AN S-CONFIGURATION IN EUCLIDEAN AND ELLIPTIC n-SPACE

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Introduction. "The remarkable analogies which exist between the complete quadrilateral and the desmic system of points suggest that it may be possible to extend the properties considered above to spaces of higher dimensions", remarks Prof. N. A. Court at the end of his paper **(2)**. Here is an attempt in that direction in Euclidean as well as in elliptic 4-space, suggesting extensions in higher spaces. The corresponding figure, called an *S-configuration*, is discussed. Its vertices lie in pairs on the edges of a simplex, separated harmonically by the respective pairs of vertices of the simplex, called its *diagonal simplex* in analogy with the diagonal triangle of a quadrilateral in a plane and a diagonal tetrahedron of a desmic system in a solid. The vertices of the dual of an S-configuration form a closed set of 2^n points w.r.t. their diagonal simplex such that all quadrics for which the simplex is selfpolar, passing through one of them pass through all of them, and each vertex is the harmonic inverse of every other w.r.t. a pair of opposite elements of the simplex. The S-configuration reduces to a cross polytope and its dual to a hypercube reciprocal to this polytope, when a cell of the diagonal simplex recedes to infinity as a selfpolar simplex for the absolute polarity, while the remaining vertex of the simplex (opposite this cell), is the common centre of the polytopes. This is analogous to a desmic system and its conjugate reducing respectively to an octahedron and a cube, when a face of the diagonal tetrahedron recedes to infinity as a selfpolar triangle for the absolute polarity while the remaining vertex of the tetrahedron (opposite this face) is the common centre of the polyhedra. The midpoints of the segments determined by the pairs of opposite vertices of an S-configuration lie in a hyperplane, called its Newtonian hyperplane.

The centres of similitude of a set of hyperspheres, with centres at the vertices of a simplex, taken in pairs, form an S-configuration with the given simplex as its diagonal simplex.

I. SPACE OF FOUR DIMENSIONS

1. Construction.

(a) Let $Q_{\lambda\mu}(\lambda, \mu = A, B, C, D, E, \lambda\mu = \mu\lambda, \lambda \neq \mu)$ be the traces of a given solid on the ten edges $\lambda\mu$ of a given simplex $(S) = ABCDE$, and $P_{\lambda\mu}$ the harmonic conjugates of $Q_{\lambda\mu}$ for the corresponding pairs of vertices of (S).

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The twenty points $P_{\lambda\mu}$, $Q_{\lambda\mu}$ are said to form an S-configuration (S-C) with $P_{\lambda\mu}$, $Q_{\lambda\mu}$ as ten pairs of its opposite vertices and (S) as its *diagonal simplex* in analogy with the diagonal triangle of a complete quadrilateral in a plane and a diagonal tetrahedron of a desmic system.

(b) $k_{\lambda\mu\nu} = Q_{\mu\nu}Q_{\nu\lambda}Q_{\lambda\mu}(\nu = A, B, C, D, E; \lambda, \mu \neq \nu)$ is a triad of collinear points on the line of intersection of the given solid with a plane face $\lambda\mu\nu$ of (S) and then $l_{\lambda\mu\nu} = Q_{\mu\nu}P_{\nu\lambda}P_{\lambda\mu}$, $m_{\lambda\mu\nu} = P_{\mu\nu}Q_{\nu\lambda}P_{\lambda\mu}$, $n_{\lambda\mu\nu} = P_{\mu\nu}P_{\nu\lambda}Q_{\lambda\mu}$ form three other triads of collinear points so that we have a complete quadrilateral **(2)** for which $\lambda\mu\nu$ is the diagonal triangle. Thus there are four triads of collinear points in each plane face of (S) and $4 \times 10 = 40$ in all, such that $(40 \times 3)/20 = 6$ of their forty lines pass through each point, two in each of the three plane faces of (S) through its edge on which lies the point considered.

(c) The ten plane faces of (S) form the diagonal triangles of the ten quadrilaterals determined by the ten tetrads of lines enumerated above. There are forty more quadrilaterals determined by these forty lines. Eight of them lie in each cell of (S) that together with the four in the four plane faces of the cell form a 12_6 configuration of Poncelet and Reye (**3**, p. 473)—twelve points lying by sixes in twelve planes, six planes through each point. They are enumerated in the following five octads of planes.

