ON THE INTEGRAL PART OF A LINEAR FORM WITH PRIME VARIABLES

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The object of this paper is to prove the following:

THEOREM. *Suppose that* λ, μ *are real non-zero numbers, not both negative,* λ *is irrational, and* k *is a positive integer. Then there exist infinitely many primes* p *and pairs of primes* p_1, p_2 *such that*

$$[\lambda p_1 + \mu p_2] = kp.$$

In particular $[\lambda p_1 + \mu p_2]$ *represents infinitely many primes.*

Here $[x]$ denotes the greatest integer not exceeding x. The proof is based on the following

PRINCIPAL LEMMA. *Suppose* λ_1, λ_2, λ_3, γ *are real numbers, the* λ*'s being non-zero and not all of the same sign, and* λ_1/λ_2 *being irrational. Then for every* $\epsilon > 0$ *there exists a sequence of integers* $N \to \infty$ *such that the number of solutions of the inequality*

$$(*) \qquad \begin{cases} |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \gamma| < \epsilon \\ p_i \leqslant N \quad (p_i \text{ prime}) \end{cases}$$

is greater than $CN^2/(\log N)^3$ *where* C *is a positive number independent of* N.

We first deduce the theorem from the lemma. We apply the lemma with

$$\lambda_1 = \lambda, \quad \lambda_2 = -k, \quad \lambda_3 = \mu, \quad \gamma = -\tfrac{1}{2}, \quad \epsilon = \tfrac{1}{2}.$$

This gives for more than $CN^2/(\log N)^3$ triples of primes p_1, p_2, p_3

$$p_i \leqslant N, \qquad kp_2 < \lambda p_1 + \mu p_3 < kp_2 + 1,$$

that is

$$(1) \qquad [\lambda p_1 + \mu p_3] = kp_2.$$

Suppose only finitely many primes p_2, say d, occur in (1). Then for some fixed p_2 (1) has more than $CN^2/d(\log N)^3$ solutions p_1, p_2 ($p_i \leqslant N$). For given p_1 there are at most $|\mu|^{-1} + 1$ primes p_3 satisfying (1) and for p_1 we have at most $C_1 N/\log N$ choices.

Thus

$$(|\mu|^{-1} + 1)\frac{C_1 N}{\log N} \geqslant \frac{CN^2}{d(\log N)^3},$$

which gives a contradiction for sufficiently large N.

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The proof of the principal lemma is based on an adaptation of Vinogradov's method for Goldbach's problem. By replacing the λ_i and γ by $\lambda_i\,\epsilon^{-1}$ and $\gamma\epsilon^{-1}$ it suffices to prove the lemma for $\epsilon = 1$. Since $\lambda_1\,\lambda_3^{-1}$ and $\lambda_2\,\lambda_3^{-1}$ are not both rational, we may further suppose that $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$.

Notation. p indicates primes, $e(\alpha) = e^{2\pi i\alpha}$.

$$S(\alpha) = S(\alpha, N) = \sum_{p \leqslant N} e(\alpha p),$$

$$I(\alpha) = I(\alpha, N) = \int_2^N \frac{e(\alpha x)}{\log x}\,dx,$$

$$K(\alpha) = \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2,$$

$$L = \log N.$$

LEMMA 1. *For real θ,*

$$2\,\mathrm{Re} \int_0^\infty e(\theta\alpha)K(\alpha)\,d\alpha = \begin{cases} 0 & \text{if } |\theta| \geqslant 1, \\ 1 - |\theta| & \text{if } |\theta| \leqslant 1. \end{cases}$$

A proof is given by Davenport and Heilbronn (**1**, Lemma 4).

LEMMA 2. *The number of solutions of the inequality* (*) (*with* $\epsilon = 1$) *is not less than*

$$\mathrm{Re} \int_0^\infty S(\lambda_1\,\alpha)S(\lambda_2\,\alpha)S(\lambda_3\,\alpha)e(\gamma\alpha)K(\alpha)\,d\alpha.$$

Proof. This follows from Lemma 1 in the usual way, by multiplying out the trigonometric sums in the integrand and then interchanging summation and integration.

The remaining lemmas are devoted to establishing a lower bound for the expression in Lemma 2. For ease of reference we shall refer to Prachar's book (**3**).

Throughout ρ will denote a positive constant, which will be fixed later. $C, C_1, \ldots$ denote positive numbers independent of N.

