b,b'))u(\psi(b)\psi(b'))\\ &= \phi(a)(b\cdot\phi(a'))g(\psi(b),\psi(b'))u(\psi(b)\psi(b'))\\ &= \phi(a)(u(\psi(b))\phi(a')u(\psi(b))^{-1})u(\psi(b))u(\psi(b'))\\ &= \phi(a)u(\psi(b))\phi(a')u(\psi(b'))\\ &= \theta(a,b)\theta(a',b'), \quad \text{for all} \quad a,a'\in A, \quad b,b'\in B.\end{aligned}$$

Thus θ is a homomorphism.
That $\theta j = i\phi$ and $\psi\beta = \alpha\theta$ are clear.

References

[1] S. Eilenberg and S. MacLane, 'Algebraic cohomology groups and loops', *Duke Math. J.* 14 (1947), 435–463.
[2] G. J. Hauptfleisch, 'A note on central group extensions', *J. Austral. Math. Soc.* 15 (1973), 428–429.

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