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BOOLEAN ALGEBRA RETRACTS

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A Boolean algebra B is a retract of an algebra A if there exist homomorphisms $f: B \to A$ and $g: A \to B$ such that gf is the identity map B. Some important properties of retracts of Boolean algebras are stated in [**3**, §§ 30, 31, 32]. If A and B are α-complete, and A is α-generated by B, Dwinger [**1**, p. 145, Theorem 2.4] proved necessary and sufficient conditions for the existence of an α-homomorphism $g: A \to B$ such that g is the identity map on B. Note that if α is not an infinite cardinal, B must be equal to A. The dual problem was treated by Wright [**6**]; he assumed that A and B are σ-algebras, and that $g: A \to B$ is a σ-homomorphism, and gave conditions for the existence of a homomorphism $f: B \to A$ such that gf is the identity map.

In this paper we state necessary and sufficient conditions for a subalgebra B of A to be a retract of A. If an algebra B is a retract of every algebra A in which it can be embedded, then B is complete [**3**, p. 143, Corollary 2]. As might be expected from this fact, if B is a subalgebra of A, then B is a retract of A if and only if it satisfies certain completeness properties in A. This is not to say that B need be in any sense complete. For example, for any algebras B and C, B is a retract of the sum of B and C. We conclude the paper with some applications of the main theorem, and an example.

Throughout the paper, we shall assume that B is some fixed subalgebra of the Boolean algebra A. The join, meet, and complement of elements in A will be denoted by $\vee$, $\wedge$, and $(')$, respectively; inclusion will be denoted by $\leqq$. If a and b are elements of A, we set $a + b = (a \wedge b') \vee (b \wedge a')$; that is, the symmetric difference. The zero element is designated by 0, and the unit element by 1. The ideal generated by the element a in A is denoted (a). We abbreviate (0) by $\mathbf{0}$. If $X \subset B$, then $\bigvee_A X$ designates the least upper bound of X in A, while $\bigvee_B X$ stands for the least upper bound of X in B.

We shall be concerned throughout this paper with the set $Z_B = \{a \in A: b \leqq a$ and $b \in B$ implies $b = 0\}$. If a Boolean homomorphism $\theta: A \to B$ is the identity map on B, then $a + \theta(a) \in Z_B$ for all $a \in A$; for if $b \leqq a + \theta(a)$ and $b \in B$, then $b = \theta(b) \leqq \theta(a + \theta(a)) = 0$. Also, given $a \in A$ and $b \in B$, $a + b \in Z_B$ if and only if for each $c \in B$, $c \leqq a$ implies $c \leqq b$, and $c \geqq a$ implies $c \geqq b$. To see this, assume that $a + b \in Z_B$ and $c \leqq a$. Then $c \wedge b' \leqq a \wedge b' \leqq a + b$, proving that $c \wedge b' = 0$; i.e., that $c \leqq b$. Similarly, $c \geqq a$ implies $c \geqq b$. Conversely, suppose that $0 \neq c \leqq a + b$ with $c \in B$. If $a \wedge c \neq 0$, then $c \wedge b' \leqq a$ while $c \wedge b' \nleqq b$; if $a \wedge c = 0$, then $a \leqq (b \wedge c)'$, while $b \nleqq (b \wedge c)'$.

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Again, suppose that there is a homomorphism $\theta: A \to B$ whose restriction to B is the identity. If $X \subseteq B$ and $\bigvee_A X$ exists, then $\bigvee_B X$ must also exist, since for $a = \bigvee_A X$, $a + \theta(a) \in Z_B$ and $\theta(a) = \bigvee_B X$ by the results of the last paragraph. The well-known result [**2**, problem 16.2] that if B is a retract of A and A is complete, B must be complete, follows easily.

We now define another important notation. If D is an ideal of A, let $D_B = \{b \in B: b \wedge d' \in Z_B$ for some $d \in D\}$. It follows easily that D_B is an ideal in B. By this definition, then, $(x)_B = \{b \in B: b \wedge x' \in Z_B\}$.