$$
\begin{aligned}
p_\phi &= k_{\lambda\mu\nu}k_{\mu\nu\theta}k_{\nu\theta\lambda}k_{\theta\lambda\mu}, & p_{\phi'} &= k_{\lambda\mu\nu}n_{\mu\nu\theta}m_{\nu\theta\lambda}l_{\theta\lambda\mu},\\
q_\phi &= l_{\lambda\mu\nu}n_{\mu\nu\theta}n_{\nu\theta\lambda}n_{\theta\lambda\mu}, & q_{\phi'} &= l_{\lambda\mu\nu}k_{\mu\nu\theta}l_{\nu\theta\lambda}m_{\theta\lambda\mu},\\
r_\phi &= m_{\lambda\mu\nu}m_{\mu\nu\theta}m_{\nu\theta\lambda}m_{\theta\lambda\mu}, & r_{\phi'} &= m_{\lambda\mu\nu}l_{\mu\nu\theta}k_{\nu\theta\lambda}n_{\theta\lambda\mu},\\
t_\phi &= n_{\lambda\mu\nu}l_{\mu\nu\theta}l_{\nu\theta\lambda}l_{\theta\lambda\mu}, & t_{\phi'} &= n_{\lambda\mu\nu}m_{\mu\nu\theta}n_{\nu\theta\lambda}k_{\theta\lambda\mu},
\end{aligned}
$$

where $\theta, \phi = A, B, C, D, E; \lambda, \mu, \nu \neq \theta \neq \phi$.

It may be observed here that four of these forty planes pass through each of the forty lines (§1(b)) and twelve through each vertex of the (S-C).

(d) The ten points $Q_{\lambda\mu}$(§1(a)), the ten lines $k_{\lambda\mu\nu}$(§1(b)) and the five planes p_λ(§1(c)) form a Desargues' 10_3 configuration as the intersection of the given solid with the different elements of the simplex (S). There are sixteen such solids containing such configurations, one each. The following list shows that a pair of them pass through each of the forty planes (§1(c)):

$$
\begin{aligned}
s_0 &= p_Ap_Bp_Cp_Dp_E, & s_{0'} &= p_{A'}r_{B'}t_{C'}p_{D'}p_E,\\
s_1 &= q_{A'}t_Br_Cq_{D'}q_E, & s_{1'} &= q_Aq_{B'}q_{C'}q_Dq_E,\\
s_2 &= r_{A'}r_{B'}r_{C'}r_Dr_E, & s_{2'} &= r_Ap_{B'}q_Cr_{D'}r_E,\\
s_3 &= t_Aq_Bp_{C'}t_{D'}t_E, & s_{3'} &= t_{A'}t_{B'}t_Ct_Dt_E,\\
s_4 &= q_{A'}p_{B'}p_{C'}p_Dp_{E'}, & s_{4'} &= q_Ar_Bt_Cp_{D'}p_{E'},\\
s_5 &= p_At_{B'}r_{C'}q_{D'}q_{E'}, & s_{5'} &= p_{A'}q_Bq_Cq_Dq_{E'},\\
s_6 &= t_Ar_{B'}r_Cr_Dr_{E'}, & s_{6'} &= t_{A'}p_Bq_{C'}r_{D'}r_{E'},\\
s_7 &= r_{A'}q_{B'}p_Ct_{D'}t_{E'}, & s_{7'} &= r_At_Bt_{C'}t_Dt_{E'}.
\end{aligned}
$$

It is readily seen that four of these solids pass through each of the forty lines (§1(b)) and eight through each vertex of the (S-C).

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(e) Thus the S-configuration is

$$20(.,6,12,8)40(3,.,4,4)40(6,4,.,2)16(10,10,5,.)$$

in Baker's notation for configurations (1).

2. The dual of an S-configuration.

(a) The dual ($R.S$-C) of an S-configuration is thus

$$16(.,\ 5,\ 10,\ 10)40(2,\ .,\ 4.6)40(4,\ 4,\ .,\ 3)20(8,\ 12,\ 6,\ .),$$

that is, it constitutes sixteen points, forty lines, forty planes, and twenty solids such that:

Through each point pass five lines, ten planes, and ten solids.

Each line contains two points, and four planes and six solids pass through it.

Each plane contains four points and four lines, and three solids pass through it.

Each solid contains eight points, twelve lines, and six planes.

(b) In fact, we start with a point, say $U(1, 1, 1, 1, 1)$, the unit point referred to the given simplex (S), and join it to the ten plane faces of (S) giving the ten solids $u_i = u_j (i \neq j;\ i, j = a, b, c, d, e)$, where $u_i = 0$, $u_j = 0$ represent the five solid faces of (S). These ten solids together with their ten harmonic conjugates $u_i = -u_j$ w.r.t. $u_i = 0$ and $u_j = 0$, two through each plane face of (S) separated harmonically by two of its solid faces through this plane face, constitute the twenty solids of the ($R.S$-C) for which (S) is said to be the *diagonal simplex*.