LEMMA 3. *If a and q are integers, α is real, and*

$$|\alpha - a/q| < 1/qT_1$$

where

$$T_1 = C_1\,NL^{-\rho}, \qquad (a, q) = 1, \qquad L^\rho < q \leqslant T_1,$$

then

$$|S(\alpha)| < CNL^{(9/2)-\frac{1}{2}\rho}.$$

Proof. This follows at once from (**3**, Chapter VI, Theorem 6.1) (Vinogradov's estimate).

LEMMA 4. *If a and q are integers, $(a, q) = 1$, $1 \leqslant q \leqslant L^{\rho}$, α is real, and*

$$\alpha = (a/q) + \beta, \qquad |\beta| < q^{-1}T_1^{-1}, \qquad T_1 = C_1 NL^{-\rho},$$

then

$$|S(\alpha) - \{\mu(q)/\phi(q)\}I(\beta)| < CN\exp(-AL^{\frac{1}{2}}),$$

where A is an absolute positive constant, and $\mu(q)$ and $\phi(q)$ are the Möbius and Euler functions respectively.

Proof. This follows at once from (**3**, Chapter VI, Theorem 3.3) on replacing

$$\sum_{2\leqslant n\leqslant N} \frac{e\,(n\beta)}{\log n}$$

by $I(\beta)$. The error introduced by this is $O(1 + L^{\rho-1})$, which is negligible.

LEMMA 5.

$$|I(\alpha)| < \begin{cases} CNL^{-1} & \text{all } \alpha, \\ CN^{\frac{1}{2}}L^{-1} & \text{for } |\alpha| \geqslant N^{-\frac{1}{2}}, \\ |\alpha|^{-1}L^{-1} & |\alpha| < N^{-\frac{1}{2}}. \end{cases}$$

Proof. The first estimate follows from $|I(\alpha)| \leqslant |I(0)|$; the second and third are proved by integrating by parts.

LEMMA 6. *The volume of the 3-dimensional body B defined by*

$$|\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3| < \tfrac{1}{2}, \qquad 2 \leqslant x_i \leqslant N \ (i = 1, 2, 3)$$

is greater than CN^2.

Proof. We chose a positive constant δ which satisfies

$$(1 + \tfrac{1}{4})\lambda_1|\lambda_2|^{-1}\delta < 1, \qquad \tfrac{1}{2}\lambda_1|\lambda_3|^{-1}\delta < 1, \qquad (1 + \tfrac{1}{2} + \tfrac{1}{4})\delta < \tfrac{1}{2}.$$

The three intervals

$$\tfrac{1}{4}\delta N - \tfrac{1}{2}\lambda_1^{-1} < x_1 < (1 + \tfrac{1}{4} + \tfrac{1}{2})\delta N + \tfrac{1}{2}\lambda_1^{-1},$$
$$\tfrac{3}{4}\lambda_1|\lambda_2|^{-1}\delta N < x_2 < (1 + \tfrac{1}{4})\lambda_1|\lambda_2|^{-1}\delta N,$$
$$\tfrac{1}{4}\lambda_1|\lambda_3|^{-1}\delta N < x_3 < \tfrac{1}{2}\lambda_1|\lambda_3|^{-1}\delta N$$

lie in $(2, N)$ if N is sufficiently large, and have lengths at least λ_1^{-1}, $C_1 N$, and $C_2 N$ respectively. These inequalities therefore define a box of volume greater than CN^2. Remembering that $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, it is at once verified that for such x_i we also have

$$|\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3| < \tfrac{1}{2},$$

which proves the lemma.

LEMMA 7. *Let*

$$J_1 = \operatorname{Re} \int_0^{T_1^{-1}} S(\lambda_1 \alpha)S(\lambda_2 \alpha)S(\lambda_3 \alpha)e(\gamma\alpha)K(\alpha)\, d\alpha,$$

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where $T_1 = C_1 NL^{-\rho}$. *Then* $J_1 > CN^2L^{-3}$.