Recall that an ideal D in B is *complete* if $\bigvee_B X$ exists for each $X \subseteq D$.

LEMMA 1. *Suppose that x_1 and x_2 are elements of A, and that $(x_1)_B$ and $(x_2)_B$ are complete. Then*

(i) *if $x_3 \leqq x_1$, $x_3 \in A$, then $(x_3)_B$ is complete,*

(ii) *$(x_1 \vee x_2)_B$ is complete, and*

(iii) *$(x_1 + x_2)_B$ is complete.*

Proof. (i) Assume that $x_3 \leqq x_1$ and that $X \subseteq (x_3)_B$. For each $b \in X$, $b \wedge x_3' \in Z_B$, and thus $b \wedge x_1' \in Z_B$, proving that $X \subseteq (x_1)_B$. There exists $b_0 \in (x_1)_B$ such that $b_0 = \bigvee_B X$. If there is some $c \in B$ with $c \leqq b_0 \wedge x_3'$, then $b_0 \wedge c' \geqq b$ for all $b \in X$, because $b \wedge c \in B$ and $b \wedge x_3' \in Z_B$. Therefore, $c = 0$; i.e., $b_0 \wedge x_3' \in Z_B$ and $b_0 \in (x_3)_B$.

(ii) Suppose that $X \subseteq (x_1 \vee x_2)_B$, and let $X_i = \{b \in B: b \leqq x$ for some $x \in X$ and $b \wedge x_i' \in Z_B\}$, for $i = 1, 2$. Since $X_i \subseteq (x_i)_B$, let $y_i = \bigvee_B X_i$, $i = 1, 2$. We will show that $y_1 \vee y_2 = \bigvee_B X$. First, assume that $b \in X$. We know that $(b \wedge y_1') \wedge x_2' \in Z_B$ since if for some $c \in B$, $c \leqq (b \wedge y_1') \wedge x_2'$, then $c \leqq b$ implies $c \wedge x_1' \wedge x_2' = c \wedge x_1' \in Z_B$. Therefore, $c \in X_1$, and we must have $c \leqq y_1$, proving $c = 0$. But, since $b \wedge y_1' \in B$, $b \wedge y_1' \in X_2$; i.e., $b \wedge y_1' \leqq y_2$ so that $b \leqq y_1 \vee y_2$. Next, assume that for some $b_0 \in B$, $b_0 \wedge x = 0$ for all $x \in X$. Certainly, $b_0 \wedge z_i = 0$ for all $z_i \in X_i$, and thus $b_0 \wedge y_i = 0$, $i = 1, 2$. We have now shown that $y_1 \vee y_2 = \bigvee_B X$.

Finally, if b_1 is a non-zero element of B, and $b_1 \leqq (y_1 \vee y_2) \wedge (x_1 \vee x_2)'$, then $b_1 \wedge y_i \neq 0$ for some i, say $i = 1$. Thus there is an $x \in X$ such that

$$b_1 \wedge y_1 \wedge x \neq 0 \qquad \text{and} \qquad (b_1 \wedge y_1 \wedge x) \wedge (x_1 \vee x_2)' \in Z_B.$$

Since $b_1 \leqq (x_1 \vee x_2)'$, $b_1 \wedge y_1 \wedge x = 0$, which is impossible.

(iii) This follows immediately from (i) and (ii).

Given $X_1, X_2 \subseteq A$, we say that X_1 is *B-equivalent* to X_2, and write $X_1 \equiv X_2$, if for each $x_1 \in X_1$ there is an $x_2 \in X_2$ and an element $b \in B$ such that $(x_1 + x_2 + b)_B$ is complete, and if for each $x_2 \in X_2$ there is an $x_1 \in X_1$ and an element $b \in B$ such that $(x_2 + x_1 + b)_B$ is complete. It follows from Lemma 1 that this relation is an equivalence relation.