The sixteen points or rather the sixteen vertices of the ($R.S$-C) are then no other than $(\pm 1,\ \pm 1,\ \pm 1,\ \pm 1,\ 1)$ referred to (S), which form a closed set **(7)** w.r.t. it in the sense that all quadrics, for which (S) is selfpolar, passing through one of them pass through all of them, and every vertex of this configuration is an harmonic inverse of another w.r.t. a pair of opposite elements of (S).

Thus: *The vertices of the dual of an S-configuration form a closed set of sixteen points w.r.t. their diagonal simplex.*

(c) Let the secant from U to the edge DE and the opposite plane face ABC of (S) meet them in P_{DE} and $P_{DE'}$ respectively. P_{DE} is then $(0, 0, 0, 1, 1)$ and $P_{DE'}$ is $(1, 1, 1, 0, 0)$ referred to (S). Let U' be such that $(UP_{DE}U'P_{DE'}) = -1$. U' is then $(-1, -1, -1, 1, 1)$. The harmonic conjugate $Q_{DE}(0, 0, 0, -1, 1)$ of P_{DE} w.r.t. D, E lies in the polar **(9)** solid of U, viz. $s_0 = \sum u_i = 0$ w.r.t. (S). Thus we can construct points like P_{DE} on the remaining nine edges of (S) other than DE and their harmonic conjugates w.r.t. the respective pairs of its vertices, and identify these twenty points with the ten pairs of opposite vertices of the (S-C) (§1(a)). Thus: *The sixteen solids of an S-configuration* (S-C) *are the sixteen polar solids of the sixteen vertices of its dual* ($R.S$-C) *w.r.t. their diagonal simplex such that through every vertex of the* (S-C)

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pass the joins of four pairs of vertices of the (*R.S-C*) *lying in one of its twenty solids corresponding to the vertex of the* (*S-C*) *considered.* For example, through P_{DE} pass the joins of four pairs of vertices $(\pm 1, \pm 1, \pm 1, 1, 1)$—$UU'$ is one—lying in the solid $ABCP_{DE}$.

3. A cross polytope and hypercube.

(a) The six pairs of points $P_{\lambda\mu}$, $Q_{\lambda\mu}(\lambda, \mu \neq E)$ (§1(a)) form a desmic system **(2)** of three tetrahedra such that any two of them are quadruply perspective from the vertices of the third, in the solid face $ABCD$ of the simplex (S). Let the two tetrads of planes ξ_E and $\xi_{E'}$ $(\xi = p, q, r, t)$ (§1(c)) form respectively two tetrahedra $T_{E'}$ and $T_{E''}$. They are readily seen to be the other two diagonal tetrahedra **(10)** of the desmic system considered besides the one $T_E = ABCD$ such that T_E, $T_{E'}$, $T_{E''}$ form the conjugate system **(6)**.

Let the plane ABC recede to infinity, in which case the four lines ψ_{ABC} $(\psi = k, l, m, n)$ (§1(b)) and the six points $P_{\lambda\mu}$, $Q_{\lambda\mu}(\lambda, \mu = A, B, C)$, lying in the same plane, likewise recede to infinity, thus leaving the three pairs of points $P_{D\lambda}$, $Q_{D\lambda}$ respectively on the three edges DA, DB, DC of T_E with D as the common midpoint of their segments.

Let ABC be so chosen that it forms a selfpolar triangle for the circle at infinity. DA, DB, DC then form a rectangular system of axes. Let further the three points $P_{D\lambda}$ be equidistant from D. $P_{D\lambda}$, $Q_{D\lambda}$ then form the three pairs of opposite vertices of an octahedron with ξ_E, $\xi_{E'}$ as the four pairs of its parallel opposite triangular faces. $T_{E'}$, $T_{E''}$ form a *stella octangula* **(3**, p. 378**)** whose vertices then form a cube reciprocal to this octahedron.

Thus: *A desmic system of points and its conjugate one reduce to an octahedron and a cube respectively, when a face of its diagonal tetrahedron recedes to infinity as a selfpolar triangle for the circle at infinity there, with centre at the vertex of the tetrahedron opposite this face.*

(b) Now let the solid $ABCD$ recede to infinity, in which case the eight planes ξ_E, $\xi_{E'}$ (§3(a)), the sixteen lines $\psi_{\lambda\mu\nu}(\lambda, \mu, \nu \neq E)$ (§1(b)) and the twelve points $P_{\lambda\mu}$, $Q_{\lambda\mu}$ (§3(a)), lying in the same solid, likewise recede to infinity, thus leaving the four pairs of points $P_{E\lambda}$, $Q_{E\lambda}$ respectively on the four edges EA, EB, EC, ED of (S) with E as the common midpoint of their segments. The dual (*R.S-C*) of the *S*-configuration (§2(c)) becomes a parallelotope **(5**, p. 122**)** with E as its centre, and $P_{E\lambda}$, $Q_{E\lambda}$ become the centres of the eight parallelepiped faces of it.