Proof. Let $\lambda = \max_i |\lambda_i|$. In the interval of the integral we have

$$|\lambda_i \alpha| \leqslant |\lambda_i| T_1^{-1} \leqslant \lambda C_2 N^{-1} L^{\rho}.$$

Hence, by Lemma 4 with $a = 0$, $q = 1$,

$$|S(\lambda_i \alpha) - I(\lambda_i \alpha)| < CN \exp(-AL^{\frac{1}{2}}) \qquad (i = 1, 2, 3).$$

Since $0 \leqslant K(\alpha) \leqslant 1$, we may replace each $S(\lambda_i \alpha)$ in J_1 by $I(\lambda_i \alpha)$ with an error at most

$$3N^2 T_1^{-1} CN \exp(-AL^{\frac{1}{2}}) < C_3 N^2 L^{\rho} \exp(-AL^{\frac{1}{2}}),$$

since trivially $|S(\lambda_i \alpha)| \leqslant N$, $|I(\lambda_i \alpha)| \leqslant N$.

In the resulting expression

$$\mathrm{Re} \int_0^{T_1^{-1}} I(\lambda_1 \alpha) I(\lambda_2 \alpha) I(\lambda_3 \alpha) e(\gamma\alpha) K(\alpha)\, d\alpha$$

we replace $e(\gamma\alpha)$ by 1 with an error less than

$$\int_0^{T_1^{-1}} N^3 |e(\gamma\alpha) - 1| d\alpha \leqslant |\gamma| N^3 \int_0^{T_1^{-1}} \alpha\, d\alpha < CN^3 T_1^{-2} = C_2 NL^{2\rho}.$$

In the resulting expression

$$J_2 = \mathrm{Re} \int_0^{T_1^{-1}} I(\lambda_1 \alpha) I(\lambda_2 \alpha) I(\lambda_3 \alpha) K(\alpha)\, d\alpha$$

we replace the upper limit of integration by ∞ with an error

$$\int_{T_1^{-1}}^{\infty} |I(\lambda_1 \alpha) I(\lambda_2 \alpha) I(\lambda_3 \alpha)| K(\alpha)\, d\alpha.$$

By Lemma 5 this is less than

$$C\left\{ \int_{T_1^{-1}}^{N^{-\frac{1}{2}}} L^{-3} \alpha^{-3}\, d\alpha + \int_{N^{-\frac{1}{2}}}^{1} N^{3/2} L^{-3}\, d\alpha + \int_1^{\infty} N^{3/2} L^{-3} K(\alpha)\, d\alpha \right\}$$
$$< C\{L^{-3} T_1^2 + L^{-3} N^{3/2} + L^{-3} N^{3/2}\} < CN^2 L^{-3-2\rho}.$$

Thus

$$(2) \qquad |J_1 - J_3| < CN^2 L^{-3-2\rho},$$

where

$$J_3 = \mathrm{Re} \int_0^{\infty} I(\lambda_1 \alpha) I(\lambda_2 \alpha) I(\lambda_3 \alpha) K(\alpha)\, d\alpha.$$

Now

$$J_3 = \operatorname{Re} \int_0^\infty K(\alpha)\left\{ \int_2^N \int_2^N \int_2^N \frac{e(\alpha(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3))}{\log x_1 \log x_2 \log x_3} dx_1\, dx_2\, dx_3 \right\} d\alpha$$

$$= \operatorname{Re} \int_2^N \int_2^N \int_2^N (\log x_1 \log x_2 \log x_3)^{-1}$$

$$\times \int_0^\infty e\{\alpha(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)\} K(\alpha)\, d\alpha\, dx_1\, dx_2\, dx_3$$

$$= \int_2^N \int_2^N \int_2^N (\log x_1 \log x_2 \log x_3)^{-1}$$

$$\times \tfrac{1}{2}\max(0, 1 - |\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3|)\, dx_1\, dx_2\, dx_3$$

by Lemma 1. Hence

$$J_3 \geqslant \frac{1}{4} \int_2^N \int_2^N \int_2^N (\log x_1 \log x_2 \log x_3)^{-1} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3,$$

where f is the characteristic function of the body B defined in Lemma 6. By that lemma

$$J_3 \geqslant \frac{1}{4} L^{-3} \int_2^N \int_2^N \int_2^N f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 > CL^{-3}N^2.$$

This together with (2) gives

$$J_1 = J_3 + (J_1 - J_3) > C_2 N^2 L^{-3}.$$

The next lemma is similar to Lemma 13 in the paper **(2)** by Davenport and Roth.