LEMMA 2. *Suppose that $X \subseteq A$. There is a subset $X_1 \subseteq A$ such that $X_1 \equiv X$ and $(X_1) \cap B = 0$, if and only if there is a subset $X_2 \subseteq A$ such that $X_2 \equiv X$ and $(x)_B \cap (X_2)$ is complete for each $x \in X_2$.*

Proof. Note that if $(X_1) \cap B = 0$, then $(x)_B \cap (X_1) = 0$ for all $x \in X_1$, and, of course, the zero ideal is complete.

Assume that there is some $X_2 \subseteq A$ such that $X_2 \equiv X$ and $(x)_B \cap (X_2)$ is complete for each $x \in X_2$. Given some $x \in X_2$, define $b_x = \bigvee_B [(x)_B \cap (X_2)]$, and let $X_1 = \{x \wedge b_x' : x \in X_2\}$. For each $y \in X_2$, define $y^* = y \wedge b_y'$. We shall show that $(X_1) \cap B = 0$. Suppose, instead, that there is a non-zero element $b_0 \in (X_1) \cap B$. Then $b_0 \leqq x$ for some $x \in (X_1)$, with

$$x = x_1^* \vee \ldots \vee x_n^*,$$

where $x_i \in X_2$, $i = 1, \ldots, n$. We proceed by induction on n. If $x = x_1^*$ for some $x_1 \in X_2$, then $b_0 = 0$. Assume that $x_1^* \vee \ldots \vee x_n^* \in Z_B$ for all sets $\{x_i: i = 1, \ldots, n\} \subseteq X_2$, but that $b_0 \leqq x_1^* \vee \ldots \vee x_n^* \vee y^*$ for some set $\{x_i: i = 1, \ldots, n\} \subseteq X_2$ and some element $y \in X_2$. But

$$b_0 \wedge y' \leqq x_1^* \vee \ldots \vee x_n^*,$$

and therefore, $b_0 \wedge y' \in Z_B$ by the induction hypothesis. This shows that $b_0 \leqq b_y$ and thus that $b_0 \leqq x_1^* \vee \ldots \vee x_n^*$; i.e., that $b_0 = 0$.

However, $X_1 \equiv X_2$ because there is a one-to-one correspondence between the elements $y^* \in X_1$ and $y \in X_2$, and $y^* + y = (y \wedge b_y') + y = b_y \wedge y$. Hence $(y^* + y + 0)_B = (b_y \wedge y)_B$ is complete by Lemma 1 (i), because $b_y \wedge y \leqq b_y$ and $(b_y)_B = (y)_B \cap (X_2)$. Since $X_2 \equiv X$, and our relation is transitive, $X_1 \equiv X$. This completes the proof.

We are now ready to state the main theorem. Define the set Y_B to be $\{a \in Z_B: (a)_B \text{ is incomplete}\}$.

THEOREM 3. *The subalgebra B of A is a retract of A if and only if*
(i) *for each $a \in A$ there is some $b \in B$ such that $a + b \in Z_B$, and*
(ii) *there is some $Y \subseteq A$ such that $Y \equiv Y_B$ and $(y)_B \cap (Y)$ is complete for each $y \in Y$.*

Proof. By Lemma 2, (ii) is equivalent to: (ii)′ there is some $Y \subseteq A$ such that $Y \equiv Y_B$ and $(Y) \cap B = 0$.

Suppose that B is a retract of A. There must be a homomorphism $\theta: A \to B$ such that θ is the identity on B. We have already noted that $a + \theta(a) \in Z_B$ for all $a \in A$, and this implies (i).

Let $Y = \{x + \theta(x): x \in Y_B\}$, so that $Y \subseteq \ker \theta$ and $(Y) \cap B = 0$. But $y \in Y$ if and only if $y = x + \theta(x)$ for some $x \in Y_B$, and $(y + x + \theta(x))_B = 0$. Therefore, $Y \equiv Y_B$, proving (ii)′.