Let $ABCD$ be so chosen that it forms a selfpolar tetrahedron for the absolute polarity. EA, EB, EC, ED then form a rectangular system of axes. The (*R.S-C*) is then an orthotope **(5**, p. 123**)**. Further, let the four points $P_{E\lambda}$ be equidistant from E. $P_{E\lambda}$, $Q_{E\lambda}$ then form the four pairs of opposite vertices of a cross polytope β_4**(3**, p. 376**)** with s_i, $s_{i'}$ $(i = 0, \ldots, 7)$ (§1(d)) as the eight pairs of its parallel opposite tetrahedral faces, ξ_λ, $\xi_{\lambda'}$ as its sixteen pairs of parallel opposite triangular faces and $\psi_{E\lambda\mu}$ as its twenty-four edges. The (*R.S-C*) is now a hypercube γ_4 **(5**, p. 123**)** reciprocal to this polytope.

Thus: *The S-configuration reduces to a cross polytope and its dual to a hypercube reciprocal to this polytope, when a solid face of their diagonal simplex recedes to infinity as a selfpolar tetrahedron for the absolute polarity, with centre at the vertex of the simplex opposite this face.*

(c) A hypercube has eight pairs of opposite vertices, sixteen pairs of opposite parallel edges, twelve pairs of opposite parallel plane faces (i.e., twenty-four squares), and four pairs of opposite parallel solid faces (i.e., eight cubes). It may be asked here what happens to the other eight lines, sixteen planes, and twelve solids of the ($R.S$-C) that becomes a hypercube. The answer to this query lies in the enumeration of its eight diagonals joining the eight pairs of its opposite vertices, sixteen central rectangles determined by the sixteen pairs of its opposite parallel edges, and twelve central rectangular parallelepipeds determined by the twelve pairs of its opposite parallel square faces.

(d) When the ($R.S$-C) becomes a hypercube (§3(b)), the four pairs of tetrahedra T_λ, $T_{\lambda'}$ formed by the tetrads of planes ξ_λ, $\xi_{\lambda'}$ (§§3(b), 1(c)) form four *stellae octangulae* (§3(a)) inscribed in four cubes with their common centre at E. These cubes are reciprocal to the four octahedra formed by the four sets of three diagonals of the cross polytope (§3(b)) reciprocal to the hypercube. The twenty-four square faces of these four cubes are readily recognized to be the eight triads of the central square sections of the eight cube faces of the hypercube.

4. The Newtonian solid.

(a) *The ten midpoints of the ten segments determined by the pairs of opposite vertices of an S-configuration lie in a solid, referred to as its Newtonian solid, and form a Desargues'* (10_3) *configuration there.*

We shall refer to this as a *Newton's Theorem.*

(b) Conversely: *If on the edges of a simplex pairs of points are marked harmonic to the respective pairs of its vertices and so that the midpoints of the ten segments so marked lie in a solid, the ten pairs of points marked form an S-configuration.*

This follows from the converse of the Newton's Theorem **(2)** in space. For the midpoints of the six such segments marked on the six edges of the tetrahedron of a solid face of (S) lie in a plane, common to this solid and the solid of the ten points under consideration, leading to the desmic system of the six pairs of points marked in the solid face of (S) considered. Five such systems in the five solid faces of (S) constitute an S-configuration and hence the proposition.

(c) *An S-configuration is determined by its diagonal simplex and its Newtonian solid.*

For a desmic system of points is determined **(2)** by its diagonal tetrahedron in a solid face of (S) and its Newtonian plane common to this solid and the given Newtonian solid.

(d) *The ten harmonic conjugates, w.r.t. the pairs of opposite vertices of an S-configuration, of the points of intersection of the edges of its diagonal simplex with a given transversal solid, lie in a solid.*

This is a projective form of the Newton's Theorem (§4(a)).

(e) *The pairs of points of contact of the pairs of hyperspheres coaxal with a given hypersphere and the circumhypersphere of a given simplex touching its edges form the pairs of opposite vertices of an S-configuration.*

The pairs of points of contact under consideration on each edge of (S) form respectively the united elements or foci of the involution determined by the pairs of its intersections with the family of coaxal hyperspheres considered. They therefore are separated harmonically by the respective pairs of vertices of (S), being the intersections of its edges with its circumhypersphere that belongs to the family. Again the midpoints of their segments evidently lie on the radical solid of the family and hence the proposition (§4(b)).