LEMMA 8. *If M runs through those real numbers for which $M(\log M)^{-(5/2)\rho}$ runs through the denominators of the continued fraction for $\lambda_1/|\lambda_2|$, $N = [M]$, then for*

$$\lambda N^{-1} L^\rho < \alpha < L^3, \qquad \lambda = \max(\lambda_1^{-1}, |\lambda_2|^{-1}),$$

we have $\min(|S(\lambda_1 \alpha)|, |S(\lambda_2 \alpha)|) < CNL^{(9/2)-\frac{1}{2}\rho}$.

Proof. If Q is a denominator of the continued fraction for $\lambda_1/|\lambda_2|$, then by Legendre's law of best approximation

$$|(\lambda_1/\lambda_2)q - a| > \tfrac{1}{2}Q^{-1}$$

for all integers a, q with $1 \leqslant q < Q$; cf. footnote in **(2**, p. 94**)**. For every α there are integers a_i, q_i such that

$$(a_i, q_i) = 1, \qquad |\lambda_i \alpha - (a_i/q_i)| < q_i^{-1} L^\rho N^{-1}, \qquad 1 \leqslant q_i \leqslant NL^{-\rho} \quad (i = 1, 2).$$

Put

$$\beta_i = \lambda_i \alpha - (a_i/q_i).$$

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For the α in the interval under consideration we cannot have $a_1 = 0$ or $a_2 = 0$, for otherwise for $i = 1$ or 2

$$N^{-1}L^{\rho} < |\lambda_i|\alpha < L^{\rho}N^{-1},$$

a contradiction. Further, if $q_i > L^{\rho}$ for $i = 1$ or 2, then the required estimate follows from Lemma 3. Hence we may suppose that $q_i \leqslant L^{\rho}(i = 1, 2)$. Hence by Lemma 4

$$\begin{aligned}|S(\lambda_i \alpha)| &< (1/\phi(q_i))|I(\beta_i)| + CN\exp(-AL^{\frac{1}{2}})\\ &< |\beta_i|^{-1}L^{-1} + CN\exp(-AL^{\frac{1}{2}}) < C_1|\beta_i|^{-1}L^{-1},\end{aligned}$$

by Lemma 5 since $|\beta_i| < q_i^{-1}L^{\rho}N^{-1} < N^{-\frac{1}{2}}$. If the result to be proved is false, it follows from this that

$$NL^{(9/2)-\frac{1}{2}\rho} < C_1|\beta_i|^{-1}L^{-1} \qquad (i = 1, 2). \tag{3}$$

Now

$$(\lambda_1/\lambda_2)a_2 q_1 - a_1 q_2 = q_1 q_2(\beta_1 + \beta_2).$$

Since $q_i \leqslant L^{\rho}$, (3) gives

$$\begin{aligned}|(\lambda_1/\lambda_2)a_2 q_1 - a_1 q_2| &\leqslant q_1 q_2(|\beta_1| + |\beta_2|)\\ &< 2C_1 q_1 q_2 N^{-1}L^{-(11/2)+\frac{1}{2}\rho} < CN^{-1}L^{-(11/2)+(5/2)\rho}.\end{aligned} \tag{4}$$

We put $q = |a_2 q_1|$, $a = \pm a_1 q_2$, the sign being chosen so that

$$|(\lambda_1/\lambda_2)q - a| = |(\lambda_1/\lambda_2)a_2 q_1 - a_1 q_2|.$$

By (4),

$$|(\lambda_1/\lambda_2)q - a| < CN^{-1}L^{-(11/2)+(5/2)\rho},$$

and

$$1 \leqslant q = (|a_2|/q_2)q_1 q_2 < C_2 L^{3+2\rho}$$

since $|\lambda_2 \alpha| < |\lambda_2| L^3$ and a_2/q_2 is near $\lambda_2 \alpha$. Putting

$$Q = M(\log M)^{-(5/2)\rho},$$

we have in particular

$$1 \leqslant q < Q,\ |(\lambda_1/\lambda_2)q - a| < 1/2Q.$$

Since Q is a denominator of the continued fraction for $\lambda_1/|\lambda_2|$, this contradicts Legendre's law of best approximation. Hence (3) is false and the lemma is proved.