Conversely, suppose that (i) and (ii)′ hold. By Zorn's lemma, there is an ideal D maximal with respect to the properties $D \supseteq (Y)$ and $D \cap B = 0$. Clearly, if $a \in A$ and for each $d \in D$, $a \vee d \in Z_B$, then $a \in D$. We wish to prove that for each $x \in Z_B$ there is some $d \in D$ such that $x + d \in B$.

Assume first that $(x)_B$ is complete. Define $X = \{b \in B: b \leqq x \vee d, d \in D\}$, and since $X \subseteq (x)_B$, let $b_0 = \bigvee_B X$. Now, suppose that for some $b \in B$,

$b \leqq (x \wedge b_0') \vee d$, with $d \in D$. Since $b \leqq x \vee d$, $b \leqq b_0$, which shows that $b \leqq d$; therefore $b = 0$. This happens for each $d \in D$; hence $x \wedge b_0' \in D$. Also, suppose that $b \leqq (b_0 \wedge x') \vee d_1$ with $b \in B$, $d_1 \in D$. Take any $y \in X$, and then $y \leqq x \vee d_2$, for some $d_2 \in D$. But

$$\begin{aligned} b \wedge y &\leqq ((b_0 \wedge x') \vee d_1) \wedge (x \vee d_2) \\ &= (b_0 \wedge x' \wedge x) \vee (x \wedge d_1) \vee (b_0 \wedge x' \wedge d_2) \vee (d_1 \wedge d_2) \leqq d_1 \vee d_2, \end{aligned}$$

and therefore $b \wedge y = 0$. This must imply that $b \wedge b_0 = 0$, $b \leqq d_1$, and thus that $b = 0$. Since $b_0 \wedge x' \in D$, we have $b_0 + x \in D$, and therefore it is clear that $x + d \in B$ for some $d \in D$.

Assume now that $(x)_B$ is incomplete. Because of this, $x \in Y_B$, and there is some $y \in Y$ with $(x + y + b)_B$ complete, for an element $b \in B$. Applying the argument of the last paragraph to $(x + y + b)_B$ instead of $(x)_B$, we see that there is some $d \in D$ such that $(x + y + b) + d \in B$; that is, $x + (y + d) \in B$, and $y + d \in D$.

To conclude the proof of the theorem, let $\theta: A \to A/D$ be the natural homomorphism. Since $D \cap B = 0$, θ is one-to-one on B. We claim that $\theta(A) = \theta(B)$. Suppose that $a \in A$. There must be an element $b \in B$ such that $a + b \in Z_B$, and therefore, there must be some $d \in D$ such that $(a + b) + d \in B$. But then, letting $c = a + b + d$, we have

$$\theta(a) = \theta(a + d) = \theta(b + c),$$

with $b + c \in B$.

As an application of the main theorem, we prove that every complete subalgebra of a Boolean algebra A is a retract of A. This is also an immediate consequence of the fact, found in [**5**, Theorem 33.1], that every complete Boolean algebra is injective.

COROLLARY 4. *If a subalgebra B of A is complete, then B is a retract of A.*

Proof. Given $a \in A$, let $b_0 = \bigvee_B\{b \in B: b \leqq a\}$. Then $a + b_0 \in Z_B$. Moreover, $Y_B = \emptyset$.

It is interesting to note that Luxemburg [**4**] showed that Corollary 4 is logically equivalent to the statement that every complete Boolean algebra is injective. He conjectured that these results were independent from both the axiom of choice and the Boolean prime ideal theorem, i.e., that every proper ideal in a Boolean algebra can be extended to a prime ideal.

If Y_B is countable, then we may determine that B is a retract of A without having to construct the entire set Y, defined in Theorem 3, before we check the individual elements.

THEOREM 5. *Suppose that Y_B is countable. The subalgebra B of A is a retract of A if*

(i) *for each* $a \in A$ *there is some* $b \in B$ *such that* $a + b \in Z_B$, *and*
(ii) *for each pair* $x, y \in Z_B$ *either* $\bigvee_B\{b \in B: b \leqq x \vee y\}$ *or*

$$\bigvee_B\{b \in B: b \wedge x = 0\}$$

exists.