5. Centres of similitude of five hyperspheres.

(a) *The centres of similitude of five hyperspheres taken two at a time form an S-configuration the diagonal simplex of which has for its vertices the centres of the given hyperspheres.*

The centres of similitude of a pair of hyperspheres are defined in **(8)**, by analogy with those of a pair of spheres, as a pair of points dividing the segment between their centres in the ratio of their radii or as the double points **(10)** of the involution determined by their centres and their limiting points that represent the two zero-hyperspheres belonging to their family of coaxals. Thus:

(i) The centres of similitude of a pair of hyperspheres are the same as those of their great spheres or great circles lying in a solid or a plane through the line of their centres as its sections with them.

(ii) The centres of similitude of three hyperspheres taken two at a time are the same as those of their great spheres lying in a solid through the plane of their centres, or of their great circles in this plane itself, as its sections with them, and therefore form the three pairs of vertices **(10)** of a quadrilateral in this plane such that their centres form its diagonal triangle.

(iii) The centres of similitude of four hyperspheres taken two at a time are the same as those of the great spheres in the solid of their centres as its sections with them, and therefore form the six pairs of vertices **(2)** of a desmic system such that their centres form its diagonal tetrahedron.

(iv) The centres of similitude of five hyperspheres taken two at a time form a configuration such that those of four of them form a desmic system as its section with the solid of the centres of the four hyperspheres considered and hence the proposition (§3(a)).

(b) Conversely: *With the vertices of the diagonal simplex of an S-configuration as centres five hyperspheres may be drawn so that the pairs of its opposite vertices will be the centres of similitude of the five hyperspheres taken in pairs.*

We can draw four spheres with centres at the vertices of the diagonal tetrahedron of a desmic system so that the pairs of opposite vertices of the system are the centres of similitude of the four spheres **(2)** taken in pairs. Therefore we can draw four such hyperspheres and consequently five hyperspheres satisfying the necessary conditions of the proposition.

(c) *The ten hyperspheres of similtude of five hyperspheres taken in pairs belong to a coaxal net, that is, they are orthogonal to a coaxal family of hyperspheres.*

The hypersphere of similitude **(8)** of a pair of hyperspheres is the one drawn on the join of their centres of similitude as diameter and therefore their centres are a pair of inverse points **(10)** w.r.t. this hypersphere. Hence any hypersphere through the centres of a pair of given hyperspheres is orthogonal to their hypersphere of similitude which is coaxal with them and therefore orthogonal to any hypersphere orthogonal to them. Thus the ten hyperspheres of similitude of five given hyperspheres taken in pairs are orthogonal to the one orthogonal to the given five and to the other through their five centres.

6. Elliptic space.[1]

(a) When deriving the elliptic 3-space from a 3-sphere in Euclidean 4-space by identifying antipodal points, it is observed by Coxeter **(3**, p. 478**)** that: 'When antipodal points are identified, the four hexagonal central sections of a cuboctahedron yield the sides of a complete quadrilateral, and the twelve cuboctahedral central sections of $\{3, 4, 3\}$ yield the twelve planes of Reye's configuration.' The Reye's configuration (§1(c)) is identical with a desmic system (§3(a)) that constitutes the regular honeycomb $\partial[3\beta]$ of Coxeter **(3**, p. 478**)** in the elliptic space.

It may be observed here that, when antipodes are identified, the three squares $t_1\beta_2$, which are truncations of the three central squares β_2 of an octahedron β_3, give the twelve diagonals of the six square faces of the cuboctahedron $t_1\beta_3$ which is a truncation of β_3, and yield the three diagonals of the quadrilateral yielded by the four hexagonal central sections of $t_1\beta_3$. In short, the cuboctahedron $t_1\beta_3$ of a Euclidean space yields a complete quadrilateral and its diagonal triangle in an elliptic plane.

Thus the four cuboctahedra which are truncations of the four central octahedra β_3 of a cross polytope β_4 (§3(d)) yield four quadrilaterals whose four diagonal triangles constitute a diagonal tetrahedron of the desmic system yielded by the 24-cell $\{3, 4, 3\}$ or $t_1\beta_4$ **(5**, p. 148**)** while the other eight central cuboctahedrons of $t_1\beta_4$ yield the other eight quadrilaterals of the Reye's configuration, whose diagonal triangles form the eight faces of the other two diagonal tetrahedra (§3(a)) of the desmic system.

In fact, the 24-cell $\{3, 4, 3\} = t_1\beta_4$ of a Euclidean 4-space yields, in elliptic 3-space, a desmic system of three quadruply perspective tetrahedra and its conjugate formed by its three such diagonal tetrahedra.

[1]The idea of elliptic space was suggested by the referee.