LEMMA 9. *If $\lambda \neq 0$, then*

$$\int_0^{\infty} |S(\lambda\alpha)|^2 K(\alpha)\, d\alpha < CNL^{-1}.$$

Proof.

$$|S(\lambda\alpha)|^2 = \sum_{p_1 \leqslant N} \sum_{p_2 \leqslant N} e\{\lambda\alpha(p_1 - p_2)\}.$$

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Hence by Lemma 1, the integral is less than the number of solutions of

$$|\lambda(p_1 - p_2)| < 1, \qquad p_1, p_2 \leqslant N.$$

For given p_1 this gives at most $2|\lambda|^{-1} + 1$ choices for p_2 and hence the number of solutions is at most

$$\pi(N)(2|\lambda|^{-1} + 1) < CNL^{-1}.$$

LEMMA 10. *If N, ρ, and λ are the same as in Lemma 8, then*

$$\int_{\lambda N^{-1}L^{\rho}}^{L^3} |S(\lambda_1 \alpha)S(\lambda_2 \alpha)S(\lambda_3 \alpha)|K(\alpha)\, d\alpha < CN^2L^{(7/2)-\frac{1}{2}\rho}.$$

Proof. By Lemma 8, the integral is less than

$$CNL^{(9/2)-\frac{1}{2}\rho}\left\{\int_0^{\infty} |S(\lambda_2 \alpha)|S(\lambda_3 \alpha)|K(\alpha)\, d\alpha + \int_0^{\infty} |S(\lambda_1 \alpha)S(\lambda_3 \alpha)|K(\alpha) d\alpha\right\}.$$

By Cauchy's inequality, the first of the two integrals is

$$\leqslant \left\{\int_0^{\infty} |S(\lambda_2 \alpha)|^2 K(\alpha)\, d\alpha \int_0^{\infty} |S(\lambda_3 \alpha)|^2 K(\alpha)\, d\alpha\right\}^{\frac{1}{2}} < CNL^{-1}$$

by Lemma 9. The same estimate holds for the second integral and the result follows.

LEMMA 11. *If $A > 4$, then*

$$\int_A^{\infty} |S(\lambda\alpha)|^2 K(\alpha)\, d\alpha \leqslant \frac{16}{A}\int_0^{\infty} |S(\lambda\alpha)|^2 K(\alpha)\, d\alpha.$$

Proof. This is a special case of (**2**, Lemma 2).

LEMMA 12.

$$\int_{L^3}^{\infty} |S(\lambda_1 \alpha)S(\lambda_2 \alpha)S(\lambda_3 \alpha)|K(\alpha)\, d\alpha < CN^2L^{-5}.$$

Proof. By Hölder's inequality the integral is $\leqslant (L_1 L_2 L_3)^{1/3}$, where

$$L_i = \int_{L^3}^{\infty} |S(\lambda_i \alpha)|^3 K(\alpha)\, d\alpha.$$

Using the trivial estimate $|S(\lambda_i \alpha)| < CNL^{-1}$ and Lemma 11,

$$L_i \leqslant CNL^{-1}\int_{L^3}^{\infty} |S(\lambda_i \alpha)|^2 K(\alpha)\, d\alpha \leqslant C_1 NL^{-4}\int_0^{\infty} |S(\lambda_i \alpha)|^2 K(\alpha)\, d\alpha$$

and this by Lemma 9 is $\leqslant CN^2L^{-5}$.

Proof of the Principal Lemma. We shall prove that if N runs through the sequence specified in Lemma 8 and $\rho = 15$, then

$$J \equiv \text{Re} \int_0^{\infty} S(\lambda_1 \alpha)S(\lambda_2 \alpha)S(\lambda_3 \alpha)e(\gamma\alpha)K(\alpha)\, d\alpha > CN^2L^{-3}.$$

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We divide the interval of integration into three subintervals E_1, E_2, E_3 where

$$E_1\colon 0 < \alpha < \lambda N^{-1} L^{\rho},$$

$$E_2\colon \lambda N^{-1} L^{\rho} < \alpha < L^3,$$

$$E_3\colon L^3 < \alpha,$$

where λ is defined in Lemma 8. By Lemma 7, the integral over E_1 is $> CN^2L^{-3}$. The integral over E_2 is in absolute value

$$< CN^2L^{(7/2)-(15/2)} = CN^2L^{-4},$$

by Lemma 10. By Lemma 12 the integral over E_3 is in absolute value less than CN^2L^{-5}. Hence

$$J > CN^2\mathrm{L}^{-3}$$

and this with Lemma 2 proves the Principal Lemma.

One might conjecture that if λ is positive and irrational, then $[\lambda p]$ represents infinitely many primes.

References

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