Proof. Enumerate Y_B as $\{y_n: n \in \omega\}$, where ω is the first infinite ordinal. Define $z_0 = y_0$, and assume that $\{z_i: i \leqq n\}$ has been constructed such that $z_i = y_i + b_i$, $b_i \in B$, for all $i \leqq n$, and $z_0 \vee \ldots \vee z_n \in Z_B$. Let

$$x_n = z_0 \vee \ldots \vee z_n.$$

If $b_{n+1} = \bigvee_B\{b \in B: b \leqq x_n \vee y_{n+1}\}$ exists, let $z_{n+1} = y_{n+1} + b_{n+1}$, and clearly $z_0 \vee \ldots \vee z_{n+1} \in Z_B$.

Otherwise, let $b_{n+1} = \bigvee_B\{b \in B: b \wedge y_{n+1} \wedge x_n' = 0\}$, and define $z_{n+1} = y_{n+1} + b_{n+1}'$. For each $b \in B$, $b \leqq z_{n+1} \vee x_n$ implies $b \leqq (y_{n+1}' \wedge b_{n+1}') \vee x_n$, and therefore, $b \wedge y_{n+1} \wedge x_n' = 0$. This shows that $b \leqq b_{n+1}$, $b = 0$, and thus $z_0 \vee \ldots \vee z_{n+1} \in Z_B$.

Let $Y = \{z_n: n \in \omega\}$. Clearly, $Y \equiv Y_B$, and $(Y) \cap B = 0$.

Example 6. Let X be the Cartesian product of the ordinals 5 and ω. Define $a(i, n)$ as the pair $\{(i, n), (i + 1, n)\}$, and let B be the finite-cofinite Boolean algebra on the set $\{a(i, n): i = 0, 2, 4; n \in \omega\}$. Let A be the Boolean algebra generated by $B \cup \{x\} \cup \{y\}$, where

$$x = \bigcup\{a(1, n): n \in \omega\} \qquad \text{and} \qquad y = \bigcup\{a(3, n): n \in \omega\}.$$

Then for each $a \in A$ there is some $b \in B$ such that $a + b \in Z_B$, and Y_B is countable. However, B is not a retract of A.

Suppose that $\theta: A \rightarrow B$ is a homomorphism which is the identity map on B. If $\theta(x)$ is finite, then all but a finite number of the elements $\{a(2, n): n \in \omega\}$ are contained in $\theta(y)$, because $\{a(2, n): n \in \omega\} \subseteq \theta(x) \cup \theta(y)$. However, $\{a(0, n): n \in \omega\}$ must be contained in the complement of $\theta(y)$. There is no element in B satisfying both of these conditions. But if $\theta(x)$ is infinite, then $\theta(y)$ must be finite, which also is impossible.

COROLLARY 7. *If* A *is countable and* $\bigvee_B\{b \in B: b \leqq a\}$ *exists for each* $a \in A$, *then* B *is a retract of* A.

REFERENCES

1. Ph. Dwinger, *Retracts in Boolean algebras*, Proc. Sympos. Pure Math., Vol. II, pp. 141–151 (Amer. Math. Soc., Providence, R. I., 1961).
2. ——— *Introduction to Boolean algebras*, Hamburger Mathematische Einzelschriften, Heft 40 (Physica-Verlag, Würzburg, 1961).
3. P. R. Halmos, *Lectures on Boolean algebras* (Van Nostrand, Princeton—Toronto—New York—London, 1963).

4. W. A. J. Luxemburg, *A remark on Sikorski's extension theorem for homomorphisms in the theory of Boolean algebras*, Fund. Math. *55* (1964), 239–247.
5. R. Sikorski, *Boolean algebras*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, New Series, Band 25 (Springer-Verlag, Berlin—Göttingen—Heidelberg—New York, 1964).
6. J. D. M. Wright, *A lifting theorem for Boolean σ-algebras*, Math. Z. *112* (1969), 326–334.

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