(b) Following the above chain of argument we may now expect and observe that: *The truncation $t_1\beta_5$* **(4)** *of a Euclidean* 5-*dimensional cross polytope β_5 yields an S-configuration* (*S-C*) *and its diagonal simplex* (*S*) *in an elliptic* 4-*space, when antipodal points are identified.* For the five 24-cells $\{3, 4, 3\}$ which are truncations of the five central cross polytopes β_4 of β_5(**5**, p. 136**)** yield five desmic systems (§3(a)) which constitute the (*S-C*) derived from $t_1\beta_5$, and the ten cuboctahedra which are truncations of the ten central octahedra β_3 of β_5 yield ten quadrilaterals whose ten diagonal triangles (§1(c)) constitute the simplex (*S*) (§1(a)), and whose forty sides give the forty triads (§1(b)) of collinear points of the (*S-C*). The said ten central cuboctahedra of $t_1\beta_5$ lie by fours in each central $t_1\beta_4$ of $t_1\beta_5$, the other five octads of central cuboctahedra (one octad in each central $t_1\beta_4$) of $t_1\beta_5$ yield the forty quadrilaterals (§1(c)) of the (*S-C*). Finally the twenty diagonals of $t_1\beta_5$ yield the twenty vertices (§1(a)) of the (*S-C*) and the sixteen pairs of parallel truncations $t_1\alpha_4$ of the sixteen pairs of parallel opposite cells α_4 of β_5 yield the sixteen (§1(d)) Desargues' 10_3 configurations of the (*S-C*). For one $t_1\alpha_4$ consists of five octahedra which are truncations of the five tetrahedral faces of an α_4, that contain twenty triangles and fifteen squares; thus the elements of a pair of parallel $t_1\alpha_4$ will constitute a pentad of cuboctahedra which yield a pentad of quadrilaterals of the Desargues' configuration.

For further elucidation we work out some details of the 5-dimensional polytope $t_1\beta_5$ following Coxeter (**5**, pp. 145-48, 158, 197-202**)** as follows. It is denoted by

•——⊙——•——•—4—• or $\left\{\begin{matrix}3 \\ 3, 3, 4\end{matrix}\right\}$.

Its elements consists of:

3840/(2.48)	= 40 vertices,	• •——•—4—•
3840/(2.8)	= 240 edges,	⊙ •—4—•
3840/(6.8)	= 80 triangles,	•——⊙ •—4—•
3840/(6.2)	= 320 triangles,	⊙——• •
3840/24	= 160 tetrahedra $\{3, 3\}$,	⊙——•——•
3840/(24.2)	= 80 octahedra $\left\{\begin{matrix}3 \\ 3\end{matrix}\right\}$,	•——⊙——• •
3840/120	= 32 $t_1\alpha_4$ or $\left\{\begin{matrix}3 \\ 3, 3\end{matrix}\right\}$,	•——⊙——•——•
3840/384	= 10 β_4 or $\{3, 3, 4\}$,	⊙——•——•—4—•

The 40 vertices form its 20 diagonals.

The 240 edges form its 40 central hexagons yielding the forty triads of collinear points of the (*S-C*).

The 80 triangles and the 60 central squares of the 10 β_4 form its first central 10 cuboctahedra yielding the diagonal simplex of the $(S\text{-}C)$.

The 320 triangles and the 240 central squares of the 80 octahedra form its other 40 central cuboctahedra.

The 160 tetrahedra form its 10 β_4.

II. SPACE OF n DIMENSIONS

7. S-configuration.

(a) Analogously (§1(a)) we can define an S-configuration $(S\text{-}C)n$ in an n-dimensional space as one constituted by

$$\binom{n+1}{2}$$

traces of a hyperplane s on the edges of a given n-dimensional simplex $(S)n$ and their harmonic conjugates w.r.t. the respective pairs of vertices on each edge of $(S)n$, which therefore is called analogously the *diagonal* simplex of the $(S\text{-}C)n$.

Evidently: *The section of an n-dimensional S-configuration by an r-dimensional face of its diagonal simplex is an r-dimensional S-configuration there with the r-dimensional simplex of the face considered as its diagonal simplex.*

Thus: *The S-configuration in a solid and a plane face of the diagonal simplex of an n-dimensional S-configuration are the desmic system and the complete quadrilateral there respectively, as their sections with it, of which the diagonal tetrahedron and the diagonal triangle are those in the faces considered.*

(b) On each edge of $(S)n$ there are two points of the $(S\text{-}C)n$, called a pair of its opposite vertices, separated harmonically by the pair of vertices of $(S)n$ on the edge considered. In each plane face of $(S)n$ there are then three pairs of opposite vertices of the $(S\text{-}C)n$ on the three edges of this face, as the three pairs of opposite vertices of its quadrilateral section of the $(S\text{-}C)n$. Thus: *The*

$$2\binom{n+1}{2}$$

vertices of an n-dimensional S-configuration lie by twos on the edges of its diagonal simplex separated harmonically by the corresponding pairs of vertices of the simplex, by threes on four lines in each plane face of the simplex as the six vertices of the quadrilateral formed by them, therefore by threes on

$$4\binom{n+1}{3}$$

lines in all which then constitute the S-configuration. Through each vertex of it there pass two of its lines in each plane face of the simplex and $2(n-1)$ *in all.* For through each edge of $(S)n$ there pass $(n-1)$ of its plane faces and each vertex of the $(S\text{-}C)n$ lies on an edge of $(S)n$.

(c) n independent lines through a point determine an n-dimensional space and $(n - 1)$ of them a hyperplane. If through a vertex of the $(S\text{-}C)n$ we take one of two lines (§7(b)) in each plane face of $(S)n$ through that edge of $S(n)$ on which lies the vertex of the $(S\text{-}C)n$ considered, we obtain 2^{n-1} sets of $(n - 1)$ independent lines that determine 2^{n-1} hyperplanes.

Now if we take any two lines of the said $(n - 1)$ lines determining a hyperplane, we observe two more such lines completing a quadrilateral in their plane, intersecting in a vertex of the $(S\text{-}C)n$ on that edge of $(S)n$ which is the sixth edge of the tetrahedral face of $(S)n$ determined by the two of its plane faces that contain the two lines considered, one in each, and five of its edges or those of its tetrahedral face under consideration (§7(a)). Thus each of the 2^{n-1} hyperplanes through a vertex of the $(S\text{-}C)n$ meets every edge of $(S)n$ in a vertex of the $(S\text{-}C)n$, or in other words, each such hyperplane contains

$$\binom{n+1}{2}$$

vertices of the $(S\text{-}C)n$. Therefore there are

$$2\binom{n+1}{2} \times 2^{n-1} \Big/ \binom{n+1}{2} = 2^n$$

hyperplanes in all, of the type of the given one, viz. (§7(a)), we started with, that constitute the $(S\text{-}C)n$. Thus: *The n-dimensional S-configuration consists of*

$$2\binom{n+1}{2}$$

vertices and 2^n hyperplanes such that through each vertex there pass 2^{n-1} (half the number) of its hyperplanes and each hyperplane contains

$$\binom{n+1}{2}$$

(half the number) of its vertices.

8. Dual of an S-configuration.

(a) The dual $(R.S\text{-}C)n$ of an S-configuration is then one constituted by

$$\binom{n+1}{2}$$

hyperplanes through the $(n - 2)$-dimensional faces of a given simplex $S(n)$ joined to a given point and their harmonic conjugates w.r.t. the respective pairs of cells through each $(n - 2)$- dimensional face of $(S)n$ which therefore is called analogously the *diagonal simplex* of the $(R.S\text{-}C)n$.

(b) We can arrive at the 2^n vertices $(\pm 1, \ldots, \pm 1, 1)$ of the $(R.S\text{-}C)n$, referred to $(S)n$, in the manner we did for a four dimensional $(R.S\text{-}C)$ (§2(b)).

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Thus: *The vertices of an n-dimensional reciprocal of an S-configuration form a closed set of 2^n points, w.r.t. their diagonal simplex, such that all quadrics, for which this simplex is selfpolar, passing through one of them passes through all of them, and each vertex is an harmonic inverse of every other w.r.t. a pair of opposite elements of the simplex* **(7)**.

(c) Extending the idea of polarity **(7)** w.r.t. an n-dimensional simplex we can state (cf. §2(c)) that: *The 2^n hyperplanes of an n-dimensional S-configuration are the polar hyperplanes of the vertices of the reciprocal configuration w.r.t. their diagonal simplex such that through each vertex of the S-configuration there pass the joins of 2^{n-2} pairs of vertices, of the reciprocal, lying in one of its*

$$2\binom{n+1}{2}$$

hyperplanes corresponding to the vertex of the S-configuration considered.

9. Cross polytopes and hypercubes. Following the line of argument above (§3(b)) we are now in a position to state that: *The n-dimensional S-configuration reduces to an n-dimensional cross polytope and its reciprocal to an n-dimensional hypercube reciprocal to this polytope, when a cell of their diagonal simplex recedes to infinity as a selfpolar* $(n - 1)$*-dimensional simplex for the absolute polarity, with centre at the vertex of the simplex opposite this cell.*

The apparent deficiencies of elements of the hypercube as a reduction of an $(R.S\text{-}C)n$ are supplemented by its diagonal elements (§3(c)), those of the cross polytope as a reduction of an $(S\text{-}C)n$ lie at infinity (§3(b)).

10. Newtonian hyperplanes.

(a) *The midpoints of the*

$$\binom{n+1}{2}$$

segments determined by the pairs of opposite vertices of an n-dimensional S-configuration lie in a hyperplane, referred to as its Newtonian hyperplane.

We shall refer to this as a Newton's Theorem. This and the following results of this article and those of the next two articles follow by the method of induction (§4).

(b) Conversely: *If on the edges of an n-dimensional simplex pairs of points are marked harmonic to the respective pairs of its vertices so that the midpoints of the*

$$\binom{n+1}{2}$$

segments so marked lie in a hyperplane, the

$$\binom{n+1}{2}$$

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pairs of points marked form the pairs of opposite vertices of an n-dimensional S-configuration.

(c) *An n-dimensional S-configuration is determined by its diagonal simplex and its Newtonian hyperplane.*

(d) *The*

$$\binom{n+1}{2}$$

harmonic conjugates, w.r.t. the pairs of vertices of an S-configuration, of the points of intersection of the edges of its diagonal simplex with a given transversal hyperplane lie in a hyperplane (projective form of the Newton's Theorem above).

(e) As an immediate application of the above Theorem (§10(b)) we have: *The pairs of points of contact of the pairs of the hyperspheres, coaxal with a given hypersphere and the circumhypersphere of a given simplex (n-dimensional), touching its edges form the pairs of opposite vertices of an n-dimensional S-configuration, the Newtonian hyperplane of which is the radical hyperplane of the family of the coaxal hyperspheres considered.*

11. Centres of similitude of $n + 1$ hyperspheres.

(a) *The centres of similitude of $n + 1$ hyperspheres taken in pairs form an S-configuration, the diagonal simplex of which is the central simplex of the given hyperspheres.*

(b) Conversely: *With the vertices of the diagonal simplex of an n-dimensional S-configuration as centres $(n + 1)$ hyperspheres may be drawn so that the pairs of its opposite vertices will be the centres of similitude of the $n + 1$ hyperspheres taken in pairs.*

(c) *The*

$$\binom{n+1}{2}$$

hyperspheres of similitude of $n + 1$ hyperspheres taken in pairs belong to a coaxal net, that is, they are orthogonal to a coaxal family of hyperspheres.

Extending the argument above (§5(c)) to an n-dimensional space, we may note that the

$$\binom{n+1}{2}$$

hyperspheres of similitude of $(n + 1)$ given hyperspheres taken in pairs are orthogonal to the hypersphere, orthogonal to the given hyperspheres, and to the circumhypersphere of their central simplex, and therefore to the family of coaxals determined by these two hyperspheres.

12. Elliptic space. *When elliptic n-space is derived from an n-sphere in Euclidean $(n + 1)$-space by identifying antipodal points, the $(n + 1)$-dimensional*

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polytope $t_1\beta_{n+1}$ *(which is a truncation of the* $(n+1)$*-dimensional crosspolytope* β_{n+1}*) will yield an n-dimensional S-configuration* (§(6)).

Here $t_1\beta_{n+1}$ is denoted by

$$\cdot\!\!-\!\!\!-\!\!\!-\!\!\odot\!\!-\!\!\!-\!\!\!-\!\!\cdot\;\cdot\;\cdot\!\!-\!\!\!-\!\!\!-\!\!\cdot\!\!-\!\!\!-\!\!\!-\!\!\cdot \quad \text{or} \quad \begin{Bmatrix} 3 & \\ 3, \ldots, 3, 4 \end{Bmatrix}$$

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References

1. H. F. Baker, *Principles of geometry*, vol. 4 (Cambridge, 1940), p. 104.
2. N. A. Court, *Theorems, their converses and their extensions*, Nat. Math. Mag., 17, *5* (1943), 1–7.
3. H. S. M. Coxeter, *Regular honeycombs in elliptic space*, Proc. Lond. Math. Soc. (3), *4* (1954), 471–501.
4. ———, *The polytopes with regular-prismatic vertex figures*, Phil. Trans. Royal Soc. A *229* (1931), 359–60.
5. ———, *Regular polytopes* (London, 1948).
6. Sahib Ram Mandan, *Umbilical projection*, Proc. Ind. Acad. Scs., Sec. A, 15, *1* (1942), 16–17.
7. ———, *Harmonic inversion* (in press).
8. ———, *Altitudes of a simplex in four dimensional space* (in press).
9. ———, *Mutually selfpolar pentads in* S_4, Panjab Uni. Res. Bul., *14* (1951), 31–32.
10. ———, *On four intersecting spheres* (in press).

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